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# ON NEARLY PARAKÄHLER MANIFOLDS

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Abstract. The purpose of the present paper is to study on nearly paraKähler manifolds. Firstly, to investigate some properties of the Ricci tensor and the Ricci* tensor of nearly paraKähler manifolds. Secondly, to define a special metric connection with torsion on nearly paraKähler manifolds and present its some properties.

## 1. Introduction

An almost product structure on a $2k$-dimensional smooth manifold $M$ is a $(1,1)$-tensor field $P$ squaring to the identity. In this case, the pair $(M,P)$ is called an almost product manifold. An almost paracomplex manifold is an almost product manifold $(M,P)$ such that the two eigenbundles $T^{+}M$ and $T^{-}M$ associated with the two eigenvalues $\pm 1$ of $P$ have the same rank. The Nijenhuis tensor $N$ of an almost paracomplex structure $P$ is given by

$$N_P(X,Y) = [PX,PY] - P[PX,Y] - P[X,PY] + [X,Y].$$

It is well known that an almost paracomplex structure is integrable if and only if the corresponding Nijenhuis tensor $N$ vanishes. An integrable almost paracomplex structure is a paracomplex structure. For a survey on paracomplex geometry we refer to [1].

An almost paraHermitian manifold consists of a smooth manifold $M$ endowed with an almost paracomplex structure $P$ and a pseudo-Riemannian metric $g$ compatible in the sense that

$$g(PX,Y) = -g(X,PY) \text{ or equivalently } g(PX,PY) = -g(X,Y). \tag{1.1}$$

Note that the metric $g$ is neutral, i.e., it has signature $(k,k)$ and the eigenbundles $T^{\pm}M$ are totally isotropic with respect to $g$. The condition (1.1) also implies that $g$ is hybrid with respect to $P$. The 2-covariant skew-symmetric tensor field $F$ defined by $F(X,Y) = g(PX,Y)$ is the fundamental 2-form of the almost paraHermitian manifold $(M,g,P)$. Recall the defining conditions of some of the classes:

$-N_P = 0$, paraHermitian manifolds,

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$-dF = 0$, almost paraKähler manifolds,

$-\nabla^g P = 0 \Leftrightarrow dF = 0$ and $N_P = 0$, paraKähler manifolds,

$-(\nabla^g_X P)X = 0$, nearly paraKähler manifolds. Here and in the following, let $\nabla^g$ always denote the Levi-Civita connection of the pseudo-Riemannian metric $g$ of an almost paraHermitian manifold.

Nearly paraKähler manifolds have been introduced in [5]. These manifolds naturally appear as one class in the classification of almost paraHermitian manifolds [2, 6], which generalizes type $\mathcal{W}_1$ in the well-known Gray–Hervella classification [4] of almost Hermitian structures. In the present note, we study some properties concerning nearly paraKähler structures on a differentiable manifold endowed with a naturally associated metric which is necessarily of neutral signature. We give some of local and global results in nearly paraKähler manifolds. Throughout this paper, manifolds, tensor fields and connections under consideration are all assumed to be differentiable and of class $C^\infty$.

## 2. Nearly paraKähler manifolds

An almost paraHermitian manifold $(M, g, P)$ is called nearly paraKähler if the almost paraHermitian structure is not paraKähler and satisfies the identity

$$(\nabla^g_X F)(Y, Z) + (\nabla^g_Y F)(X, Z) = 0 \tag{2.1}$$

for any vector fields $X, Y, Z$ on $M$, where $F$ is the fundamental $2-$form of the almost paraHermitian manifold $(M, g, P)$. It is easy to see that the condition (2.1) reduces to

$$(\nabla^g_X P)Y + (\nabla^g_Y P)X = 0 \tag{2.2}$$

for the almost paracomplex structure $P$.

