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MEROMORPHIC FUNCTIONS SHARING SOME FINITE SETS IM

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Abstract. We show that if two nonconstant meromorphic functions f and g on $\mathbb{C}$ sharing some finite sets IM, then there is a nonconstant rational function $R(z)$ such that $R(f) = R(g)$.

1. Introduction

For nonconstant meromorphic functions f and g on $\mathbb{C}$ and a finite set S in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, we say that f and g share S CM (counting multiplicities) if $f^{-1}(S) = g^{-1}(S)$ and if for each $z_0 \in f^{-1}(S)$ two functions $f - f(z_0)$ and $g - g(z_0)$ have the same multiplicity of zero at z_0, where the notations $f - \infty$ and $g - \infty$ mean $1/f$ and $1/g$, respectively. Also, if $f^{-1}(S) = g^{-1}(S)$, then we say that f and g share S IM (ignoring multiplicities). In particular if S is a one-point set $\{a\}$, then we say also that f and g share a CM or IM.

In [3] and [4], R. Nevanlinna showed the following two theorems:

Theorem 1.1. *Let f and g be two distinct nonconstant meromorphic functions on $\mathbb{C}$ and $a_1, \ldots, a_4$ four distinct points in $\overline{\mathbb{C}}$. If f and g share $a_1, \ldots, a_4$ CM, then f is a Möbius transform of g, i.e., $f = (ag + b)/(cg + d)$ for some complex numbers a, b, c, d with $ad - bc \neq 0$, and there exists a permutation σ of $\{1, 2, 3, 4\}$ such that $a_{\sigma(3)}, a_{\sigma(4)}$ are Picard exceptional values of f and g and the cross ratio $(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, a_{\sigma(4)}) = -1$. Furthermore, the Möbius transformation fixes $a_{\sigma(1)}$ and $a_{\sigma(2)}$, and $a_{\sigma(3)}$ and $a_{\sigma(4)}$ interchanges under the Möbius transformation.*

Theorem 1.2. *Let f and g be two nonconstant meromorphic functions on $\mathbb{C}$ sharing distinct five points in $\overline{\mathbb{C}}$ IM. Then $f = g$.*

Remark 1.3. Let $T(z) = (az + b)/(cz + d)$ be a Möbius transformation of order 2, *i.e.*, $T^2 = T \circ T$ is the identity. Then $d = -a$ and $a^2 + bc \neq 0$. This Möbius transformation has two distinct fixed points ξ_1, ξ_2 in $\overline{\mathbb{C}}$. Let T_0 be a Möbius transformation such that $T_0(0) = \xi_1, T_0(\infty) = \xi_2$. Then the Möbius

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transformation $T_1 = T_0 \circ T \circ T_0{}^{-1}$ fixes 0 and ∞ and it is of order 2, and hence $T_1(z) = -z$. Put $w = T(z)$, then we have $T_0(w) = T_0 \circ T(z) = T_1 \circ T_0(z) = -T_0(z)$, and hence $\{T_0(w)\}^2 = \{T_0(z)\}^2$. Since the Möbius transformation of Theorem 1.1 is of order 2, we see that the existence of a nonconstant rational function $R(z)$ such that $R(f) = R(g)$ under the assumption of Theorem 1.1. Of course, the existence of such a rational function is trivial if $f = g$.

In [6] the author showed the following:

Theorem 1.4. *Let $S_1, \ldots, S_5$ be pairwise disjoint one-point or two-point sets in $\overline{\mathbb{C}}$. If two nonconstant meromorphic functions f and g on $\mathbb{C}$ share $S_1, \ldots, S_5$ IM, then f is a Möbius transform of g.*

The Möbius transformation in the conclusion of Theorem 1.4 is also of order 2 since the composition of it and itself has at least three fixed points. So, we see the existence of a rational function as in remark above.

By the results of [7–10], if two nonconstant meromorphic functions f and g on $\mathbb{C}$ share pairwise disjoint one-point or two-point sets S_1, S_2, S_3, S_4 CM, then f is a Möbius transform of g, and hence there is a nonconstant rational function $R(z)$ such that $R(f) = R(g)$.

These raise the following problems:

Problem 1. Let q be an integer not less than 5. Let $S_1, \ldots, S_q$ be pairwise disjoint finite sets in $\overline{\mathbb{C}}$. If two nonconstant meromorphic functions f and g share $S_1, \ldots, S_q$ IM, then does there exist a nonconstant rational function $R(z)$ such that $R(f) = R(g)$?

Problem 2. Let q be an integer not less than 4. Let $S_1, \ldots, S_q$ be pairwise disjoint finite sets in $\overline{\mathbb{C}}$. If two nonconstant meromorphic functions f and g share $S_1, \ldots, S_q$ CM, then does there exist a nonconstant rational function $R(z)$ such that $R(f) = R(g)$?

Both problems are affirmatively answered, as shown above, for the case that the all finite sets are one-point sets or two-points sets, and also we can find similar results for polynomials in [1] and [5]. In this paper, we give a partial solution for Problem 1.

