Bull. Korean Math. Soc. **55** (2018), No. 3, pp. 799–808
https://doi.org/10.4134/BKMS.b170258
pISSN: 1015-8634 / eISSN: 2234-3016

KAUFFMAN POLYNOMIAL OF PERIODIC KNOTTED TRIVALENT GRAPHS

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Abstract. We generalize some of the congruences in [20] to periodic knotted trivalent graphs. As an application, a criterion derived from one of these congruences is used to obstruct periodicity of links of few crossings for the odd primes $p = 3, 5, 7$, and 11. Moreover, we derive a new criterion of periodic links. In particular, we give a sufficient condition for the period to divide the crossing number. This gives some progress toward solving the well-known conjecture that the period divides the crossing number in the case of alternating links.

1. introduction

Periodicity is reflected in many examples of polynomial invariants for links such as the Jones polynomial [17] and its 2-variable generalization [5, 6, 20, 22–25], the Alexander polynomial [8,15,16], and the twisted Alexander polynomials [10]. Different criteria of periodicity have been found in terms of hyperbolic structures on knot complements [1], homology groups of cyclic branched covers [18,19], concordance invariants of Casson and Gordan [19], Khovanov homology [7], link Floer homology [9] and the Heegaard Floer correction terms of the finite cyclic branched covers of knots [11].

Our purpose in this paper is to study periodicity of knotted trivalent graphs in terms of the rational function defined in [4] to recover the Kauffman polynomial as a state summation. The trivalent graphs considered are of two types planar and knotted embedded in $\mathbb{R}^3$. Each of which has two types of edges called standard and wide such that each trivalent vertex is incident to one wide edge and two standard edges. We refer the reader to [4] for further details about this topic.

The congruences of periodic knotted trivalent graphs are given in terms of this rational function. The restriction of these congruences to links yields the congruences of periodic links in terms of the Kauffman polynomial given in

Received March 23, 2017; Accepted November 22, 2017.
2010 *Mathematics Subject Classification.* 57M27, 57M15.
Key words and phrases. Kauffman polynomial, periodic links, knotted trivalent graphs, crossing number, adequate links.

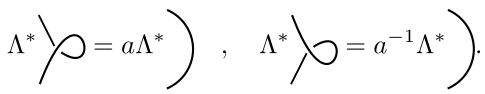
[20]. We apply a criterion derived from these congruences to study periodicity of some examples of knots and links. In particular, we give two tables summarizing the application of this criterion for all knots of 11 crossings or less and links of 8 crossings or less for the odd primes $p = 3, 5, 7, 11$.

Finally, we provide a new criterion of periodic links in terms of the breadth of a specialized Kauffman polynomial modulo the odd prime. This criterion implies that the period of a periodic link divides its crossing number under some condition. If this sufficient condition holds for alternating links, then we obtain a positive solution to the well-known conjecture that the period of a periodic alternating link divides its crossings number.

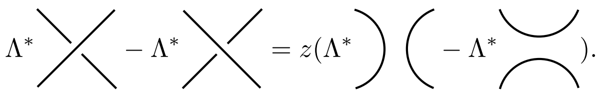
2. The Kauffman polynomial of knotted trivalent graphs

Kauffman in [12] introduced a new two-variable Laurent polynomial $\Lambda^*_D = \Lambda^*_D(a, z)$ of regular isotopy of unoriented diagrams of links. If w is the writhe of the oriented diagram D, then $F^*_L(a, z) = a^{-w}\Lambda^*_D(a, z)$ is an invariant of the oriented link L. The polynomial $\Lambda^*_D(a, z)$ is defined recursively by the following relations.

(a) If D is a simple closed curve, then $\Lambda^*_D = 1$.
(b) If D_1 and D_2 are related by a finite sequence of Reidemeister moves of type II or III, then $\Lambda^*_{D_1} = \Lambda^*_{D_2}$.
(c)



(d)



Throughout this paper, diagrams that appear in one equation are identical except as indicated in a small disk.

