Bull. Korean Math. Soc. **55** (2018), No. 3, pp. 809–818
https://doi.org/10.4134/BKMS.b170265
pISSN: 1015-8634 / eISSN: 2234-3016

DOMAINS WITH INVERTIBLE-RADICAL FACTORIZATION

Malik Tusif Ahmed and Tiberiu Dumitrescu

Abstract. We study those integral domains in which every proper ideal can be written as an invertible ideal multiplied by a nonempty product of proper radical ideals.

In [15] Vaughan and Yeagy introduced and studied the notion of *SP-domain*, i.e., an integral domain whose ideals are products of radical (also called semi-prime) ideals. They proved that an SP-domain is always almost Dedekind (i.e., every localization at a maximal ideal is a rank one discrete valuation domain (DVR)). They also gave an example of an SP-domain which is not Dedekind. For examples of almost Dedekind domains which are not SP, see [16] and [6, Example 3.4.1]. The study of SP-domains was continued by Olberding (in [12]) who gave several characterizations for SP-domains inside the class of almost Dedekind domains and also gave a method to construct SP-domains starting from Boolean topological spaces.

In a sequence of papers ([10], [11], [13]) Olberding introduced and studied the concept of *ZPUI* (*Zerlegung Prim und Umkehrbaridealen*) *domain*, i.e., a domain for which every proper nonzero ideal can be factored as a product of an invertible ideal times a nonempty product of pairwise comaximal prime ideals (Olberding did his study for commutative rings, but we are interested here only in domain case). He showed that a domain A is ZPUI if and only if every proper nonzero ideal can be factored as a product of a finitely generated ideal times a nonempty finite product of prime ideals if and only if A is a strongly discrete h-local Prüfer domain [13, Theorem 1.1]. Let A be a domain. We recall that A is *h-local* if the factor ring A/I is local (resp. semilocal) for each nonzero prime ideal (resp. nonzero ideal) I of A. Also A is a *Prüfer domain* if its nonzero finitely generated ideals are invertible. A Prüfer domain is *strongly discrete* if it has no idempotent prime ideal except zero.

In this paper we study a new class of domains. Call a domain A an *ISP-domain* (*invertible semiprime domain*) if each proper ideal of A is can be written as an invertible ideal multiplied by a nonempty product of proper radical ideals. So any SP-domain (resp. ZPUI-domain) is an ISP-domain.

Received March 24, 2017; Accepted September 14, 2017.
2010 *Mathematics Subject Classification.* Primary 13A15; Secondary 13F15.
Key words and phrases. ZPUI-domain, SP-domain, Prüfer domain.

In Section 1 we prove the following results. If A is an ISP-domain, then any factor domain of A and any (flat) overring of A are also ISP-domains (Propositions 2 and 3, see also Proposition 9). Any one-dimensional ISP-domain is almost Dedekind and, consequently, any Noetherian ISP-domain is a Dedekind domain (Corollary 4). In Section 2 we prove that if A is an ISP-domain, then A is a strongly discrete Prüfer domain and every nonzero prime ideal of A is contained in a unique maximal ideal (Theorem 5). Consequently, an ISP-domain such that every ideal has finitely many minimal prime ideals is a ZPUI-domain (Corollary 10). In Section 3 we consider the question whether every one-dimensional ISP-domain is an SP-domain. We provide a positive answer for domains in which every nonzero element is contained in at most finitely many noninvertible maximal ideals (Theorem 13). In particular, a one-dimensional ISP-domain having only finitely many noninvertible maximal ideals is an SP-domain (Corollary 14). In Section 4 we give an example of a two-dimensional ISP-domain A which is not h-local. Hence A is neither an SP-domain nor a ZPUI-domain.

Throughout this paper, our rings are commutative and unitary. For any undefined terminology, we refer the reader to [8] or [9].

1. Basic results

We recall the key definition of our paper.

Definition 1. We say that a domain A is an *ISP-domain* (*invertible semiprime domain*) if every proper nonzero ideal I of A can be written as $JQ_1 \cdots Q_n$ where $n \geq 1$, J is an invertible ideal and each Q_i is a proper radical ideal.