Recall that the Nijenhuis tensor of the almost paracomplex structure $P$ satisfies

$$\begin{aligned} N_P(X, Y) &= [PX, PY] - P[PX, Y] - P[X, PY] + [X, Y] \\ &= -P[(\nabla^g_{PX} P)\, PY - (\nabla^g_{PY} P)\, PX] - P[(\nabla^g_X P)\, Y - (\nabla^g_Y P)\, X] \end{aligned}$$

for vector fields $X, Y, Z$ on $M$. Here we use the formula:

$$\nabla^g_X Y - \nabla^g_Y X = [X, Y] \text{ and } (\nabla^g_X P)\, PY = -P\, (\nabla^g_X P)\, Y.$$

On the nearly paraKähler manifold $(M, g, P)$, the Nijenhuis tensor $N_P$ simplifies to

$$N_P(X, Y) = -4P(\nabla^g_X P)Y \text{ (see also [5])}.$$

When the Nijenhuis tensor $N_P$ vanishes, we obtain $\nabla^g P = 0$ which gives the following proposition.

**Proposition 2.1.** *An integrable nearly paraKähler manifold is always a paraKähler manifold.*

## 2.1. Properties concerning Ricci and Ricci* tensors

Let $(M, g, P)$ be a $2k$-dimensional nearly paraKähler manifold. Coordinate systems in $M$ are denoted $(U, x^i)$, where $U$ is the coordinate neighbourhood and $x^i$, $i = 1, 2, \dots, 2k$ are the coordinate functions. On putting $X = \frac{\partial}{\partial x^i}$ and $Y = \frac{\partial}{\partial x^j}$, the conditions (2.1) and (2.2) can be respectively written in local coordinates as

$$\nabla^g_i F_{jm} + \nabla^g_j F_{im} = 0$$

and

$$\nabla^g_i P^h_j + \nabla^g_j P^h_i = 0.$$

Contraction with respect to $i$ and $h$ in the last equation gives $\nabla^g_i P^i_j = 0$.

**Proposition 2.2.** *In a nearly paraKähler manifold $(M, g, P)$, the Ricci tensor is hybrid with respect to the almost paracomplex structure $P$.*

*Proof.* In a nearly paraKähler manifold $(M, g, P)$, if we apply the Ricci identity to the tensor $P^i_j$, we get

$$\nabla^g_k \nabla^g_j P^h_i - \nabla^g_j \nabla^g_k P^h_i = R^g_{kjm}{}^h P^m_i - R^g_{kji}{}^m P^h_m,$$

where $R^g_{kji}{}^h$ are components of the Riemannian curvature tensor $R^g$. Contracting the above equation with respect to $k$ and $h$, with help of $\nabla^g_i P^i_j = 0$, we obtain

$$\begin{aligned} \nabla^g_h \nabla^g_j P^h_i &= R^g_{jm} P^m_i - R^g_{hji}{}^m P^h_m = R^g_{jm} P^m_i - R^g_{hjil} g^{lm} P^h_m \\ &= R^g_{jm} P^m_i - R^g_{hjil} F^{lh} = R^g_{jm} P^m_i - H_{ji}. \qquad (2.3) \end{aligned}$$

Here $R^g_{jm}$ are the components of the Ricci tensor of $g$, $F^{lh}$ are the contravariant components of the fundamental 2-form $F$ and $H_{ji} = R^g_{hjil} F^{lh}$. The tensor $H_{ji}$ is an anti-symmetric tensor, i.e., $H_{(ji)} = 0$. In fact, by means of $F^{lh} = -F^{hl}$, $R^g_{(hj)il} = R_{hj(il)} = 0$, we have

$$H_{ji} = \frac{1}{2}\left(R^g_{hjil} + R^g_{ljih}\right) F^{lh} = \frac{1}{2}\left(R^g_{hjil} - R^g_{ihlj}\right) F^{lh}$$

and similarly

$$H_{ij} = \frac{1}{2}\left(R^g_{hijl} - R^g_{jhli}\right) F^{lh}$$

from which,

$$H_{ji} + H_{ij} = \left(R^g_{hjil} - R^g_{ihlj} + R^g_{hijl} - R^g_{jhli}\right) F^{lh} = 0.$$