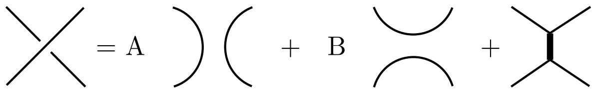
The above polynomial is sometimes called the Dubrovnik version of the two-variable Kauffman polynomial of unoriented links with oriented version $F_L(a, z)$ defined by $F_L(a, z) = a^{-w}\Lambda_D(a, z)$, where w is the writhe of the diagram D. If D represents a link with r components, then the original Kauffman polynomials $\Lambda_D(a, z), F_L(a, z)$ for unoriented and oriented links respectively are related to the above polynomials $\Lambda^*_D(a, z), F^*_L(a, z)$ by the following formulas. The first one is due to Lickorish; see [12, Page 466], [13, Page 104] and the second one follows from the first one.

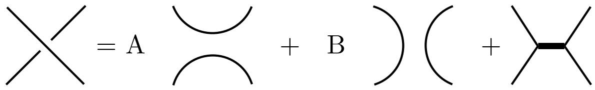
$$\Lambda_D(a, z) = (-1)^{r+1}(-\sqrt{-1})^{-w(D)}\Lambda^*_D(-\sqrt{-1}a, \sqrt{-1}z). \tag{1}$$

$$F_L(a, z) = F^*_L(-\sqrt{-1}a, \sqrt{-1}z). \tag{2}$$

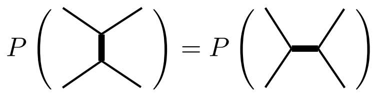
In [4], the authors constructed a three-variable rational function which is an invariant of certain type of knotted unoriented trivalent graphs whose restriction to links yields a version of the Kauffman polynomial. We give a quick overview of this construction.

This polynomial is obtained as a state summation, the states are planar trivalent graphs. The states of any knotted trivalent graph are obtained by applying the recurrence formulas in Equations 3 and 4 with commuting variables A and B. These two equations associate to each knotted trivalent graph Υ a formal linear combination of planar trivalent graphs.

(3) 

(4) 

The authors of [4] assigned a unique polynomial $P(\Gamma) \in \mathbb{Z}[a^{\pm 1}, A, B, (A-B)^{\pm 1}]$ to each planar trivalent graph that takes the value 1 for the unknot and satisfies many identities mentioned in [4]. One of these identities that will be used in our paper is as follows:

(5) 

After resolving all the crossings of the knotted trivalent graph, we express it as a finite formal linear combination of the states whose coefficients are monomials in the commuting variables A and B. Now we associate to the knotted trivalent graph Υ the polynomial $\mathbb{P}_\Upsilon$ that is obtained by summing up the graph polynomials $P(\Gamma)$ weighted by their coefficients. That is,

$$\mathbb{P}_\Upsilon = \sum_\Gamma A^m B^n P(\Gamma),$$

where m, n is the number of times of A-resolution, B-resolution respectively to obtain the graph state Γ.

We refer the reader to [4, Theorem 1] for the proof of the existence and uniqueness of the polynomial $P(\Gamma)$ for any planar trivalent graph. The polynomial $\mathbb{P}_\Upsilon$ restricted to links gives the Kauffman polynomial using the following formula that first appeared in [4, Corollary 1]:

$$\Lambda^*_D(a, z) = \Lambda^*_D(a, A-B) = \mathbb{P}_D(a, A, B).$$

3. Main results

Hereafter, we let $\mathcal{R}$ to be the subring of $\mathbb{Z}[\omega, a^{\pm 1}, A, B, (A-B)^{\pm 1}]$ generated by $\omega, a^{\pm 1}, A, B$ and $\frac{a-a^{-1}}{A-B}$ for any primitive $(p-1)$-th root of unity ω. Observe that $A-B$ is not invertible in the subring $\mathcal{R}$. Now we define two ideals I

and J of the subring $\mathcal{R}$. The ideal I generated by p and $(A-B)^p$ and the ideal J generated by p and $\left(\frac{a-a^{-1}}{A-B}\right)-\omega$. We need the following lemma that is motivated by [20, Lemma 1.1] and generalize it to the case of knotted trivalent graphs.