Clearly a ZPUI-domain or an SP-domain is an ISP-domain. The well-known Bezout domain $A = \mathbb{Z} + X\mathbb{Q}[X]$ (see [4] for its basic properties) is not an ISP-domain. Indeed, consider the ideal $I = X\mathbb{Z}[1/2] + X^2\mathbb{Q}[X]$. The radical ideals containing I are $X\mathbb{Q}[X]$ and $nA = n\mathbb{Z} + X\mathbb{Q}[X]$ with n a positive square-free integer. So there is no element $f \in A$ such that $I \subseteq fA$ and If^{-1} is a product of radical ideals. Note that every proper nonzero principal ideal gA can be written in the form required by Definition 1. Indeed, if $g \notin X\mathbb{Q}[X]$, then g is a product of principal primes and if $g \in X\mathbb{Q}[X]$, then $g = 2(g/2)A$. Note also that A is strongly discrete.

In this section we prove a few basic properties of ISP-domains.

Proposition 2. *If A is an ISP-domain and P a prime ideal of A, then A/P is an ISP-domain.*

Proof. Let $I \supset P$ be a proper ideal of A. As A is an ISP-domain, we can write $I = JH_1 \cdots H_n$ with J an invertible ideal, $n \geq 1$ and each H_i a proper radical ideal. Since all ideals $I, H_1, \ldots, H_n$ contain P, we get

$$I/P = (J/P)(H_1/P) \cdots (H_n/P)$$

with J/P invertible and each H_i/P a proper radical ideal. □

Proposition 3. *Let A be an ISP-domain and B a flat overring of A. Then B is an ISP-domain.*

Proof. Let H be a proper nonzero ideal of B and $I = H \cap A$. By [2, Theorem 2], $IB = H$. As A is an ISP-domain, we can write $I = JQ_1 \cdots Q_n$ with J an invertible ideal, $n \geq 1$ and all Q_i's proper radical ideals. Then $H = IB = (JB)(Q_1B) \cdots (Q_nB)$, where JB is invertible and each Q_iB is a radical ideal. Indeed, since $A_{M\cap A} = B_M$ for every $M \in \mathrm{Max}(B)$ (cf. [2, Theorem 2]), it is easy to check locally that a radical ideal of A extends to a radical ideal of B. If every Q_iB is equal to B, then $H = JB$ and $WB = B$ where $W = Q_1 \cdots Q_n$. Hence $J \subseteq JB \cap A = H \cap A = I = JW \subseteq J$, so $J = JW$, thus $W = A$ (because J is invertible), a contradiction. □

We give a simple application of Proposition 3.

Corollary 4. *Any one-dimensional ISP-domain is almost Dedekind. Consequently, a Noetherian ISP-domain is a Dedekind domain.*

Proof. Let A be a one-dimensional ISP-domain. By Proposition 3, we may assume that A is local with maximal ideal M. Let $x \in M - \{0\}$. Since the radical ideals of A are 0 and M, we get $xA = yM^k$ for some $y \in A$ and $k \geq 1$, so M is invertible, hence A is a DVR. For the "Consequently" part, assume, by the contrary, that A is a Noetherian ISP-domain which is not Dedekind. By the first part, $\dim(A) \geq 2$, so, using Proposition 3, we may assume that A is a two-dimensional local domain (with maximal ideal M). Let $x \in M - M^2$, P a height one prime ideal containing x and let $y \in M - P$. Since $P \not\subseteq M^2$, M is minimal over (P, y^2) and A is an ISP-domain, we get $(P, y^2) = M$. Modding out by P, we get a contradiction. □

2. ISP domains are Prüfer strongly discrete

The following theorem is the main result of this paper.

Theorem 5. *If A is an ISP-domain, then*

(a) *A is a strongly discrete Prüfer domain, and*
(b) *every nonzero prime ideal of A is contained in a unique maximal ideal.*

In particular, a local domain is an ISP-domain if and only if it is a strongly discrete valuation domain.

