Bull. Korean Math. Soc. **55** (2018), No. 3, pp. 837–849
https://doi.org/10.4134/BKMS.b170300
pISSN: 1015-8634 / eISSN: 2234-3016

PROPERTIES OF HURWITZ POLYNOMIAL AND HURWITZ SERIES RINGS

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Abstract. In this paper, we study the closedness such as seminomality and t-closedness, and Noetherian-like properties such as piecewise Noetherianness and Noetherian spectrum, of Hurwitz polynomial rings and Hurwitz series rings. To do so, we construct an isomorphism between a Hurwitz polynomial ring (resp., a Hurwitz series ring) and a factor ring of a polynomial ring (resp., a power series ring) in a countably infinite number of indeterminates.

1. Introduction

Throughout this paper, all rings are assumed to be commutative with unity.

The ring of Hurwitz series has been studied by many authors, in particular by Keigher and Pritchard [15] and Benhissi and Koja [4]. Keigher and Pritchard demonstrated that, closely related to the power series ring, the ring of Hurwitz series over a commutative ring with identity has many interesting properties, including applications to differential algebra. Then Benhissi and Koja studied the transfer of some ring properties from the ground ring to the ring of Hurwitz series. In [17, 18], Lim and Oh studied some chain conditions on composite Hurwitz rings. In this paper, we study the transfer of closedness and Noetherian-like properties between the ground ring and the Hurwitz polynomial ring and the Hurwitz power series ring. To do so, we construct an isomorphism between the Hurwitz polynomial ring (resp., the Hurwitz series ring) and a factor ring of a polynomial ring (resp., the power series ring) in a countably infinite number of indeterminates.

Hurwitz series rings and Hurwitz polynomial rings are defined as follows. Let R be a commutative ring. We let $R[\![X]\!]_{\mathsf{H}}$ ("H" stands for "Hurwitz") be the set of formal expressions of the form $f = \sum_{i=0}^{\infty} f_i X^{[i]}$, where $f_i \in R$. Let $g = \sum_{i=0}^{\infty} g_i X^{[i]}$, where $g_i \in R$. Then define $f + g = \sum_{i=0}^{\infty}(f_i + g_i)X^{[i]}$ and $f * g = \sum_{n=0}^{\infty} h_n X^{[n]}$, where $h_n = \sum_{k=0}^{n} \binom{n}{k} f_k g_{n-k}$ for all n. With these

Received April 4, 2017; Accepted September 14, 2017.
2010 *Mathematics Subject Classification.* Primary 13F20, 13F25; Secondary 13A15, 13B21, 13E99.
Key words and phrases. Hurwitz series ring, t-closed, seminormal, weakly normal, piecewise Noetherian ring, Noetherian spectrum.

two operations $R[\![X]\!]_{\mathsf{H}}$ is a commutative ring with identity containing R called the ring of *Hurwitz series over* R. Similarly we can define the ring $R[X]_{\mathsf{H}}$ of *Hurwitz polynomials over* R, which is a subring of $R[\![X]\!]_{\mathsf{H}}$. The association $f \longmapsto \sum_{n=0}^{\infty} \frac{f_n}{n!} X^n$ defines ring homomorphisms $R[\![X]\!]_{\mathsf{H}} \longrightarrow (R \otimes_{\mathbb{Z}} \mathbb{Q})[\![X]\!]$ and $R[X]_{\mathsf{H}} \longrightarrow (R \otimes_{\mathbb{Z}} \mathbb{Q})[X]$. If R is a $\mathbb{Z}$-torsion-free ring, then these two homomorphisms are inclusions, $X^{[n]}$ can be identified with $X^n/n!$ for all n, and $R[\![X]\!]_{\mathsf{H}}$ (resp., $R[X]_{\mathsf{H}}$) can be identified with its image in $(R \otimes_{\mathbb{Z}} \mathbb{Q})[\![X]\!]$ (resp., $(R \otimes_{\mathbb{Z}} \mathbb{Q})[X]$).

All notation will be standard, unless otherwise noted. In particular, denote by $\mathbb{Z}$ the ring of integers and $\mathbb{Q}$ the field of rational numbers.

2. Generalized power series rings

Let $(S, \leq)$ be an ordered set. The ordered set $(S, \leq)$ is *artinian* if every strictly decreasing sequence of elements of S is finite, and $(S, \leq)$ is *narrow* if every subset of pairwise order-incomparable elements of S is finite. It is easy to see that $(S, \leq)$ is artinian if and only if every non-empty subset of S has a minimal element. Moreover, if $\leq$ is a total order, then $(S, \leq)$ is artinian if and only if it is well-ordered. $(S, \leq)$ is a *strictly ordered monoid* if $s, s', t \in S$ and $s < s'$ imply $s + t < s' + t$. Note that if S is cancellative or if $\leq$ is the trivial order (i.e., $x \leq y$ implies $x = y$), then $(S, \leq)$ is a strictly ordered monoid.