Lemma 3.1. *For any knotted trivalent graph Υ, we have $\mathbb{P}_\Upsilon(a,A,B)\in\mathcal{R}$. In particular, we have $\Lambda^*_D(a,A-B)\in\mathcal{R}$.*

Proof. We use induction on the number of pairs of trivalent vertices in the knotted trivalent graph Υ. In the case that Υ has zero pairs of trivalent vertices, then Υ simply represents a link diagram D of some link. In this case, we use induction on the number of crossings in the diagram D. The result holds for the trivial link of n components simply since $\Lambda^*_D(a,A-B)=\mathbb{P}_D(a,A-B)=\left(\frac{a-a^{-1}}{A-B}+1\right)^{n-1}\in\mathcal{R}$. Now suppose that the result holds for any link diagram of m crossings or less. Let D be a link diagram of $m+1$ crossings and pick a crossing in this link diagram. We assume that the result holds for the link diagram D after changing a crossing. To simplify notation, we use D_1 for the link diagram D and D_2 for the link diagram of D after changing a crossing. Now we apply Equations 3 and 4 on D_1 and D_2 respectively under the assumption that the crossing is of the same type as in Equation 3. We subtract these two equations, to get

$$\Lambda^*_{D_1}(a,A,B)-\Lambda^*_{D_2}(a,A,B)=(A-B)(\Lambda^*_{)\,(}-\Lambda^*_{\asymp}).$$

Finally the result holds using the induction hypothesis on the right hand side of the above equation and the assumption that the result holds for D_2.

Now if the knotted trivalent graph Υ contains $(m+1)$ pairs of trivalent vertices then with the aid of the recurrence formulas given in Equations 3 and 4, we obtain

$$\Upsilon=\Gamma-A\Upsilon_0-B\Upsilon_1$$

for some knotted trivalent graphs Γ,Υ_0 and Υ_1 of m pairs of trivalent vertices. Finally, the result follows from the induction hypothesis on each term. $\square$

Now we generalize [20, Theorem 1.4] to the case of knotted trivalent graphs.

Theorem 3.2. *For any p-periodic knotted trivalent graph Υ, we have*

$$\mathbb{P}_\Upsilon(a,A,B)\equiv\mathbb{P}_{\Upsilon_\nu}(a,A,B)\mod I,$$

where Υ_ν is the knotted trivalent graph Υ after changing all crossings of the orbit ν. Therefore, we conclude $\mathbb{P}_\Upsilon(a,A,B)\equiv\mathbb{P}_{\Upsilon'}(a,A,B)=\mathbb{P}_\Upsilon(a^{-1},A,B)$ mod I, *where Υ' is the mirror image of Υ.*

Proof. We apply Equation 3 and Equation 4 to all crossings of the orbit ν in Υ and the corresponding crossings in Υ_ν. Using Equation 5, we can pair the terms in both summations in a way that the graph state in both terms is identical but with possibly different coefficients. This can be achieved by using A-resolution for the crossings of the orbit ν in Υ and B-resolution for the corresponding crossings in Υ_ν or vice versa. By considering the state summations modulo p, we need only to examine the contributions from states that are p-periodic.

Finally, we consider difference between $\mathbb{P}_\Upsilon$ and $\mathbb{P}_{\Upsilon'}$ that is

$$\begin{aligned}\mathbb{P}_\Upsilon(a,A,B)-\mathbb{P}_{\Upsilon_\nu}(a,A,B) &\equiv (A^p-B^p)\left(\mathbb{P}_{\Upsilon_0}(a,A,B)-\mathbb{P}_{\Upsilon_1}(a,A,B)\right)\\ &\equiv (A-B)^p\left(\mathbb{P}_{\Upsilon_0}(a,A,B)-\mathbb{P}_{\Upsilon_1}(a,A,B)\right) \mod p\\ &\equiv 0 \mod I,\end{aligned}$$

where Υ_0 and Υ_1 are the knotted trivalent graph Υ after applying A- and B-resolutions respectively on the crossings of the orbit ν. The second statement follows after we apply the first result to all orbits. □