Changing the role $j$ and $i$ in (2.3), we write

$$\nabla_h \nabla_i P^h_j = R^g_{im} P^m_j - H_{ij}. \qquad (2.4)$$

Adding (2.3) to (2.4), we have

$$\nabla_h (\nabla_j P^h_i + \nabla_i P^h_j) = R^g_{jm} P^m_i + R^g_{im} P^m_j - 2H_{(ij)},$$

which gives

$$R^g_{mj} P^m_i = -R^g_{im} P^m_j$$

by virtue of $R^g_{jm} = R^g_{mj}$. □

**Theorem 2.3.** *In a nearly paraKähler manifold $(M, g, P)$, in order that $R^g = -R^{g*}$, it is necessary and sufficient that:*

$$\nabla^{gm} \nabla^g_m F_{ji} = 0,$$

*where $R^g$ and $R^{g*}$ are respectively the Ricci tensor and the Ricci* tensor of $g$ and $F$ is the fundamental 2-form of $(M, g, P)$.*

*Proof.* Let $(M, g, P)$ be a nearly paraKähler manifold. The tensor $R^{g*}$ which is locally expressed as

$$R^{g*}_{ji} = -H_{jm} P^m_i = -R^g_{hjml} F^{lh} P^m_i$$

is called the Ricci* tensor of $M$ [8]. It is easy to see that $R^{g*}_{jm} P^m_i = -H_{ji}$. In the case, using $\nabla^g_j F_{im} = -\nabla^g_j F_{mi} = \nabla^g_m F_{ji}$, (2.3) becomes

$$\begin{aligned} \nabla^g_h \nabla^g_j P^h_i &= R^g_{jm} P^m_i + R^{g*}_{jm} P^m_i, \\ \nabla^g_h \nabla^g_j (g^{mh} F_{im}) &= \left(R^g_{jm} + R^{g*}_{jm}\right) P^m_i, \\ g^{mh} \nabla^g_h \nabla^g_j F_{im} &= \left(R^g_{jm} + R^{g*}_{jm}\right) P^m_i, \\ \nabla^{gm} \nabla^g_m F_{ji} &= \left(R^g_{jm} + R^{g*}_{jm}\right) P^m_i, \end{aligned}$$

which completes the proof. □

As a direct result of the Theorem 2.3, we have:

**Corollary 2.4.** *In an integrable nearly paraKähler manifold $(M, g, P)$, $R^g = -R^{g*}$.*

**Proposition 2.5.** *In a nearly paraKähler manifold $(M, g, P)$, the Ricci* tensor is hybrid with respect to the almost paracomplex structure $P$.*

*Proof.* For the Ricci* tensor in a nearly paraKähler manifold $(M, g, P)$, we have

$$\begin{aligned} R^{g*}_{ji} &= -H_{jm} P^m_i, \\ -R^{g*}_{jn} P^n_m &= H_{jm}. \end{aligned}$$

From this, with help of $H_{(ij)} = 0$, we get

$$\begin{aligned} H_{jm} + H_{mj} &= -R^{g*}_{jn} P^n_m - R^{g*}_{mn} P^n_j, \\ 0 &= R^{g*}_{jn} P^n_m + R^{g*}_{mn} P^n_j. \end{aligned}$$

Since $R^{g*}_{mn} = R^{g*}_{nm}$, we have

$$R^{g*}_{jn} P^n_m = -R^{g*}_{nm} P^n_j.$$

□

Now, consider an almost paraHermitian manifold $(M, g, P)$, if the square norm $\|\nabla^g P\|^2$ of $\nabla^g P$, locally expressed by

$$\|\nabla^g P\|^2 = g^{ij} g^{mn} g_{lk} (\nabla^g P)^l_{im} (\nabla^g P)^k_{jn}$$

is zero, then the almost paraHermitian manifold is an isotropic paraKähler manifold. The isotropic paraKähler structure is analogue of the isotropic Kähler structure originally introduced in [3]. In [7], authors studied the isotropy property of antiKähler Codazzi manifolds. The similar problem can be taken into account for the case of nearly paraKähler manifolds.