The following definition is due to Ribenboim [8]: Let $(S, \leq)$ be a strictly ordered monoid and let R be a commutative ring. Let $[\![R^{S,\leq}]\!]$ be the set of all functions $f : S \longrightarrow R$ such that $\mathrm{Supp}(f) = \{s \in S \mid f(s) \neq 0\}$ is artinian and narrow. It is clear that $[\![R^{S,\leq}]\!]$ is an additive abelian group with pointwise addition. For every $s \in S$ and $f_1, \ldots, f_n \in [\![R^{S,\leq}]\!]$, let $X_s(f_1, \ldots, f_n) = \{(u_1, \ldots, u_n) \in S^n \mid s = u_1 + \cdots + u_n,\ u_i \in \mathrm{Supp}(f_i) \text{ for each } i\}$. It follows from [8, (e) p. 368] that $X_s(f_1, \ldots, f_n)$ is finite. This fact allows one to define the operation of convolution $*$ as following;

$$(f * g)(s) = \sum_{(u,v) \in X_s(f,g)} f(u)g(v).$$

With this operation, and pointwise addition, $[\![R^{S,\leq}]\!]$ becomes a commutative ring with identity element e, where

$$e(s) = \delta_{0s} = \begin{cases} 1 & \text{if } s = 0, \\ 0 & \text{if } 0 \neq s \in S. \end{cases}$$

We call $[\![R^{S,\leq}]\!]$ the ring of *generalized power series*. It should be noted that the definition of $[\![R^{S,\leq}]\!]$ depends on the order $\leq$; for example, see [8, p. 371]. It follows from [8, p. 368] that R is canonically embedded as a subring of $[\![R^{S,\leq}]\!]$, and that S is canonically embedded as a submonoid of the monoid $([\![R^{S,\leq}]\!], *)$ via the map $s \longmapsto (e_s : t \longmapsto \delta_{st})$.

Let $\mathbb{N}^{(\aleph_0)}$ denote the monoid $\bigoplus_{i=1}^{\infty} \mathbb{N}$, where $\mathbb{N}$ denotes the monoid of non-negative integers under addition. Let $o(\mathbf{n}) = \sum_{i=1}^{\infty} i n_i \in \mathbb{N}$ for all $\mathbf{n} = (n_i)_{i=1}^{\infty} \in \mathbb{N}^{(\aleph_0)}$, so that $o : \mathbb{N}^{(\aleph_0)} \longrightarrow \mathbb{N}$ is a monoid homomorphism. Define an ordering

$\leq$ on $\mathbb{N}^{(\aleph_0)}$ by declaring $\mathbf{n} < \mathbf{m}$ if and only if $o(\mathbf{n}) < o(\mathbf{m})$. Then $(\mathbb{N}^{(\aleph_0)}, \leq)$ is a strictly ordered monoid, and, since for any $r \in \mathbb{N}$ there are only finitely many $\mathbf{n} \in \mathbb{N}^{(\aleph_0)}$ with $o(\mathbf{n}) \leq r$, the ordered monoid $(\mathbb{N}^{(\aleph_0)}, \leq)$ is artinian and narrow. We let $R[\![X_1, X_2, \ldots]\!]$ denote the ring $[\![R^{\mathbb{N}^{(\aleph_0)}, \leq}]\!]$. Elements of $R[\![X_1, X_2, \ldots]\!]$ are functions $f : \mathbb{N}^{(\aleph_0)} \longrightarrow R$, where

$$(f + g)(\mathbf{s}) = f(\mathbf{s}) + g(\mathbf{s})$$

and

$$(fg)(\mathbf{s}) = \sum_{\substack{\mathbf{l}+\mathbf{m}=\mathbf{s} \\ \mathbf{l} \in \mathrm{Supp}(f) \\ \mathbf{m} \in \mathrm{Supp}(g)}} f(\mathbf{l}) g(\mathbf{m}) \quad \text{for any } \mathbf{s} \in \mathbb{N}^{(\aleph_0)}.$$

For $\mathbf{n} = (n_i)_{i=1}^{\infty} \in \mathbb{N}^{(\aleph_0)}$, one denotes by $X^{\mathbf{n}}$ the element $e_{\mathbf{n}}$ of $R[\![X_1, X_2, \ldots]\!]$. Elements of $R[\![X_1, X_2, \ldots]\!]$ can be written as formal sums $\sum_{\mathbf{n} \in \mathbb{N}^{(\aleph_0)}} f_{\mathbf{n}} X^{\mathbf{n}}$.

Remark 2.1. In the literature, the ring $R[\![X_1, X_2, \ldots]\!]$ is often denoted $R[\![X_1, X_2, \ldots]\!]_3$, to distinguish it from two of its subrings, $R[\![X_1, X_2, \ldots]\!]_1$ and $R[\![X_1, X_2, \ldots]\!]_2 \supseteq R[\![X_1, X_2, \ldots]\!]_1$. The ring $R[\![X_1, X_2, \ldots]\!]_1$ is defined to be $\bigcup_{n=1}^{\infty} R[\![X_1, X_2, \ldots, X_n]\!]$, which is the direct limit of

$$R[\![X_1]\!] \longrightarrow R[\![X_1, X_2]\!] \longrightarrow R[\![X_1, X_2, X_3]\!] \longrightarrow \cdots.$$

The ring $R[\![X_1, X_2, \ldots]\!]_2$ is the ring $[\![R^{\mathbb{N}^{(\aleph_0)}, \leq_2}]\!]$, where $\mathbb{N}^{(\aleph_0)}$ is ordered by relation $\leq_2$ determined by the usual degree map $\mathbf{n} \longmapsto \sum_{i=1}^{\infty} n_i$, instead of the map o. Elements of $R[\![X_1, X_2, \ldots]\!]_2$ are series in $R[\![X_1, X_2, \ldots]\!]$ of the form $\sum_{i=0}^{\infty} f_i$ with $f_i \in R[X_1, X_2, \ldots]$ homogeneous of degree i for all i.