We need a string of three lemmas.

Lemma 6. *If A is an ISP-domain and $P \subset M$ are nonzero prime ideals of A, then $P \subseteq M^2A_M$.*

Proof. By Proposition 3, we may assume that A is local with maximal ideal M. Assume that $P \not\subseteq M^2$ and take $x \in M - P$. Since A is an ISP-domain and $P \not\subseteq M^2$, we get that (P, x^2) is a radical ideal, so $(P, x^2) = (P, x)$ which gives a contradiction after modding out by P. □

Lemma 7. *Let A be an ISP-domain, $P \subset M$ prime ideals and $x \in M - P$ such that M is minimal over (P, x). Then MA_M is a principal ideal.*

Proof. By Proposition 3, we may assume that A is local with maximal ideal M. We show first that M is not idempotent. On contrary assume that $M^2 = M$. Note that $\sqrt{(P, x)} = M$ is the only radical ideal containing (P, x). As A is an ISP-domain and $M = M^2$, we get $(P, x) = yM$ for some $y \in A$. As $P \subseteq yM$, we get $y \notin P$ (otherwise $P = yA \subseteq yM$), hence $P = Py$. From $x \in yM$, we get $x = yz$ for some $z \in M$. Now from $(Py, yz) = yM$, we get $(P, z) = M$, so M/P is a principal idempotent nonzero maximal ideal of A/P, a contradiction. Thus M is not idempotent and let us pick $w \in M - M^2$. By Lemma 6, M is the only prime ideal containing w, so $wA = M$ because A is an ISP-domain. □

Lemma 8. *If A is an ISP-domain and I an invertible radical proper ideal of A, then A/I is zero-dimensional.*

Proof. On contrary assume that $\dim(A/I) \geq 1$. Then there exist two prime ideals $P \subset M$ and $x \in M - P$ such that $I \subseteq P$ and M is minimal over (P, x). By Lemma 7, MA_M is principal. Localizing at M, we may assume that A is local with maximal ideal M. Then $I = yA$ and $M = zA$ for some $y, z \in A$. As $I \subset M$, we get $y = az^2$ for some $a \in A$, so $az \in \sqrt{yA} = yA$, hence $y = az^2 \in yzA$, thus $1 \in zA = M$, a contradiction. □

Proof of Theorem 5. (a) By [13, Lemma 3.2], it suffices to show that PA_P is a principal ideal for every nonzero prime ideal P of A. Set $B = A_P$ and $M = PA_P$. By Proposition 3, B is an ISP-domain. Given $x \in M - \{0\}$, we write $xB = yH_1 \cdots H_n$ with $y \in B$, $n \geq 1$ and H_i a proper radical ideal for $i = 1$ to n. Then each H_i is invertible hence principal, because B is local. By Lemma 8, we have $Spec(B/H_1) = \{M/H_1\}$, hence $H_1 = \sqrt{H_1} = M$.

(b) By Proposition 3, we may assume that A is semilocal. Indeed, if M_1 and M_2 are two distinct maximal ideals containing a nonzero prime ideal, then (b) fails for A_S, where $S = A - (M_1 \cup M_2)$. Now let I be a nonzero radical ideal. Since A is a semilocal Prüfer domain, it follows that I has finitely many minimal primes, say $P_1, \ldots, P_n$. Then $I = P_1 \cap \cdots \cap P_n = P_1 \cdots P_n$ because $P_1, \ldots, P_n$ are incomparable prime ideals in a Prüfer domain, hence pairwise comaximal. Since A is an ISP-domain and every nonzero radical ideal is a product of primes, A is a ZPUI-domain. By [13, Theorem 1.1], A is h-local, so (b) holds. The "in particular" assertion follows from [13, Theorem 1.1]. □

We give two corollaries of Theorem 5.