Theorem 1.5. *Let p be a non-negative integer and let q be an integer not less than 2. Let $S_1, \ldots, S_p$ be one-point sets in $\mathbb{C}$ and let $S_{p+1}, \ldots, S_{p+q}$ be n-point sets in $\mathbb{C}$, where n is an integer not less than 2. Assume that $S_1, \ldots, S_{p+q}$ are pairwise disjoint and that $p + q \geq 5$. If two distinct nonconstant meromorphic functions f and g on $\mathbb{C}$ share $S_1, \ldots, S_{p+q}$ IM, then there exists distinct j_1, j_2 in $\{p+1, \ldots, p+q\}$ such that $P_{j_2}(f)/P_{j_1}(f) = P_{j_2}(g)/P_{j_1}(g)$, where $P_j(z)$ are defining polynomials of S_j.*

By considering a suitable Möbius transformation, we have:

Corollary 1.6. *Let p be a non-negative integer and let q be an integer not less than 2. Let $S_1, \ldots, S_p$ be one-point sets in $\overline{\mathbb{C}}$ and let $S_{p+1}, \ldots, S_{p+q}$ be n-point* *sets in $\overline{\mathbb{C}}$, where n is an integer not less than 2. Assume that $S_1, \dots, S_{p+q}$ are pairwise disjoint and that $p+q \geq 5$. If two nonconstant meromorphic functions f and g on $\mathbb{C}$ share $S_1, \dots, S_{p+q}$ IM, then there exists a nonconstant rational function $R(z)$ such $R(f) = R(g)$.*

We assume that the reader is familiar with the standard notations and results of the value distribution theory (see, for example, [2]). In particular, we express by $S(r, f)$ quantities such that $\lim_{r\to\infty, r\notin E} S(r, f)/T(r, f) = 0$, where E is a subset of $(0, \infty)$ with finite linear measure and it is variable in each cases.

2. Proof of Theorem 1.5

Now we start the proof of Theorem 1.5. We may assume that $p \leq 4$ by Theorem 1.2.

By the second main theorem and the first main theorem we have

$$\begin{aligned}(p + nq - 2)T(r, f) &\leq \sum_{j=1}^{p+q} \sum_{\xi\in S_j} \overline{N}(r, \frac{1}{f - \xi}) + S(r, f)\\ &= \sum_{j=1}^{p+q} \sum_{\xi\in S_j} \overline{N}(r, \frac{1}{g - \xi}) + S(r, f)\\ &\leq (p + nq)T(r, g) + S(r, f) \qquad (1)\end{aligned}$$

and, by the same way,

$$(p + nq - 2)T(r, g) \leq (p + nq)T(r, f) + S(r, g). \qquad (2)$$

Hence, by (1) and (2), there is no need to distinguish $S(r, f)$ and $S(r, g)$, and so we denote them by $S(r)$.

By $\overline{N}_E(r, \frac{1}{f-\xi})$ and $\overline{N}_N(r, \frac{1}{f-\xi})$ we denote the counting functions which count the point z such that $f(z) = \xi = g(z)$ and $f(z) = \xi \neq g(z)$ counted once, respectively, and we define $\overline{N}_E(r, \frac{1}{g-\xi})$ and $\overline{N}_N(r, \frac{1}{g-\xi})$ by the same way. It is easy to see that $\overline{N}_N(r, \frac{1}{f-\xi}) = \overline{N}_N(r, \frac{1}{g-\xi}) = 0$ for $\xi \in S_1 \cup \dots \cup S_p$ and that

$$\begin{aligned}\sum_{\xi\in S_j} \overline{N}_E(r, \frac{1}{f - \xi}) &= \sum_{\xi\in S_j} \overline{N}_E(r, \frac{1}{g - \xi}),\\ \sum_{\xi\in S_j} \overline{N}_N(r, \frac{1}{f - \xi}) &= \sum_{\xi\in S_j} \overline{N}_N(r, \frac{1}{g - \xi}) \qquad (3)\end{aligned}$$

for $j = p + 1, \dots, q$. Since $f - g \not\equiv 0$, we have

$$\sum_{j=1}^{p+q} \sum_{\xi\in S_j} \overline{N}_E(r, \frac{1}{f - \xi}) \leq \overline{N}(r, \frac{1}{f - g}) \leq T(r, f) + T(r, g) + O(1),$$

and hence

$$\sum_{j=p+1}^{p+q}\sum_{\xi\in S_j}\overline{N}_N(r,\frac{1}{f-\xi})=\sum_{j=1}^{p+q}\sum_{\xi\in S_j}\overline{N}(r,\frac{1}{f-\xi})-\sum_{j=1}^{p+q}\sum_{\xi\in S_j}\overline{N}_E(r,\frac{1}{f-\xi})$$
$$\geq (p+nq-2)T(r,f)-T(r,f)-T(r,g)+S(r)$$
$$=(p+nq-3)T(r,f)-T(r,g)+S(r)$$

by using (1). By the same way and (3) we have

$$\sum_{j=p+1}^{p+q}\sum_{\xi\in S_j}\overline{N}_N(r,\frac{1}{f-\xi})\geq (p+nq-3)T(r,g)-T(r,f)+S(r).$$