There is a natural non-archimedean norm on the ring $R[\![X_1, X_2, \ldots]\!]$ defined as follows. Consider X_n as degree n and extend multiplicatively on elements of the form $X^{\mathbf{n}}$, so that $\deg X^{\mathbf{n}} = \sum_{i=1}^{\infty} i n_i = o(\mathbf{n})$. For all $f \in R[\![X_1, X_2, \ldots]\!]$, let

$$\deg f = \min\{\deg X^{\mathbf{n}} : \mathbf{n} \in \mathrm{Supp}(f)\} = \min\{o(\mathbf{n}) : \mathbf{n} \in \mathrm{Supp}(f)\}$$

if $f \neq 0$, and $\deg 0 = \infty$. One has the following for all $f, g \in R[\![X_1, X_2, \ldots]\!]$.

(1) $\deg f \in \mathbb{N} \cup \{\infty\}$.
(2) $\deg f = \infty$ if and only if $f = 0$.
(3) $\deg(f + g) \geq \min\{\deg f, \deg g\}$.
(4) $\deg(fg) \geq \deg f + \deg g$.

Consequently, setting $|f| = 2^{-\deg f}$ for all $f \in R[\![X_1, X_2, \ldots]\!]$, we have

(1) $|f| \in \{2^{-n} : n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{Q}_{\geq 0}$.
(2) $|f| = 0$ if and only if $f = 0$.
(3) $|f + g| \leq \max\{|f|, |g|\}$.
(4) $|fg| \leq |f||g|$.

Thus, $|\cdot|$ is a non-archimedean norm on the ring $R[\![X_1, X_2, \ldots]\!]$ and so induces a ring topology. Moreover, the subring $R[X_1, X_2, \ldots]$ is dense in $R[\![X_1, X_2, \ldots]\!]$, since

$$f = \lim_{r \to \infty} f_r$$

for all $f = \sum_{\mathbf{n}\in\mathbb{N}^{(\aleph_0)}} f_{\mathbf{n}} X^{\mathbf{n}} \in R[\![X_1, X_2, \ldots]\!]$, where

$$f_r = \sum_{\substack{\mathbf{n}\in\mathbb{N}^{(\aleph_0)} \\ o(\mathbf{n})\le r}} f_{\mathbf{n}} X^{\mathbf{n}} \in R[X_1, X_2, \ldots]$$

for all r. In fact, one has $\deg(f - f_r) > r$ for all r, while $\deg f_r \le r$ if $f_r \ne 0$.

There is also a ring norm defined on the ring $R[\![X]\!]_{\mathsf{H}}$, given by $|f| = 2^{-\deg f}$, where $\deg f = \min\{n \in \mathbb{Z}_{\ge 0} : f_n \ne 0\}$ for $f = \sum_{n=0}^{\infty} f_n X^{[n]}$. Moreover, $R[\![X]\!]_{\mathsf{H}}$ is complete with respect to this norm and contains $R[X]_{\mathsf{H}}$ as a dense subring [15, Theorem 1.1].

Theorem 2.2. *Let R be a commutative ring with identity. Let I denote the ideal of $R[X_1, X_2, \ldots]$ generated by $\left\{X_i X_j - \binom{i+j}{i} X_{i+j} \mid i, j \ge 1\right\}$, and let $\overline{I}$ denote the closure of I in the normed ring $R[\![X_1, X_2, \ldots]\!]$. Let $\theta : R[X_1, X_2, \ldots] \longrightarrow R[X]_{\mathsf{H}}$ denote the unique R-algebra homomorphism acting by $\theta : X_n \longmapsto X^{[n]}$ for all n. Then θ extends uniquely to a unique continuous R-algebra homomorphism $\overline{\theta} : R[\![X_1, X_2, \ldots]\!] \longrightarrow R[\![X]\!]_{\mathsf{H}}$, and θ and $\overline{\theta}$ induce R-algebra isomorphisms*

$$R[X]_{\mathsf{H}} \cong R[X_1, X_2, \ldots]/I$$

and

$$R[\![X]\!]_{\mathsf{H}} \cong R[\![X_1, X_2, \ldots]\!]/\overline{I}.$$

Proof. One has $I \subseteq \ker\theta$, so there is an induced surjective R-algebra homomorphism $\widetilde{\theta} : R[X_1, X_2, \ldots]/I \longrightarrow R[X]_{\mathsf{H}}$. By [9, Remark 3.3], $\widetilde{\theta}$ is an R-algebra isomorphism. Now, one has $\deg X_n = n = \deg X^{[n]}$ for all n and therefore, for all $f \in R[X_1, X_2, \ldots]$, one has $\deg\theta(f) \ge \deg f$, so $|\theta(f)| \le |f|$. Therefore, since $R[\![X]\!]_{\mathsf{H}}$ is complete with respect to its absolute value and $R[X_1, X_2, \ldots]$ is dense in $R[\![X_1, X_2, \ldots]\!]$, the map θ extends to a unique R-algebra homomorphism $\overline{\theta} : R[\![X_1, X_2, \ldots]\!] \longrightarrow R[\![X]\!]_{\mathsf{H}}$. Since $I \subseteq \ker\overline{\theta}$ and $\ker\overline{\theta}$ is closed in $R[\![X_1, X_2, \ldots]\!]$, one has $\overline{I} \subseteq \ker\overline{\theta}$. Moreover, if $f \in \ker\overline{\theta}$, then for all r one has $\deg\theta(f_r) = \deg(\overline{\theta}(f - f_r)) > r$ while $\deg\overline{\theta}(f_r) = \deg\theta(f_r) \le r$ if $\theta(f_r) \ne 0$, so $\theta(f_r) = 0$ for all r, whence $f = \lim_{r\to\infty} f_r \in \overline{I}$. Therefore $\ker\overline{\theta} = \overline{I}$. (Alternatively, note that, since every polynomial in $R[X_1, X_2, \ldots]$ is congruent modulo I to a unique linear polynomial, it is easy to see that every element of $R[\![X_1, X_2, \ldots]\!]$ is congruent modulo $\overline{I}$ to a unique linear series $c_0 + \sum_{n=1}^{\infty} c_i X_i$.) ☐