Corollary 9. *Any overring of an ISP-domain is also an ISP-domain.*

Proof. Let A be an ISP-domain and B an overring of A. By Theorem 5, A is a Prüfer domain, so B is A-flat, cf. [14, page 798]. Apply Proposition 3. □

Corollary 10. *For a domain A, the following are equivalent.*

(a) *A is a ZPUI-domain.*

(b) *A is an h-local strongly discrete Prüfer domain.*
(c) *A is an h-local ISP-domain.*
(d) *A is a generalized Dedekind ISP-domain.*
(e) *A is an ISP-domain such that* $\mathrm{Min}(I)$ *is finite for each ideal* I.

Proof. (a) $\Leftrightarrow$ (b) is a part of [13, Theorem 1.1]. Implications [(a) and (b)] $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) are well-known. For (e) $\Rightarrow$ (a), repeat the second half of the proof of Theorem 5 part (b). □

3. Almost Dedekind ISP-domains

In this section, we consider the question whether any one-dimensional ISP-domain is an SP-domain. First, we recall some terminology from [12]. Let A be an almost Dedekind domain. The maximal ideals of A containing a radical invertible ideal are called *non-critical*, while the others are called *critical.* Given I an ideal of A and $n \geq 1$, we set $V_n(I) = \{M \in \mathrm{Max}(A) \mid I \subseteq M^n\}$. Note that $V_{n+1}(I) \subseteq V_n(I)$ and $V_1(I)$ is the usual Zariski closed set $V(I)$. Next, we recall [12, Theorem 2.1] and add a new assertion (g).

Theorem 11 ([12, Theorem 2.1])**.** *For an almost Dedekind domain A, the following assertions are equivalent.*

(a) *A is an SP-domain.*
(b) *A has no critical maximal ideals.*
(c) *The radical of an invertible ideal is invertible.*
(d) *Ever principal ideal is a product of radical ideals.*
(e) *For every nonzero proper (principal) ideal* I *and* $n \geq 1$, *the set* $V_n(I)$ *is (Zariski) closed in Spec(A) and* $V_m(I)$ *is empty for some large* m.
(f) *Every nonzero proper ideal* I *can be factorized (uniquely) as* $I = J_1J_2\cdots J_n$ *with radical ideals* $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n$.
(g) *For every nonzero proper ideal* I, *we have* $I = \sqrt{I}H$ *for some ideal* H.

Proof. Since only (g) is new, it suffices to prove the equivalence of (f) and (g). (g) $\Rightarrow$ (f) We have $I = \sqrt{I}H_1$ and $H_1 = \sqrt{H_1}H_2$ for some ideals H_1 and H_2. Set $J_1 = \sqrt{I}$ and $J_2 = \sqrt{H_1}$, so $I = J_1J_2H_2$. From $I \subseteq H_1$, we get $J_1 \subseteq J_2$. Repeating, we get $I = J_1J_2\cdots J_nH_n$ with radical ideals $J_1 \subseteq \cdots \subseteq J_n$. If some H_n is A, we are done. If not, let M be a maximal ideal containing all J_i's. Then $I = J_1J_2\cdots J_nH_n \subseteq M^n$ for each $n \geq 1$, which is a contradiction because A_M is a DVR. Conversely, from $I = J_1\cdots J_n$ with $J_1 \subseteq \cdots \subseteq J_n$ radical ideals, we get $\sqrt{I} = J_1$, so we are done. □

In the next lemma, we recall two known facts.

Lemma 12. *If A is an almost Dedekind domain which is not Dedekind, then*

(a) *Every noninvertible nonzero ideal of A is contained in some noninvertible maximal ideal.*
(b) *Every infinite closed subset of* $\mathrm{Max}(A)$ *contains some noninvertible maximal ideal.*

Proof. (a) is a well-known application of Zorn's Lemma (every non finitely generated ideal is contained in a non finitely generated prime ideal).