Corollary 3.3. *For a p-periodic link L with a p-periodic diagram D, we have $\Lambda^*_D(a,A-B)\equiv\Lambda^*_{D'}(a,A-B)=\Lambda^*_D(a^{-1},A-B) \mod I$, where D' is the mirror image of D. Hence, we have $F^*_L(a,A-B)\equiv F^*_L(a^{-1},A-B) \mod I$ and $F_L(a,A-B)\equiv F_L(a^{-1},A-B) \mod I$.*

Proof. We know that the writhe of the p-periodic link diagram D is a multiple of p, so $w(D)=np$ for some $n\in\mathbb{Z}$. Now we have $a^p-a^{-p}\equiv(a-a^{-1})^p=\left(\frac{a-a^{-1}}{A-B}\right)^p(A-B)^p\equiv 0 \mod p$. Therefore, we obtain $a^{-w(D)}=a^{-np}=(a^{-p})^n\equiv(a^p)^n=a^{np}=a^{-(-(w(D)))}=a^{-w(D')} \mod I$. Now the result follows since $F^*_D(a,A-B)=a^{-w(D)}\Lambda^*_D(a,A-B)$ and $F_D(a,A-B)=a^{-w(D)}\Lambda_D(a,A-B)$ and the fact that $w(D)=-w(D')=np$ for some $n\in\mathbb{Z}$. Equation 2 implies:

$$\begin{aligned}F_L(\sqrt{-1}a,-\sqrt{-1}(A-B)) &= F^*_L(a,A-B)\equiv F^*_L(a^{-1},A-B)\\ &= F_L(-\sqrt{-1}a^{-1},-\sqrt{-1}(A-B)) \mod I.\end{aligned}$$

The third claim follows if we apply the above formula with a being $-\sqrt{-1}a$ and $A-B$ being $\sqrt{-1}(A-B)$. □

As a special case, we consider the following specialization for later an easy application.

Corollary 3.4. *If $a=B=q, A=q^{-1}$ and L is a p-periodic link with a p-periodic diagram D with mirror image D', we have*

$$\Lambda^*_D(q,q^{-1}-q)\equiv\Lambda^*_{D'}(q,q^{-1}-q) \quad (p,q^p-q^{-p}),$$

and

$$F_L(iq,iq-iq^{-1})\equiv F_{L'}(iq,iq-iq^{-1}) \quad (p,q^p-q^{-p}).$$

Recall that $F_L(a,z)$ is an invariant of oriented links, and that it differs from an invariant of unoriented links only by a factor of $a^{-w(D)}$. Thus for knots, a choice of orientation yields no new information. On the other hand, when considering links, the choice of orientation can affect the usefulness of Corollary 3.4, according to the following proposition:

Proposition 3.5. *Given an unoriented link L, if any of its oriented versions is not p-periodic, then the link is not p-periodic.*

Proof. We use the following observation from [14, Prop. 16.4]: If the oriented link L^* is obtained from L by reversing the orientation of one component K, then

$$F_{L^*}(a,z) = a^{4lk(K,L-K)}F_L(a,z).$$

After noting that the linking number of any two sublinks of L will also be divisible by p in the case of the link being p-periodic, we see that the two polynomials $F_L(iq, iq - iq^{-1})$ and $F_{L^*}(iq, iq - iq^{-1})$ will differ by a factor of $q^{2pn} \equiv 1 \quad (q^p - q^{-p})$ for some $n \in \mathbb{Z}$ since $q^{2p} \equiv 1 \quad (q^p - q^{-p})$. ☐

This is very natural since periodicity is independent of orientation. For links with $r \geq 2$ components, we thus have potentially 2^{r-1} criteria. It often happens that a link will pass the criterion of Corollary 3.4 for some choices of orientation, but not all. For example, when $p = 5$, this situation occurs for L6a3, L7a2, L8a10, L8a13, and L8n1.