Corollary 2.3. *Let R be a commutative ring with identity. There is a natural R-algebra isomorphism $R[X]_{\mathsf{H}} \cong \mathbb{Z}[X]_{\mathsf{H}} \otimes_{\mathbb{Z}} R$.*

The domain $\mathbb{Z}[X]_{\mathsf{H}}$ is not a UFD [30, Example 20] nor even a Krull domain [30, Example 23].

3. Closedness

The notions of the following closure properties come originally from K-theory. In this section we study the transfer of these properties between the Hurwitz series ring (resp., Hurwitz polynomial ring) and its base ring. To do so, we begin with the following related definitions.

Let $R \subseteq T$ be an extension of commutative rings with (the same) identity. Consider the following conditions:

(a) T is integral over R.
(b) $\mathsf{Spec}(T) \longrightarrow \mathsf{Spec}(R)$ is a bijection.
(c) The residue field extensions are isomorphisms, i.e., for each $Q \in \mathsf{Spec}(T)$ the extension $R_P/PR_P \hookrightarrow T_Q/QR_Q$ is an isomorphism, where $P = Q \cap R$.
(c′) The residue field extensions are purely inseparable.

We first recall some special extensions satisfying two or three conditions above including the condition (a).

(1) R. G. Swan called the extension $R \subseteq T$ *subintegral* if (a), (b) and (c) are satisfied ([29]).
(2) H. Yanagihara called the extension $R \subseteq T$ *weakly subintegral* if (a), (b) and (c′) are satisfied ([33]).
(3) G. Picavet and M. Picavet-L'Hermitte called the extension $R \subseteq T$ *infra-integral* if (a) and (c) are satisfied ([26]).
(4) M. Picavet-L'Hermitte called the extension $R \subseteq T$ *weakly infra-integral* if (a) and (c′) are satisfied ([28]).

Using these extensions, they defined and characterized the *seminormalization* ${}^{+}_{T}R$ (resp., *weak normalizarion* ${}^{*}_{T}R$, *t-closure* ${}^{t}_{T}R$, *strong t-closure* ${}^{\circ}_{T}R$) of R in T as the largest subintegral (resp., weakly subintegral, infra-integral, weakly infra-integral) subextension of T over R. They also defined that R is *seminormal* (resp., *weakly normal*, *t-closed*, *strongly t-closed*) *in* T if $R = {}^{+}_{T}R$ (resp., $R = {}^{*}_{T}R$, $R = {}^{t}_{T}R$, $R = {}^{\circ}_{T}R$). An integral domain R is called seminormal (resp., weakly normal, t-closed, strongly t-closed) (denoted by ${}^{c}R = R$ for $c = +, *, t$ or $\circ$) if it is so in its quotient field.

Lemma 3.1. *Let $R \subseteq T$ be an extension of commutative rings. Let J be an ideal of T, and let $I = J \cap R$. Then T/J is integral over R/I.*

Proof. Clear. □

It follows from [10, Theorem 12.10] that for an extension $R \subseteq T$ of domains and a torsion-free cancellative monoid S, the integral closure of the semigroup ring $R[S]$ in $T[S]$ is $R'[S]$, where R' is the integral closure of R in T.

Proposition 3.2. *Let $R \subseteq T$ be an extension of domains. If T is integral over R, then $T[X]_{\mathsf{H}}$ is integral over $R[X]_{\mathsf{H}}$.*

Proof. It follows from the above remark that for a torsion-free cancellative monoid S, $T[S]$ is integral over $R[S]$.

Let I be the ideal as in Theorem 2.2. Then by Theorem 2.2 and the inclusion map $R[X]_{\mathsf{H}} \hookrightarrow T[X]_{\mathsf{H}}$, we have $IT[S] \cap R[S] = I$. Now the assertion follows from Lemma 3.1. $\square$

Define $\pi : R[\![X]\!]_{\mathsf{H}} \longrightarrow R$ by $\pi(\sum_{i=0}^{\infty} f_i X^{[i]}) = f_0$ for all $\sum_{i=0}^{\infty} f_i X^{[i]} \in R[\![X]\!]_{\mathsf{H}}$. The restriction of π to the Hurwitz polynomial ring $R[X]_{\mathsf{H}}$ will be also denoted by π. The following proposition is useful in the study of Hurwitz polynomial rings and Hurwitz series rings.