**Theorem 2.6.** *In a nearly paraKähler manifold* $(M, g, P)$*, in order that* $R^g = -R^{g*}$*, it is necessary and sufficient that the nearly paraKähler manifold is an isotropic paraKähler, where* $R^g$ *and* $R^{g*}$ *are respectively the Ricci tensor and the Ricci* tensor of* $g$*.*

*Proof.* In a nearly paraKähler manifold $(M, g, P)$, transvecting $\nabla^g_j F_{im} = \nabla^g_m F_{ji}$ with $F^{ji}$, it follows that

$$\left(\nabla^g_j F_{im}\right) F^{ji} = 0$$

from which, by taking covariant derivative $\nabla^g_k$

$$\begin{aligned} \nabla^g_k \left\{ \left(\nabla^g_j F_{im}\right) F^{ji} \right\} &= 0, \\ \left(\nabla^g_k \nabla^g_j F_{im}\right) F^{ji} + \left(\nabla^g_j F_{im}\right) (\nabla^g_k F^{ji}) &= 0, \\ (\nabla^g_k \nabla^g_m F_{ji}) F^{ji} + (\nabla^g_m F_{ji})(\nabla^g_k F^{ji}) &= 0. \end{aligned}$$

Multiplying the above last equation with $g^{km}$, we find

$$\begin{aligned} g^{km} (\nabla^g_k \nabla^g_m F_{ji}) F^{ji} + g^{km} (\nabla^g_m F_{ji})(\nabla^g_k F^{ji}) &= 0, \\ (\nabla^{gm} \nabla^g_m F_{ji}) P^i_n g^{nj} + g^{km} (\nabla^g_m P^l_j g_{li})(\nabla^g_k P^i_n g^{nj}) &= 0, \\ (\nabla^{gm} \nabla^g_m F_{ji}) P^i_n g^{nj} + g^{km} g_{li} g^{nj} (\nabla^g P)^l_{mj} (\nabla^g P)^i_{kn} &= 0, \\ \left(R^g_{jm} + R^{g*}_{jm}\right) P^m_i P^i_n g^{nj} + \|\nabla^g P\|^2 &= 0, \\ \left(R^g_{jm} + R^{g*}_{jm}\right) g^{mj} + \|\nabla^g P\|^2 &= 0. \end{aligned}$$

Hence the proof is complete. □

## 2.2. A special metric connection

In this section, on a nearly paraKähler manifold $(M, g, P)$ we shall consider a special linear connection $\nabla_X Y = \nabla^g_X Y + S(X, Y)$ satisfying some special conditions, where $S$ is a $(1, 2)$-tensor field.

Applying the covariant derivative $\nabla_X$ to the fundamental 2-form $F$, we obtain

$$\begin{aligned} (\nabla_X F)(Y, Z)) &= X\left(F\left(Y, Z\right)\right) - F(\nabla_X Y, Z) - F(Y, \nabla_X Z) \\ &= X\left(F\left(Y, Z\right)\right) - F(\nabla^g_X Y + S(X, Y), Z) \\ &\quad - F(Y, \nabla^g_X Z + S(X, Z)) \end{aligned}$$

$$\begin{aligned}&= X\left(F\left(Y,Z\right)\right) - F(\nabla^g_X Y, Z)\\&\quad - F(S(X,Y),Z) - F(Y,\nabla^g_X Z) - F(Y,S(X,Z))\\&= (\nabla^g_X F)(Y,Z) - g(PS(X,Y),Z) - g(PY,S(X,Z))\\&= (\nabla^g_X F)(Y,Z) + g(S(X,Y),PZ) - g(S(X,Z),PY)\\&= (\nabla^g_X F)(Y,Z) + S_P(X,Y,Z) - S_P(X,Z,Y)\end{aligned}$$

for any vector fields $X, Y, Z$ on $M$, where $S_P(X,Y,Z) = g(S(X,Y),PZ)$. From on now, we consider a special linear connection $\nabla$ which satisfies the following conditions: i) $\nabla F = 0$ and ii) $S_P(X,Y,Z) + S_P(Z,Y,X) = 0$. Hence, we can write the followings:

$$\begin{aligned}(\nabla^g_X F)(Y,Z) + S_P(X,Y,Z) - S_P(X,Z,Y) &= 0,\\(\nabla^g_Y F)\left(Z,X\right) + S_P\left(Y,Z,X\right) - S_P\left(Y,X,Z\right) &= 0,\\(\nabla^g_Z F)\left(X,Y\right) + \ S_P\left(Z,X,Y\right) - S_P\left(Z,Y,X\right) &= 0,\end{aligned}$$

from which

$$\begin{aligned}2S_P\left(X,Y,Z\right) &= (\nabla^g_X F)\left(Y,Z\right) - (\nabla^g_Y F)\left(Z,X\right) + (\nabla^g_Z F)\left(X,Y\right),\\2S_P\left(X,Y,Z\right) &= (\nabla^g_X F)\left(Y,Z\right),\\-2g(PS(X,Y),Z) &= g((\nabla^g_X P)\,Y,Z),\\S(X,Y) &= -\tfrac{1}{2}P\,(\nabla^g_X P)\,Y.\end{aligned}$$

Thus, we get the following theorem.