(b) Let I be a nonzero ideal such that $V(I)$ is infinite. By (a), we may assume that I is invertible, so the assertion follows from [6, Proposition 3.2.2]. We give an alternative proof. For each $P \in V(I)$, we have $IA_P = (PA_P)^{n_P}$ for some (unique) positive integer n_P. Consider the ideal $H = \sum_{P\in V(I)} IP^{-n_P}$. It suffices to show that H is not finitely generated, because $I \subseteq H$ implies $V(H) \subseteq V(I)$, so part (a) applies. Suppose that H is finitely generated. Then there exist distinct ideals $P_1, \ldots, P_{k+1} \in V(I)$ such that $IP_{k+1}^{-n_{k+1}} \subseteq \sum_{i=1}^{k} IP_i^{-n_i}$ where $n_j = n_{P_j}$. Since the ideals P_j are mutually comaximal, we have $IP_{k+1}^{-n_{k+1}} \subseteq I(\cap_{i=1}^{k} P_i^{n_i})^{-1}$, cf. [10, Lemma 5.1]. We cancel I and get $\cap_{i=1}^{k} P_i^{n_i} \subseteq P_{k+1}$, which is a contradiction. □

Recall that a domain A *has weak factorization*, if every nonzero nondivisorial ideal I can be factored as the product of its divisorial closure I_ν and a finite product of maximal ideals; i.e., $I = I_\nu M_1 M_2 \cdots M_n$ where $M_1, M_2, \ldots, M_n$ are maximal ideals, cf. [5]. By [6, Proposition 4.2.14], an almost Dedekind domain A has weak factorization if and only if every nonzero element of A is contained in at most finitely many noninvertible maximal ideals.

Now let A be an almost Dedekind domain A which has weak factorization. Denote by Z the set of noninvertible maximal ideals of A. We introduce an ad-hoc concept: call an ideal H of A a *clean ideal*, if H is invertible, $V(H) \cap Z = \{M\}$ and $H \not\subseteq M^2$. Let $M \in Z$ and $f \in M - \{0\}$. By our hypothesis $V(f) \cap Z$ is finite, say equal to $\{M, M_1, \ldots, M_n\}$. By Prime Avoidance Lemma (e.g. [8, Proposition 4.9]), we can pick an element $g \in M - (M^2 \cup M_1 \cup \cdots \cup M_n)$, so (f, g) is clean. Hence every $M \in Z$ contains a clean ideal. With terminology and notation above, we have:

Theorem 13. *For an almost Dedekind domain A which has weak factorization, the following assertions are equivalent.*

(a) *A is an SP-domain.*
(b) *A is an ISP-domain.*
(c) *For every clean ideal H, the set $V_2(H)$ is finite.*
(d) *Every $M \in Z$ contains a clean ideal H such that $V_2(H)$ is finite.*

Proof. We may assume that A is not a Dedekind domain. Set $F = \mathrm{Max}(A) - Z$. (a) ⇒ (b) is obvious. (b) ⇒ (c) Assume, to the contrary, that H is a clean ideal and $V_2(H)$ contains an infinite set $\{P_n \mid n \geq 1\} \subseteq F$. Set $V(H) \cap Z = \{M\}$. Let I be the (integral) ideal $\sum_{n\geq 0} HP_{2n+1}^{-1}$. Since $H \subseteq I$ and $V(H) \cap Z = \{M\}$, we get $V(I) \cap Z = \{M\}$, because $M \supseteq H = P_{2n+1}HP_{2n+1}^{-1}$ implies $M \supseteq HP_{2n+1}^{-1}$. As A is an ISP-domain, we can write $I = JQ$ with J an invertible ideal and $Q \neq A$ a product of radical ideals. Since $M \in V(I) - V_2(I)$, we have one of the two cases below.

Case 1: $M \supseteq J$ and $M \not\supseteq Q$. Then $V(Q) \cap Z$ is empty, so Q is invertible, cf. Lemma 12. So $I = JQ$ is invertible, hence finitely generated. Then $HP_{2n+1}^{-1} \subseteq HP_1^{-1} + \cdots + HP_{2n-1}^{-1}$ for some $n \geq 1$. Since H can be cancelled and the other ideals involved are invertible and comaximal, we get $P_{2n+1}^{-1} \subseteq (P_1 \cap \cdots \cap P_{2n-1})^{-1}$ (cf. [10, Lemma 5.1]), hence $P_{2n+1} \supseteq P_1 \cap \cdots \cap P_{2n-1}$, which is a contradiction.