Proposition 3.3. *Let R be a ring. One has the following.*

(1) ([2, Proposition 1.1]) $R[X]_{\mathsf{H}}$ (*resp.*, $R[\![X]\!]_{\mathsf{H}}$) *is an integral domain if and only if R is an integral domain with* $\mathrm{char}(R) = 0$.
(2) ([2, Proposition 1.2]) *Let I be an ideal of R. Then $R[\![X]\!]_{\mathsf{H}}/\pi^{-1}(I) \cong R/I$ and $R[X]_{\mathsf{H}}/\pi^{-1}(I) \cong R/I$.*
(3) ([14, Proposition 2.4] *and* [4, Corollary 1.5]) *If R contains $\mathbb{Q}$, then $R[\![X]\!]_{\mathsf{H}} \cong R[\![X]\!]$ and $R[X]_{\mathsf{H}} \cong R[X]$.*

Recall that a ring R is said to be:

(1) *seminormal* if for any $b, c \in R$ such that $b^3 = c^2$ there exists $r \in R$ with $r^2 = b, r^3 = c$ ([29]).
(2) *weakly normal* if for any $b, c \in R$ such that $b^3 = c^2$ there exists $r \in R$ with $r^2 = b, r^3 = c$ and for any $b, c, e \in R$ and any non-zero divisor $d \in R$ such that $c^p = bd^p$ and $pc = de$ for some prime p there exists $r \in R$ with $r^p = b, pr = e$ ([33]).
(3) *t-closed* if for any $b, c, a \in R$ such that $b^3 + abc - c^2 = 0$ there exists $r \in R$ with $r^2 - ar = b, r^3 - ar^2 = c$ ([27, Définition 1.1]).
(4) *strongly t-closed* if R is a weakly normal and t-closed ring ([28]).

Then we have the following connection.

Proposition 3.4. *Let R be a domain. Then the following statements are equivalent.*

(1) *R is a seminormal* (*resp., weak normal, t-closed, strongly t-closed*) *domain.*
(2) $^{+}_{R}R = R$ (*resp.*, $^{*}_{R}R = R$, $^{t}_{R}R = R$, $^{\circ}_{R}R = R$).
(3) $^{+}R = R$ (*resp.*, $^{*}R = R$, $^{t}R = R$, $^{\circ}R = R$).

Proof. The weak normality follows from [31, Corollary 3.16], while the rest follow from [25, Proposition 1.6]. Also see [31] for the seminormality and [27, Theorem 2.3] for the t-closedness. $\square$

Denote by $\mathsf{qf}(R)$ the quotient field of a domain R.

Theorem 3.5. *Let R be a domain of characteristic zero.*

(1) *$R[\![X]\!]_{\mathsf{H}}$ is a seminormal domain if and only if R is a seminormal domain and R contains $\mathbb{Q}$.*

(2) *$R[\![X]\!]_{\mathsf{H}}$ is a weakly normal domain if and only if R is a weakly normal domain and R contains $\mathbb{Q}$.*

(3) *Let R be a Noetherian domain such that the integral closure of R is also Noetherian (for example, a Noetherian domain with finite integral closure). Then $R[\![X]\!]_{\mathsf{H}}$ is a t-closed domain if and only if R is a t-closed domain and R contains $\mathbb{Q}$.*

(4) *Let R be a Noetherian domain such that the integral closure of R is also Noetherian. Then $R[\![X]\!]_{\mathsf{H}}$ is a strongly t-closed domain if and only if R is a strongly t-closed domain and R contains $\mathbb{Q}$.*

(5) *$R[\![X]\!]_{\mathsf{H}}$ is completely integrally closed if and only if R is completely integrally closed and R contains $\mathbb{Q}$.*

Proof. (1) Assume that R is a seminormal domain and R contains $\mathbb{Q}$. Then $R[\![X]\!]_{\mathsf{H}} \cong R[\![X]\!]$ is a seminormal domain by [6, Theorem]. For the converse, assume that $R[\![X]\!]_{\mathsf{H}}$ is a seminormal domain. Let $\alpha \in K := \mathsf{qf}(R)$ such that $\alpha^2, \alpha^3 \in R$. Then $\alpha^2, \alpha^3 \in R[\![X]\!]_{\mathsf{H}}$. Note that $\alpha \in K \subset \mathsf{qf}(R[\![X]\!]_{\mathsf{H}})$. Then by hypothesis $\alpha \in R[\![X]\!]_{\mathsf{H}} \cap K = R$. Hence R is a seminormal domain. Let p be a prime number. Note that for any positive integer n, the exponent of the highest power of p that divides $\binom{3n}{n,n,n}$ (resp., $\binom{2n}{n,n}$) is $e_3(p,n) := \sum_{k=1}^{\infty}(\lfloor \frac{3n}{p^k} \rfloor - 3\lfloor \frac{n}{p^k} \rfloor)$ (resp., $e_2(p,n) := \sum_{k=1}^{\infty}(\lfloor \frac{2n}{p^k} \rfloor - 2\lfloor \frac{n}{p^k} \rfloor)$). Set $m := p^2 - 1$. Then $e_3(p,m) \geq 3$ and $e_2(p,m) \geq 2$. Thus we have $(\frac{X^{[m]}}{p})^2, (\frac{X^{[m]}}{p})^3 \in R[\![X]\!]_{\mathsf{H}}$. By hypothesis, $\frac{X^{[m]}}{p} \in R[\![X]\!]_{\mathsf{H}}$. Thus p is a unit in R. Therefore R contains $\mathbb{Q}$.

(2) Assume that R is a weakly normal domain and R contains $\mathbb{Q}$. Then $R[\![X]\!]_{\mathsf{H}} \cong R[\![X]\!]$ is a weakly normal domain by [7, Corollary 4]. For the converse, assume that $R[\![X]\!]_{\mathsf{H}}$ is a weakly normal domain. Recall from [32, Theorem 1] that for an extension $A \subseteq B$ of domains, A is weakly normal in B if and only if A is seminormal in B and, whenever $b \in B$ satisfies $b^p, pb \in A$ for some prime p, then $b \in A$. Then R is a weakly normal domain by the same argument as in the seminormality case and R contains $\mathbb{Q}$ by (1).