**Theorem 2.7.** *On a nearly paraKähler manifold $(M,g,P)$, a special linear connection $\nabla$ is the linear connection having the covariant derivative given by*

$$\nabla_X Y = \nabla^g_X Y - \frac{1}{2}P\,(\nabla^g_X P)\,Y$$

*for any vector fields $X, Y$ on $M$.*

The following result follows immediately from Proposition 2.1 and Theorem 2.7.

**Corollary 2.8.** *On an integrable nearly paraKähler manifold $(M,g,P)$, the special linear connection $\nabla$ is equal to $\nabla^g$.*

**Theorem 2.9.** *Let $\nabla$ be the special linear connection on a nearly paraKähler manifold $(M,g,P)$. Then*

i) $(\nabla_X P)Y = 2(\nabla^g_X P)Y$,

ii) $T_\nabla(X,Y) = -P\,(\nabla^g_X P)\,Y$, *where $T_\nabla$ is the torsion tensor of $\nabla$. Moreover, the $(0,3)$ torsion tensor field defined by*

$$T_\nabla(X,Y,Z) = g(T_\nabla\left(X,Y\right),Z)$$

*satisfies*:

$$\begin{aligned}T\left(PX,Y,Z\right) &= T\left(X,PY,Z\right) = -T\left(X,Y,PZ\right) \text{ and}\\T\left(PX,PY,Z\right) &= T\left(PX,Y,PZ\right) = -T\left(X,PY,PZ\right),\end{aligned}$$

iii) $(\nabla_X g)(Y,Z)=0$, *i.e.,* $\nabla$ *is a metric connection with respect to* $g$,

iv) $R_\nabla(X,Y)Z = R^g(X,Y)Z + \frac{1}{2}(\nabla_X T)(Y,Z) - \frac{1}{2}(\nabla_Y T)(X,Z)$

$$+\frac{1}{4}T(X,T(Y,Z)) - \frac{1}{4}T(Y,T(X,Z))$$

*for any vector fields* $X,Y,Z$ *on* $M$.

*Proof.* i)
$$\begin{aligned}(\nabla_X P)Y &= \nabla_X(PY) - P(\nabla_X Y)\\ &= \nabla^g_X(PY) - \frac{1}{2}P(\nabla^g_X P)PY - P\nabla^g_X Y + \frac{1}{2}P^2(\nabla^g_X P)Y\\ &= \nabla^g_X(PY) - P\nabla^g_X Y + (\nabla^g_X P)Y\\ &= 2(\nabla^g_X P)Y.\end{aligned}$$

Here we used $(\nabla^g_X P)PY = -P(\nabla^g_X P)Y$.

ii)

$$\begin{aligned}T_\nabla(X,Y) &= \nabla_X Y - \nabla_Y X - [X,Y]\\ &= \nabla^g_X Y - \frac{1}{2}P(\nabla^g_X P)Y - \nabla^g_Y X\\ &\quad + \frac{1}{2}P(\nabla^g_Y P)X - [X,Y]\\ &= -\frac{1}{2}P[(\nabla^g_X P)Y - (\nabla^g_Y P)X]\\ &= -P(\nabla^g_X P)Y.\end{aligned} \tag{2.5}$$