Case 2: $M \not\supseteq J$ and $M \supseteq Q$. Since $H \subseteq Q$ and $H \not\subseteq M^2$, we have that $V_2(Q) \cap Z = \emptyset$. As Q is a product of radical ideals, [1, Lemma 1.10] shows that $V_2(Q)$ is closed, so $V_2(Q)$ is finite, cf. Lemma 12. Note that $P_{2n} \in V_2(I)$ for every $n \geq 1$. Consequently, there exists some $m \geq 1$ such that $P_{2n} \in V(J)$ for each $n \geq m$. By Lemma 12 and the fact that $H \subseteq J$, we get $V(J) \cap Z = \{M\}$, which is a contradiction.

(c) $\Rightarrow$ (d) is clear.

(d) $\Rightarrow$ (a) By [12, Theorem 2.1], it suffices to show that each $M \in Z$ contains an invertible radical ideal. By (d), M contains a clean ideal H such that $V_2(H)$ is finite, say equal to $\{P_1, \ldots, P_n\}$. For each i between 1 and n, we have $HA_{P_i} = P_i^{k_i} A_{P_i}$ for some $k_i \geq 2$. Then $HP_1^{-k_1} \cdots P_n^{-k_n}$ is an invertible radical ideal contained in M. □

The SP-domain A constructed in [12, Example 4.3] has nonzero Jacobson radical and no $M \in \mathrm{Max}(A)$ finitely generated. Thus A does not have weak factorization.

Corollary 14. *Let A be almost Dedekind domain having only finitely many noninvertible maximal ideals. Then A is an ISP-domain if and only if A is an SP-domain.*

Corollary 15. *Let A be an ISP-domain which has weak factorization and B a one-dimensional overring of A. Then B is an SP-domain.*

Proof. By Theorem 5, A is a strongly discrete Prüfer domain, so B has weak factorization, cf. [6, Corollary 4.3.3]. Now apply Corollary 9 and Theorem 13. □

The following question remains.

Question 16. Is every one-dimensional ISP-domain an SP-domain?

4. An example

In this final section we give an example of a two-dimensional ISP-domain A which is not h-local. Hence A is neither an SP-domain nor a ZPUI-domain.

Proposition 17. *Let C be an SP-domain but not Dedekind, $M = qC$ a maximal principal ideal of C and D a DVR with quotient field C/M. Assume there exists a unit p of C such that $\pi(p)$ generates the maximal ideal of D, where $\pi : C \to C/M$ is the canonical map. Then the pull-back domain $A = \pi^{-1}(D)$ is a two-dimensional ISP-domain which is not h-local.*

Proof. As $\pi(Mp^{-1}) = 0$, it follows that $M \subseteq pA$, so A/pA is the residue field of D, because $A/M = D$ and $\pi(p)$ generates the maximal ideal of D. Also, the only prime ideal of A strictly containing M is the maximal ideal pA. By standard pull-back arguments (see for instance [7, Lemma 1.1.4]), the map $P \mapsto P \cap A$ is a bijection from $Spec(C) - V(M)$ to $Spec(A) - V(M)$ and $A_{P\cap A} = C_P$. By [7, Corollary 1.1.9], A is a two-dimensional Prüfer domain. Also, by [7, Lemma 1.1.6], we have $A[p^{-1}] = C[p^{-1}] = C$. Roughly speaking, $Spec(A)$ is obtained from $Spec(C)$ by adding the maximal ideal $pA \supseteq M$. Since C is an almost Dedekind domain which is not Dedekind, there exists a nonzero element $z \in A$ belonging to infinitely many maximal ideals of A, so A is not h-local. By [7, Proposition 5.3.3], $B = A_{pA}$ is a two-dimensional strongly discrete valuation domain. It follows that $\cap_{t\geq 1} p^t A = M$.