(3) Assume that R is a t-closed domain and R contains $\mathbb{Q}$. Then $R[\![X]\!]_{\mathsf{H}} \cong R[\![X]\!]$ is a t-closed domain by [27, Proposition 2.25]. For the converse, assume that $R[\![X]\!]_{\mathsf{H}}$ is a t-closed domain. Then R is a t-closed domain by the same argument as in the seminormality case. Since every t-closed domain is seminormal, R contains $\mathbb{Q}$ by (1).

(4) This follows from the fact that a strongly t-closed domain is exactly a t-closed and weakly normal domain.

(5) This can be proved by the same argument as above. □

Let $R \subseteq T$ be an extension of domains. It is known that if R is seminormal (resp., t-closed) in T, then $R[X]$ is seminormal (resp., t-closed) in $T[X]$ [5,24]. It follows from [16, Corollary 2.6(1)], we have ${}^*_{T[X]}R[X] = ({}^*_T R)[X]$. Thus if R is weakly normal in T, then $R[X]$ is weakly normal in $T[X]$. Finally the strong t-closedness follows from the fact that R is strongly t-closed in T if and only if R is both t-closed and weakly normal in T. Also it is well known that $R[X]$

is integrally closed (resp., completely integrally closed) if and only if so is R. Therefore, $({}^{+}_{T}R)[X]$ (resp., $({}^{*}_{T}R)[X]$, $({}^{t}_{T}R)[X]$, $({}^{\circ}_{T}R)[X]$) is seminormal (resp., weakly normal, t-closed, strongly t-closed) in $T[X]$. Since $R[X] \subseteq ({}^{c}_{T}R)[X]$ for $c := +, *, t$ or $\circ$, we have ${}^{c}_{T[X]}R[X] \subseteq ({}^{c}_{T}R)[X]$ by well-known facts that the seminormalization (resp., weak normalization, t-closure, strong t-closure) of an integral domain R in a given extension domain T is the smallest ring S such that $A \subseteq S \subseteq B$ and S is seminormal (resp., weakly normal, t-closed, strongly t-closed) in T ([25, Proposition 1.6(2)] and [7, Lemma 1(iv)]).

Corollary 3.6. *Let R be a domain of characteristic zero. Then $R[X]_{\mathsf{H}}$ is a seminormal* (*resp., weakly normal, t-closed, strongly t-closed, integrally closed, completely integrally closed*) *domain if and only if R is a seminormal* (*resp., weakly normal, t-closed, strongly t-closed, integrally closed, completely integrally closed*) *domain and R contains $\mathbb{Q}$.*

Proof. For simplicity, the letter c denotes $+, *, t$ or $\circ$. First we will show that ${}^{c}R[X] \subseteq ({}^{c}R)[X]$. Then if R is a c-closed domain, then $R[X]$ is a c-closed domain.

Let $K := \mathsf{qf}(R)$. Since $K[X]$ is integrally closed, we have $\overline{R[X]} \subseteq K[X]$. By [25, Proposition 1.6(3)] and [7, Lemma 1(ii)],

$$ {}^{c}R[X] = {}^{c}_{\overline{R[X]}}R[X] \subseteq {}^{c}_{K[X]}R[X]. $$

Thus, the assertion follows by taking $T = K$ in the above remarks.

The rest can be proved by the same arguments as in Theorem 3.5. $\square$

Example 3.7. Neither $\mathbb{Z}[\![X]\!]_{\mathsf{H}}$ nor $\mathbb{Z}[X]_{\mathsf{H}}$ is a seminormal domain. Thus neither $\mathbb{Z}[\![X]\!]_{\mathsf{H}}$ nor $\mathbb{Z}[X]_{\mathsf{H}}$ is integrally closed.

4. Noetherian-like rings

Let R be a commutative ring with identity. An ideal Q of R is *primary* if each zero divisor of the ring R/Q is nilpotent, and Q is *strongly primary* if Q is primary and contains a power of its radical.

For any ideal I of R, $\mathrm{Min}(I)$ is the set of prime ideals of R which are minimal over I. Set $S(I) = R \setminus \bigcup\{P \mid P \in \mathrm{Min}(I)\}$. Then I is said to have *finite ideal-length* if the ring $R_{S(I)}/IR_{S(I)}$ is an Artinian ring.

First recall some finiteness properties weaker than Noetherian rings.

(1) R is said to be *Laskerian* if each ideal of R is a finite intersection of primary ideals.
(2) R is said to be *strongly Laskerian* if each ideal of R is a finite intersection of strongly primary ideals.
(3) R is called a *ZD-ring* if the set of zero divisors on the R-module R/A is a finite union of prime ideals for each ideal A of R.
(4) R has *Noetherian spectrum* if it satisfies the ascending conditions on radical ideals of R.

(5) R is said to be *piecewise Noetherian* if it has Noetherian spectrum and it satisfies the ACC on P-primary ideals for each prime ideal P of R [1].
(6) R is said to have *finite ideal-length* if each ideal of R has finite ideal-length [1].
(7) R is said to satisfy (*accr*) if the ascending chain $(J :_R B) \subseteq (J :_R B^2) \subseteq (J :_R B^3) \subseteq \cdots$ of ideals stabilizes for all ideals J and B of R with B finitely generated [21].