The next result generalizes [20, Equation 4.6] to the case of knotted trivalent graphs as follows:

Theorem 3.6. *For any p-periodic knotted trivalent graph Υ, we have*

$$\mathbb{P}_\Upsilon(a,A,B) \equiv (\mathbb{P}_{\Upsilon_*}(a,A,B))^p \mod J,$$

where Υ_* is the quotient knotted trivalent graph of Υ.

Proof. We use induction on the number of pairs of trivalent vertices in Υ_*. In the case, Υ_* has zero pairs of trivalent vertices, then Υ represents a link diagram D of some link. Now, there is a one-to-one correspondence between the binary resolving tree of D to compute Λ_D^* and the binary resolving tree of D_* to compute $\Lambda_{D_*}^*$. Now the result follows using induction on the number of crossings and assuming that the result holds for crossing changes.

Now if Υ contains $(m+1)$ pairs of trivalent vertices, then with the aid of the recurrence formulas given in Equations 3 and 4, we obtain

$$\Upsilon = \Gamma - A^p\Gamma_0 - B^p\Gamma_1 \mod p$$

for some p-periodic knotted trivalent graphs Γ, Γ_0, and Γ_1 of m pairs of trivalent vertices with quotient knotted trivalent graph

$$\Upsilon_* = \Gamma_* - A\Gamma_{0*} - B\Gamma_{1*},$$

where Γ_{0*} and Γ_{1*} are the quotient knotted trivalent graphs of Γ_0 and Γ_1 respectively. Finally, the result follows from the induction hypothesis on each term. ☐

Corollary 3.7. *For any p-periodic link L of periodic diagram D, we have*

$$\Lambda_D^*(a, \omega(a-a^{-1})) \equiv \left(\Lambda_{D_*}^*(a, \omega(a-a^{-1}))\right)^p \mod p.$$

This implies

$$F^*_L(a, \omega(a - a^{-1})) \equiv (F^*_{L_*}(a, \omega(a - a^{-1})))^p \mod p$$

and

$$F_L(a, \omega(a - a^{-1})) \equiv (F_{L_*}(a, \omega(a - a^{-1})))^p \mod p,$$

where L_* is the quotient link with the quotient diagram D_*.

We obtain an easy criterion of periodic links from Corollary 3.7 as follows:

Theorem 3.8. *Let L be a p-periodic link of crossing number c. Then p divides the breadth of $F_L(a, \omega(a - a^{-1}))$ modulo p. In particular, if the breadth of $F_L(a, \omega(a - a^{-1}))$ modulo p is either c or $2c$, then p divides c.*

Proof. We assume that D is a p-periodic diagram of the link L. All the terms in $\Lambda^*_D(a, \omega(a-a^{-1}))$ modulo p have powers multiple of p according to Corollary 3.7. This implies that the terms of $F^*_L(a, \omega(a-a^{-1}))$ also have powers multiple of p since the writhe of D is a multiple of p. Now the result follows since the breadth of $F^*_L(a, \omega(a - a^{-1}))$ modulo p is equal to the breadth of $\Lambda^*_D(a, \omega(a - a^{-1}))$ modulo p. □

4. Applications and further remarks

The sufficient condition of Theorem 3.8 for the period to divide the crossing number of a given link can be checked by hand. But it would be very interesting to determine all links or at least families of links that satisfy this sufficient condition. We like to point out the result of Thistlethwaite [21] that helps in reducing the work of verifying the sufficient condition for adequate links. If $\Lambda_D(a, z) = \sum_{j \geq 0, i} u_{ij} a^i z^j$ and D is an n-crossing diagram of the link L, then u_{ij} can only be nonzero if $i+j \leq n$ and $-i+j \leq n$ with equalities hold if and only if D is adequate. In this case, we conclude that the crossing number of L is n and the truncated Kauffman polynomial $\overline{\Lambda_D}(a, z) = \sum_{j+|i|=n} u_{ij} a^i z^j \neq 0$. Therefore, if $\overline{\Lambda_D}(a, \omega(a-a^{-1}))$ modulo p has breadth $2c = 2n$, then the sufficient condition holds for this link. As a special case, we can apply this argument for any alternating link.

Based on the above argument, we like to promote for the following conjecture that generalizes the conjecture in [11].