Let I be an ideal of A. We observe that $I = IB \cap IC$. Indeed, if $N \in \text{Max}(A) - \{pA\}$, then $I \subseteq IC_{A-N} = IA_N$, so $IB \cap IC \subseteq \cap_{Q\in \text{Max}(A)} IA_Q = I$. In particular, we have $A = B \cap C$. Since C is almost Dedekind and $M = qC$, we can write $IC = M^i J$ where J is an ideal of C with $M + J = C$ and $i \geq 0$, so $IC = M^i \cap J$. We also see that $H := J \cap A \not\subseteq M$. As $\cap_{t\geq 1} p^t A = M$, we can write $H = p^j L = p^j A \cap L$ where L is an ideal of A with $L \not\subseteq pA$ and $j \geq 0$. Consequently we get

$$IC \cap A = M^i \cap J \cap A = M^i \cap H = M^i \cap p^j A \cap L$$

which equals either $M^i \cap L$ if $i \geq 1$ or $p^j A \cap L$ if $i = 0$. Using basic facts on valuation domains (see [8, Section 17]), it suffices to consider the following three cases. Each time we use the equality $I = (IB \cap A) \cap (IC \cap A)$.

Case 1: $IB = p^n B$ for some $n \geq 0$. We have $IB \cap A = p^n A$. If $i \geq 1$, we get $I = p^n A \cap M^i \cap L = M^i L$. If $i = 0$, we get $I = p^n A \cap p^j A \cap L = p^k L$ with $k = \max(n, j)$.

Case 2: $IB = M^n$ for some $n \geq 1$. If $i \geq 1$, we get $I = M^n \cap M^i \cap L = M^k L$ with $k = \max(n, i)$. If $i = 0$, we get $I = M^n \cap p^j A \cap L = M^n L$.

Case 3: $IB = p^n q^m A$ for some $m \geq 1$ and $n \in \mathbb{Z}$. We have $IB \cap A = p^n q^m A$, because pA is the only maximal ideal containing q. If $i > m \geq 1$, we get $I = p^n q^m A \cap M^i \cap L = M^i L$. If $m \geq i \geq 1$, we get $I = p^n q^m A \cap M^i \cap L = p^n q^m L$. If $i = 0$, we get $I = p^n q^m A \cap p^j A \cap L = p^n q^m L$.

Consequently, to complete our proof, it suffices to show that L is a product of radical ideals. Since C is an SP-domain, we can write $LC = H_1 \cdots H_n$ with each H_i a radical ideal of C. Then each $J_i = H_i \cap A$ is a radical ideal of A. Note that none of ideals J_i is contained in pA, since $L \not\subseteq pA$. Set $R = J_1 \cdots J_n$. Then $R + pA = A$ and $L + pA = A$, so $R : p = R$ and $L : p = L$. Since $RC = H_1 \cdots H_n = LC$, we get $L = LC \cap A = RC \cap A = R$. □

Finally, we construct a specific domain satisfying the hypothesis of Proposition 17. We modify appropriately [6, Example 3.4.1]. If A is a domain and $P_1, \ldots, P_n$ are prime ideals of A, we denote by $A_{P_1\cup\cdots\cup P_n}$ the fraction ring of A with denominators in $A - (P_1 \cup \cdots \cup P_n)$. Let y and $(x_n)_{n\geq 1}$ be indeterminates over the rational field $\mathbb{Q}$. Consider the domain

$$C = \bigcup_{n \geq 1} \mathbb{Q}[x_1, \ldots, x_n, y/(x_1 \cdots x_n)]_{(x_1) \cup \cdots \cup (x_n) \cup (y/(x_1 \cdots x_n))}.$$