It is well known that:

$$
\begin{tikzcd}
\text{Noetherian} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \text{s. Laskerian} \arrow[r, Rightarrow] & \text{Laskerian} \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \text{ZD} \arrow[r, Rightarrow] & \text{CIFR} \\
\text{FIL} & \text{p. Noetherian} \arrow[l, Rightarrow] \arrow[r, Rightarrow] & \text{Noetherian spectrum} & \text{(accr)} &
\end{tikzcd}
$$

An ideal A in a ring R is said to be an *RFG-ideal* if the radical of A is the radical of a finitely generated ideal: $\sqrt{A} = \sqrt{(a_1, \dots, a_n)}$. Clearly we may then select the elements a_i to belong to A. It is also known that if R has Noetherian spectrum, then $R[X]$ does also [23, Theorem 2.5].

Lemma 4.1. *If R is a domain in which at least one prime number is not a unit, then $R[X]_{\mathsf{H}}$ does not have Noetherian spectrum.*

Proof. This follows from two facts that a commutative ring R has Noetherian spectrum if and only if every prime ideal of R is an RFG-ideal [23, Corollary 2.4], and that the ideal $\pi^{-1}(0) = (X^{[1]}, X^{[2]}, \dots)$ is a prime ideal (Proposition 3.3(2)), but not a finitely generated ideal of $R[X]_{\mathsf{H}}$ [4, Corollary 7.5]. $\square$

Corollary 4.2. *If a domain R has a prime characteristic, then $R[X]_{\mathsf{H}}$ does not have Noetherian spectrum.*

Theorem 4.3. *Let R be a domain. Then $R[X]_{\mathsf{H}}$ has Noetherian spectrum if and only if R has Noetherian spectrum and $\mathbb{Q} \subseteq R$.*

Proof. ($\Leftarrow$) By [4, Corollary 1.5], $R[X]_{\mathsf{H}}$ is isomorphic to $R[X]$, which has Noetherian spectrum.

($\Rightarrow$) Note that $R \cong R[X]_{\mathsf{H}}/\pi^{-1}(0)$. Then R has Noetherian spectrum. Now we show that $\mathbb{Q} \subseteq R$. Since $R[X]_{\mathsf{H}}$ has Noetherian spectrum, by Lemma 4.1 all the prime numbers are units in R. So all the nonzero integers are also units in R. Therefore $\mathbb{Q} \subseteq R$. $\square$

Recall from [1, Proposition 2.2] that every maximal ideal of a piecewise Noetherian ring is finitely generated.

Lemma 4.4. *If R is a domain in which at least one prime number is not a unit, then $R[X]_{\mathsf{H}}$ is not piecewise Noetherian.*

Proof. This follows from Lemma 4.1 and the fact that every piecewise Noetherian ring has Noetherian spectrum. $\square$

Alternatively we can prove Lemma 4.4 as follows. Let M be a maximal ideal of R. Then by Proposition 3.3(2), the ideal $\pi^{-1}(M) = M + (X^{[1]}, X^{[2]}, \dots)$ is a maximal ideal of $R[X]_{\mathsf{H}}$. Also by [4, Corollary 7.5], the ideal $M + (X^{[1]}, X^{[2]}, \dots)$ is not a finitely generated ideal of $R[X]_{\mathsf{H}}$. Thus it follows from [1, Proposition 2.2] that $R[X]_{\mathsf{H}}$ is not piecewise Noetherian.

Corollary 4.5. *If a domain R has a prime characteristic, then $R[X]_{\mathsf{H}}$ is not piecewise Noetherian.*

Theorem 4.6. *Let R be a domain. Then $R[X]_{\mathsf{H}}$ is piecewise Noetherian if and only if R is piecewise Noetherian and $\mathbb{Q} \subseteq R$.*

Proof. ($\Leftarrow$) By [4, Corollary 1.5], $R[X]_{\mathsf{H}}$ is isomorphic to $R[X]$, which is piecewise Noetherian.

($\Rightarrow$) Note that $R \cong R[X]_{\mathsf{H}}/\pi^{-1}(0)$. Then R is piecewise Noetherian. Now we show that $\mathbb{Q} \subseteq R$. Since $R[X]_{\mathsf{H}}$ is piecewise Noetherian, by Lemma 4.4 all the prime numbers are units in R. So all the nonzero integers are also units in R. Therefore $\mathbb{Q} \subseteq R$. $\square$

It is known that the power series ring $R[\![X]\!]$ in one variable over R is Laskerian if and only if R is Noetherian ([11, Theorem 1]). It is also shown in [21, Theorem 2] that $R[X]$ (resp., $R[\![X]\!]$) satisfies (accr) if and only if R is Noetherian. In [3, Corollary 6.5], A. Benhissi gives an example of a commutative ring (not a domain) R of zero characteristic satisfying (accr), but $R[X]_{\mathsf{H}}$ and $R[\![X]\!]_{\mathsf{H}}$ do not.

Lemma 4.7. *If R is a $\mathbb{Z}$-torsion-free commutative ring with identity in which at least one prime number is not a unit, then the rings $R[X]_{\mathsf{H}}$ and $R[\![X]\!]_{\mathsf{H}}$ do not satisfy* (accr).