Using the above equation, it follows that

$$\begin{aligned}T(PX,Y,Z) &= g(T(PX,Y),Z) = g(PT(X,Y),Z)\\ &= -g(T(X,Y),PZ) = -T(X,Y,PZ),\\ T(PX,Y,Z) &= g(T(PX,Y),Z) = g(T(X,PY),Z) = T(X,PY,Z)\end{aligned}$$

and

$$\begin{aligned}T(PX,PY,Z) &= g(T(PX,PY),Z) = g(-T(X,Y),Z) = -T(X,Y,Z),\\ T(PX,Y,PZ) &= g(T(PX,Y),PZ) = g(PT(X,Y),PZ) = -T(X,Y,Z),\\ T(X,PY,PZ) &= g(T(X,PY),PZ) = g(-PT(X,Y),PZ) = T(X,Y,Z).\end{aligned}$$

iii) With help of $g((\nabla^g_X P)Y,Z) = -g(Y,(\nabla^g_X P)Z)$, we find

$$\begin{aligned}(\nabla_X g)(Y,Z) &= X(g(Y,Z)) - g(\nabla^g_X Y,Z) - g(Y,\nabla^g_X Z)\\ &= X(g(Y,Z)) - g(\nabla^g_X Y - \frac{1}{2}P(\nabla^g_X P)Y,Z)\\ &\quad - g(Y,\nabla^g_X Z - \frac{1}{2}P(\nabla^g_X P)Z)\\ &= X(g(Y,Z)) - g(\nabla^g_X Y,Z) + \frac{1}{2}g(P(\nabla^g_X P)Y,Z)\\ &\quad + \frac{1}{2}g(Y,P(\nabla^g_X P)Z) - g(Y,\nabla^g_X Z)\end{aligned}$$

$$= (\nabla^g_X g)(Y,Z) + \frac{1}{2} g(P(\nabla^g_X P)Y,Z) + \frac{1}{2} g(Y, P(\nabla^g_X P)Z)$$
$$= -\frac{1}{2} g((\nabla^g_X P)PY,Z) - \frac{1}{2} g(PY,(\nabla^g_X P)Z)$$
$$= \frac{1}{2} g(PY,(\nabla^g_X P)Z) - \frac{1}{2} g(PY,(\nabla^g_X P)Z)$$
$$= 0.$$

iv) By virtue of (2.5), for the special metric connection $\nabla$, we have

$$\nabla_X Y = \nabla^g_X Y + \frac{1}{2} T(X,Y)$$

for any vector fields $X, Y$ on $M$. With help of this, we calculate the curvature tensor field:

$$\begin{aligned}
R_\nabla(X,Y),Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\
&= \nabla_X (\nabla^g_Y Z + \frac{1}{2} T(Y,Z)) - \nabla_Y (\nabla^g_X Z + \frac{1}{2} T(X,Z)) \\
&\quad - (\nabla^g_{[X,Y]} Z + \frac{1}{2} T([X,Y],Z)) \\
&= \nabla^g_X (\nabla^g_Y Z) + \frac{1}{2} T(X, \nabla^g_Y Z) + \frac{1}{2} \nabla^g_X (T(Y,Z)) \\
&\quad + \frac{1}{4} T(X, T(Y,Z)) - \nabla^g_Y (\nabla^g_X Z) - \frac{1}{2} T(Y, \nabla^g_X Z) \\
&\quad - \frac{1}{2} \nabla^g_Y (T(X,Z)) - \frac{1}{4} T(Y, T(X,Z) - \nabla^g_{[X,Y]} Z \\
&\quad - \frac{1}{2} T([X,Y],Z) \\
&= R^g(X,Y,Z) + \frac{1}{2} T(X, \nabla^g_Y Z) + \frac{1}{2} \nabla^g_X (T(Y,Z)) \\
&\quad + \frac{1}{4} T(X, T(Y,Z)) - \frac{1}{2} T(Y, \nabla^g_X Z) - \frac{1}{2} \nabla^g_Y (T(X,Z)) \\
&\quad - \frac{1}{4} T(Y, T(X,Z)) - \frac{1}{2} T([X,Y],Z)
\end{aligned}$$

from which, using $[X,Y] = \nabla^g_X Y - \nabla^g_Y X$

$$\begin{aligned}
R_\nabla(X,Y),Z = R^g(X,Y)Z &+ \frac{1}{2} (\nabla_X T)(Y,Z) - \frac{1}{2} (\nabla_Y T)(X,Z) \\
&+ \frac{1}{4} T(X, T(Y,Z)) - \frac{1}{4} T(Y, T(X,Z)).
\end{aligned}$$

□

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