As C is a union of an ascending chain of (semi-local) PID's, it is a one-dimensional Bezout domain. Adapting the proof of [6, Example 3.4.1], we see that the maximal ideals of C are $N = \sum_{n \geq 1}(y/(x_1 \cdots x_n))C$ and the principal ideals $(x_n C)_{n \geq 1}$. As $yC_M = MC_M$ for each $M \in \text{Max}(C)$, it follows that yC is a radical ideal, hence N is non-critical. By [12, Corollary 2.2], C is an SP-domain. The residue field C/x_1C is isomorphic to $K(y/x_1)$ where $K = \mathbb{Q}(x_n; n \geq 2)$. Then $D = K[y/x_1]_{(y/x_1)}$ is a DVR with quotient field C/x_1C. Note that $x_1 + y/x_1$ is a unit of $\mathbb{Q}[x_1, y/x_1]_{(x_1) \cup (y/x_1)}$, hence a unit of C. Moreover, the canonical map $C \to C/x_1C$ sends $x_1 + y/x_1$ to y/x_1 which is a generator of the maximal ideal of D. Thus C satisfies the hypothesis of Proposition 17.

Acknowledgements. The first author is highly grateful to Abdus Salam School of Mathematical Sciences Govt. Coll. University Lahore in supporting and facilitating this research. The second author gratefully acknowledges the warm hospitality of the same institution during his visits in the period 2006-2016.

References

[1] M. T. Ahmed and T. Dumitrescu, *SP-rings with zero-divisors*, Comm. Algebra **45** (2017), no. 10, 4435–4443.

[2] T. Akiba, *Remarks on generalized rings of quotients*, Proc. Japan Acad. **40** (1964), 801–806.

[3] D. D. Anderson, *Nonfinitely generated ideals in valuation domains*, Tamkang J. Math. **18** (1987), no. 2, 49–52.

[4] D. Costa, J. L. Mott, and M. Zafrullah, *The construction* $D + XD_s[X]$, J. Algebra **53** (1978), no. 2, 423–439.

[5] M. Fontana, E. Houston, and T. Lucas, *Factoring ideals in Prüfer domains*, J. Pure Appl. Algebra **211** (2007), no. 1, 1–13.

[6] ______, *Factoring Ideals in Integral domains*, Lecture Notes of the Unione Matematica Italiana, 14, Springer, Heidelberg, 2013.

[7] M. Fontana, J. A. Huckaba, and I. J. Papick, *Prüfer domains*, Monographs and Textbooks in Pure and Applied Mathematics, **203**, Marcel Dekker, Inc., New York, 1997.

[8] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, Inc., New York, 1972.

[9] I. Kaplansky, *Commutative Rings*, revised edition, The University of Chicago Press, Chicago, IL, 1974.

[10] B. Olberding, *Globalizing local properties of Prüfer domains*, J. Algebra **205** (1998), no. 2, 480–504.

[11] ______, *Factorization into prime and invertible ideals*, J. London Math. Soc. (2) **62** (2000), no. 2, 336–344.

[12] ______, *Factorization into radical ideals*, in Arithmetical properties of commutative rings and monoids, 363–377, Lect. Notes Pure Appl. Math., 241, Chapman & Hall/CRC, Boca Raton, FL, 2005.

[13] ______, *Factorization into prime and invertible ideals. II*, J. Lond. Math. Soc. (2) **80** (2009), no. 1, 155–170.

[14] F. Richman, *Generalized quotient rings*, Proc. Amer. Math. Soc. **16** (1965), 794–799.
[15] N. H. Vaughan and R. W. Yeagy, *Factoring ideals into semiprime ideals*, Canad. J. Math. **30** (1978), no. 6, 1313–1318.
[16] R. W. Yeagy, *Semiprime factorizations in unions of Dedekind domains*, J. Reine Angew. Math. **310** (1979), 182–186.

Malik Tusif Ahmed
Abdus Salam School of Mathematical Sciences GCU Lahore
Pakistan
Email address: tusif.ahmed@sms.edu.pk, tusif.ahmad92@gmail.com

Tiberiu Dumitrescu
Facultatea de Matematica si Informatica
University of Bucharest
14 Academiei Str., Bucharest, RO 010014, Romania
Email address: tiberiu@fmi.unibuc.ro, tiberiu_dumitrescu2003@yahoo.com