Proof. For simplicity, set $T := R[X]_{\mathsf{H}}$ or $T := R[\![X]\!]_{\mathsf{H}}$. Let p be a prime number which is not a unit in R. Now consider the following ascending sequence, where $X = X^{[1]}$:

$$((X) :_T p^2) \subseteq ((X) :_T p^3) \subseteq ((X) :_T p^4) \subseteq \cdots .$$

We claim that this sequence is strictly ascending. Note first that, since $XX^{[n]} = (1+n)X^{[n+1]}$ for all n, an element $f = \sum_{n=0}^{\infty} f_n X^{[n]}$ of T lies in (X) if and only if $f_0 = 0$ and f_n is divisible by n in R for all n. It follows that $X^{[p^n]} \in ((X) :_T p^n) - ((X) :_T p^{n-1})$ for all n, since if $p^{n-1}X^{[p^n]} \in (X)$, then p^n divides p^{n-1} in R, whence p divides 1 in R, which is a contradiction. Therefore the given sequence is strictly ascending, so T does not satisfy (accr). $\square$

Theorem 4.8. *Let R be a $\mathbb{Z}$-torsion-free commutative ring with identity. Then the ring $R[X]_{\mathsf{H}}$ (resp., $R[\![X]\!]_{\mathsf{H}}$) satisfies* (accr) *if and only if R is Noetherian and $\mathbb{Q} \subseteq R$.*

Proof. Suppose that R is Noetherian and $\mathbb{Q} \subseteq R$. By [4, Corollary 7.7], the rings $R[X]_{\mathsf{H}}$ and $R[\![X]\!]_{\mathsf{H}}$ are Noetherian and so satisfy (accr). Conversely, assume to the contrary that $R[\![X]\!]_{\mathsf{H}}$ satisfies (accr) and R is not Noetherian. Let $I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots$ be a strictly ascending chain of ideals of R. Set

$$J := \left\{ \sum_{i=0}^{\infty} a_i X^{[i]} \mid a_i \in I_i \text{ for all } i \in \mathbb{N} \right\}.$$

Then we obtain the following strictly ascending chain of ideals of $R[\![X]\!]_{\mathsf{H}}$:

$$(J : X) \subsetneq (J : X^{[2]}) \subsetneq (J : X^{[3]}) \subsetneq \cdots.$$

Indeed, if $a \in I_{i+1} \setminus I_i$, then $aX^{[i+1]} \in J$ but $aX^{[i]} \notin J$. Thus $a \in (J : X^{[i+1]})$ but $a \notin (J : X^{[i]})$. Therefore $(J : X^{[i]}) \subsetneq (J : X^{[i+1]})$ for all i. It then follows from Lemma 4.7 that $\mathbb{Q} \subseteq R$. Similarly, if R is not Noetherian, then $R[X]_{\mathsf{H}}$ does not satisfy (accr), so, again, $\mathbb{Q} \subseteq R$. □

An ideal I of a ring R is called *pseudo-irreducible* if for all ideals A and B of R, $I = AB$ and $A + B = R$ implies that $A = R$ or $B = R$. A *comaximal factorization* of a proper ideal I of R is a product $I = \prod_{i=1}^{n} I_i$ of proper ideals with $I_i + I_j = R$ for $i \neq j$. A comaximal factorization is *complete* if its factors are pseudo-irreducible. A ring R is called a *comaximal ideal factorization ring* (CIFR) whenever every proper ideal of R has a complete comaximal factorization. It is shown in [12, Proposition 3.5] that every ZD-ring is a CIFR. Denote by $J(R)$ the Jacobson radical of a ring R.

Lemma 4.9 ([12, Lemma 4.1]). *Let R be a ring. Then the following statements are equivalent.*

(1) *R is a CIFR.*
(2) *$R/J(R)$ is a CIFR.*
(3) *Every homomorphic image of R is a CIFR.*
(4) *R/I is a CIFR for some ideal $I \subseteq J(R)$.*

Theorem 4.10. *Let R be a ring. Then R is a CIFR if and only if $R[\![X]\!]_{\mathsf{H}}$ is a CIFR.*

Proof. First note that for a strictly ordered monoid $(S, \leq)$ such that for all $s \geq 0$ and $f \in [\![R^{S,\leq}]\!]$, $f \in J([\![R^{S,\leq}]\!])$ if and only if $f(0) \in J(R)$ [19, Corollary 2.2]. Thus we have an isomorphism: $[\![R^{S,\leq}]\!]/J([\![R^{S,\leq}]\!]) \cong R/J(R)$. Now the assertion follows from Theorem 2.2 and Lemma 4.9. □

Let P be a property of a class of commutative rings which is stable under the ring of generalized power series (resp., the semigroup ring) and homomorphic image. Then by Theorem 2.2, a ring R satisfies P if and only if $R[\![X]\!]_{\mathsf{H}}$ (resp., $R[X]_{\mathsf{H}}$) satisfies P, as the following showcase result shows.

In [22], W. K. Nicholson introduced the notion of a clean ring. He defined a ring R to be *clean* if every element of R can be written as a sum of a unit and an idempotent.

Theorem 4.11. *Let R be a commutative ring with identity. Then R is a clean ring if and only if $R[\![X]\!]_{\mathsf{H}}$ is a clean ring.*

Proof. Assume that R is a clean ring. Then by Theorem 2.2, the assertion follows from the facts that (1) for a strictly ordered monoid $(S, \leq)$ such that $s \geq 0$ for all $s \in S$, $[\![R^{S,\leq}]\!]$ is a clean ring if and only if R is a clean ring [20, Theorem 5.3] and (2) every homomorphic image of a clean ring is clean. The converse follows from the isomorphism $R \cong R[\![X]\!]_{\mathsf{H}}/\pi^{-1}(0)$ (Proposition 3.3(2)). $\square$

It is also known that $R[X]$ is a ZD-ring if and only if R is Noetherian [13, Theorem].

Question. When is $R[X]_{\mathsf{H}}$ a ZD-ring (resp., a ring with finite ideal-length)?

Acknowledgements. This research was supported by the Academic Research fund of Hoseo University in 2014 (2014-0425). The second author would like to thank Hoseo University and the California State University at Channel Islands for allowing him to spend a sabbatical year and wish to thank the people for their hospitality.

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