Conjecture 4.1. *If L is a periodic adequate link, then the period divides its crossing number.*

As an application, we want to see how to apply the criterion in Corollary 3.4 to obstruct periodicity of some examples of links.

Remark 4.2. The criterion in Corollary 3.4 is valid for all positive integers $p \geq 2$ not necessarily for odd primes. All links pass this criterion for $p = 2$ as a result of the following:

$$F_K(iq, iq - iq^{-1}) = F_{K'}((iq)^{-1}, iq - iq^{-1}) = F_{K'}(-iq^{-1}, iq - iq^{-1})$$

$$\equiv F_{K'}(iq, iq - iq^{-1}) \quad (2, q^2 - q^{-2}).$$

The first equation follows since taking the mirror image is equivalent to exchanging a and a^{-1} in the Kauffman polynomial and the last equivalence since $iq \equiv iq^{-1}$ modulo the ideal $(2, q^2 - q^{-2})$. Also, we believe that all links pass this criterion for $p = 3$, see the tables at the end.

Example 4.3. The knot $K = 10_{101}$ has

$$F_K(iq, iq - iq^{-1}) = 12q^{-20} - 88q^{-18} + 310q^{-16} - 718q^{-14} + 1210q^{-12} - 1550q^{-10} + 1514q^{-8} - 1088q^{-6} + 542q^{-4} - 168q^{-2} + 25$$

and

$$F_{K'}(iq, iq - iq^{-1}) = 12q^{20} - 88q^{18} + 310q^{16} - 718q^{14} + 1210q^{12} - 1550q^{10} + 1514q^{8} - 1088q^{6} + 542q^{4} - 168q^{2} + 25.$$

One can check that $F_K(iq, iq - iq^{-1}) \equiv F_{K'}(iq, iq - iq^{-1})$ $(p, q^p - q^{-p})$ only for $p = 2, 3$. Hence, we conclude that the knot 10_{101} is not periodic of period 5 or 7 according to Corollary 3.4. This confirms the result of Traczyk in [22] that this knot has no period 7. Similarly, we can also show that 10_{105} cannot be 7-periodic, a result first shown in [17].

The following two tables summarize the results for all knots and links of few crossings using the Mathematica package KnotTheory [2].

Knots Crossing Number	Total:	Obstructed: $p = 3$:	Obstructed: $p = 5$:	Obstructed: $p = 7$:	Obstructed: $p = 11$:
3	1	0	1	1	1
4	1	0	0	0	0
5	2	0	1	2	2
6	3	0	2	2	2
7	7	0	7	6	7
8	21	0	14	15	16
9	49	0	39	48	48
10	165	0	120	150	151
11	552	0	434	545	550
≤ 11	801	0	618	769	777

Links Crossing Number	Total Unoriented/Oriented:	Obstructed $p = 3$:	Obstructed $p = 5$:	Obstructed $p = 7$:
2	1/2	0	2/2	1/2
4	1/2	0	1/2	1/2
5	1/2	0	2/2	1/2
6	6/18	0	4/13	5/13
7	9/20	0	9/19	9/20
8	29/96	0	18/45	28/83
≤ 8	47/140	0	36/83	45/122

Finally, we use the criterion in Theorem 3.8 to study the possible periods of the first known example of adequate non alternating knot namely the knot 10_{152}.

Example 4.4. The knot diagram D of the knot 10_{152} in Rolfsen's table [2] has $\overline{\Lambda^*_D}(a,z) = \overline{\Lambda_D}(ia,-iz) = -z^4a^6 + 2z^5a^5 - z^6a^4 - z^8a^{-2}$. Now we check by hand that the breadth of $\overline{\Lambda^*_D}(a,\omega(a-a^{-1}))$ modulo p is 20 for all odd primes. Therefore, the only two possible prime periods of this knot is either 2 or 5 according to Theorem 3.8. Note that this result supports Conjecture 4.1 since 2 and 5 are the only prime divisors of the crossings number 10. This agrees with what is known about the periodicity of this knot in [3, Appendix C, Table I].

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