Adding these two inequalities we obtain

$$\sum_{j=p+1}^{p+q}\sum_{\xi\in S_j}\overline{N}_N(r,\frac{1}{f-\xi})\geq \frac{1}{2}(p+nq-4)(T(r,f)+T(r,g))+S(r). \tag{4}$$

Note that $q\geq 2$. From (4) we see that there exist distinct j_1 and j_2 in $\{p+1,\dots,q\}$ and a subset I of $(0,+\infty)$ of infinite linear measure such that

$$\frac{1}{q}(p+nq-4)(T(r,f)+T(r,g))+S(r)\leq \sum_{\xi\in S_{j_1}\cup S_{j_2}}\overline{N}_N(r,\frac{1}{f-\xi}) \tag{5}$$

holds for $r\in I$. Put $Q(z,w)=(P_{j_1}(z)P_{j_2}(w)-P_{j_1}(w)P_{j_2}(z))/(z-w)$ and $\Phi=Q(f,g)$. Assume that $\Phi\not\equiv 0$. If $f(z),g(z)\in S_{j_1}\cup S_{j_2}$ and $f(z)\neq g(z)$, then $\Phi(z)=0$. Therefore we have

$$\sum_{\xi\in S_{j_1}\cup S_{j_2}}\overline{N}_N(r,\frac{1}{f-\xi})\leq N_0(r,\frac{1}{\Phi}) \tag{6}$$

holds for $r\in I$, where $N_0(r,\frac{1}{\Phi})$ denotes the counting functions corresponding to the zeros of Φ that are not the poles of f and g. We see that $Q(z,w)$ is a symmetric polynomial of z and w and it has degree at most $n-1$ with respect to each of z and w. By using the first fundamental theorem and the definition of counting function and that of proximity function, we have

$$N_0(r,\frac{1}{\Phi})\leq N(r,Q(f,g))+m(r,Q(f,g))$$
$$\leq (n-1)(N(r,f)+N(r,g)+m(r,f)+m(r,g))+O(1)$$
$$=(n-1)(T(r,f)+T(r,g))+O(1).$$

By connecting (5), (6) and this,

$$\frac{1}{q}(p+nq-4)(T(r,f)+T(r,g))+S(r)\leq (n-1)(T(r,f)+T(r,g))+O(1)$$

holds for $r\in I$. Here I may be different from that in (5). We obtain $p+nq-4\leq q(n-1)$, which contradicts hypothesis $p+q\geq 5$. Therefore we conclude that $\Phi\equiv 0$, which induces that $P_{j_2}(f)/P_{j_1}(f)=P_{j_2}(g)/P_{j_1}(g)$.

3. An application to the uniqueness

In this section, we apply the above results to the uniqueness of meromorphic functions. Let n be an integer not less than 2, and let $S_1, \ldots, S_5$ be pairwise disjoint n-point sets in $\mathbb{C}$. For each $j = 1, \ldots, 5$, we take a defining polynomial $P_j(z)$ of S_j, and let $\xi_{j1} \cdots \xi_{jn}$ be the distinct elements of S_j.

Theorem 3.1. *Assume that*

$$\frac{P_j(\xi_{l\mu})}{P_k(\xi_{l\mu})} \neq \frac{P_j(\xi_{i\nu})}{P_k(\xi_{l\nu})} \tag{7}$$

for distinct $j, k, l \in \{1, \ldots, 5\}$ *and* $1 \leq \mu < \nu \leq n$. *If two nonconstant meromorphic functions* f *and* g *on* $\mathbb{C}$ *share* $S_1, \ldots, S_5$ *IM, then* $f = g$.

Proof. Assume that $f \neq g$. From Theorem 1, we may assume that

$$\frac{P_1(f)}{P_2(f)} = \frac{P_1(g)}{P_2(g)},$$

by renumbering $S_1, \ldots, S_5$, if necessary. By (7), there is no z such that $f(z), g(z)$ are distinct values in $S_3 \cup S_4 \cup S_5$. Therefore, f and g share each values in $S_3 \cup S_4 \cup S_5$. This fact yields, by Theorem 1.2, $f = g$, which is a contradiction. Hence we conclude $f = g$. $\square$

Remark 3.2. In the case of $n = 2$, the assumption (7) becomes to

$$\begin{vmatrix} 1 & a_j & b_j \\ 1 & a_k & b_k \\ 1 & a_l & b_l \end{vmatrix} \neq 0,$$

where $P_j(z) = z^2 + a_j z + b_j$ and so on. This is a necessary and sufficient condition for the absence of a Möbius transformation exchanging two elements of each S_j, S_k, S_l.

For $n \geq 3$, we can weaken the assumption about (7). It is enough to hold (7) for distinct two l, in the case of $n = 3, 4$, and for one l, in the case of $n \geq 5$, different from any given $1 \leq j < k \leq 5$.

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