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A CHARACTERIZATION OF n-POSETS OF LD $n-k$ WITH SIMPLE POSETS

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Abstract. A simple poset is a poset whose linear discrepancy increases if any relation of the poset is removed. In this paper, we investigate more important properties of simple posets such as its width and height which help to construct concrete simple poset of linear discrepancy l. The simplicity of a poset is similar to the ld-irreducibility of a poset. Hence, we investigate which posets are both simple and ld-irreducible. Using these properties, we characterize n-posets of linear discrepancy $n-k$ for $k=2,3$, and, lastly, we also characterize a poset of linear discrepancy 3 with simple posets and ld-irreducible posets.

1. Introduction

For a poset $\mathbf{P}=(X,\leq_{\mathbf{P}})$, the *linear discrepancy* of a poset $\mathbf{P}$, denoted by $\mathrm{ld}(\mathbf{P})$, is defined as

$$\mathrm{ld}(\mathbf{P})=\min_{f\in F}\max_{x||y\in X}|f(x)-f(y)|,$$

where F is the set of all injective order preserving maps from X to integers, and $x||y$ denotes that x and y are incomparable in $\mathbf{P}$ [6].

An *ℓ-ld-irreducible poset* is the poset whose linear discrepancy is ℓ and decreases by at least one if any element is removed from it. The idea of characterization of posets of linear discrepancy l is as follows. The linear discrepancy of a given poset is l if it contains no $(l+1)$-ld-irreducible poset as its subposet, but it contains an l-ld-irreducible poset as its subposet. This means, to characterize posets of linear discrepancy of l, the complete lists of all l- and $(l+1)$-ld-irreducible posets should be necessarily obtained. In 2006, this characterization method was firstly suggested by G.-B. Chae, M. Cheong, and S.-M. Kim [1], in which posets of linear discrepancy 1 was characterized by providing 1- and 2-ld-irreducible posets. To characterize posets of linear discrepancy 2, all of the 3-ld-irreducible posets of width 2 were given by D. M. Howard, G.-B. Chae, M. Cheong, and S.-M. Kim [3], and then, together with previously

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obtained all the 2-ld-irreducible posets in [1], a characterization of posets of ld 2 was given by D. M. Howard, M. T. Keller and S. J. Young [4] in 2008, from presenting the complete list of 3-ld-irreducible posets. However, preceding the characterization of posets of linear discrepancy 3 and more, this method of *irreducibility* does not seem to be efficient any more since the complexity for obtaining the list of all the 4-ld-irreducible posets will rapidly increase.

In this paper, we give a general poset characterization with respect to linear discrepancy by applying this irreducibility idea to order relations. We change the previous irreducible point of view of a removal of an element in the poset to that of a removal of a relation of it. This kind of irreducible poset, called a *simple poset*, is defined as the n-poset whose linear discrepancy increases by removing any relations of the poset, i.e., an (n, l)-simple poset $\mathbf{S}$ is the n-poset of linear discrepancy l consisting of the smallest number of order relations among the n-posets of linear discrepancy l. In other word, a simple poset $\mathbf{S}$ of linear discrepancy l is the 'simplest' poset among posets of the linear discrepancy l.

In Section 2, we first define a simple poset with the viewpoint of removal of its order relation. And then, in order to observe simple posets efficiently, we introduce the matrix representation of a poset, seen in [2], which has a great benefit for observing the poset after a removal of order relations in a posets rather than its covering graph representation, called a Hesse diagram. Next, using this matrix representation and our previous results in [2], we express simple posets in terms of matrix. In Section 3, we observe some basic properties of posets represented by matrices and simple posets, and then we obtain a generalized construction of all simple n-posets with respect to individual linear discrepancies. Finally, we give some characterizations of n-posets of linear discrepancy l by simple posets, and by simple posets together with l-ld-irreducible posets. As an application of simple posets, we give a new characterization of n-posets of linear discrepancy $n-2$ as an alternative characterization due to S.-L. Ng [5] in 2004. Moreover, we provide a characterization of a poset of linear discrepancy $(n-3)$ with simple posets. As a corollary of our main theorem, we also give the first characterization of n-posets of linear discrepancy 3 by the list of 3-ld-irreducible posets from [4] and our constructed $(n, 3)$-simple posets.

2. A simple poset and its matrix representation

Let $\mathbf{P} = (X, \leq_{\mathbf{P}})$ be a poset, where X has n elements. Then $\mathbf{P}$ is called an n-poset. If there is no possibility of confusion, we write $x \in \mathbf{P}$ instead of $x \in X$. For a partial order relation $\leq_{\mathbf{P}} \subseteq X \times X$ of $\mathbf{P}$, and $(x, y) \in \leq_{\mathbf{P}}$, we write this as $x \leq_{\mathbf{P}} y$ for convenience. If $x \leq_{\mathbf{P}} y$ or $y \leq_{\mathbf{P}} x$, then we say that x and y are comparable, denoted by $x \perp y$. For x and $y \in X$, the notation $x <:_{\mathbf{P}} y$ denotes that x is covered by y, and y covers x, i.e., $x \leq_{\mathbf{P}} y$, and there is no element z in $\mathbf{P}$ such that $x \leq_{\mathbf{P}} z \leq_{\mathbf{P}} y$. If x and y are incomparable, i.e.,

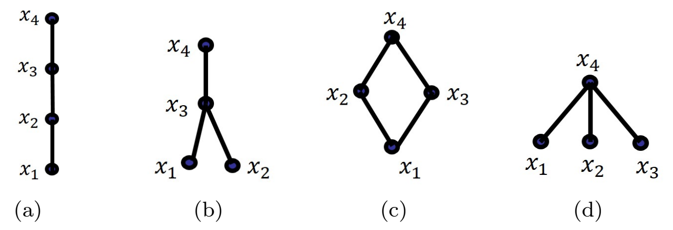

FIGURE 1. Posets obtained by removing some relations from $\mathbf{C}$.

$(x, y) \notin \leq_{\mathbf{P}}$ and $(y, x) \notin \leq_{\mathbf{P}}$, then we write it as $x \|_{\mathbf{P}} y$. If there is no confusion, we just write it as $x \| y$.

For an n-poset $\mathbf{P}$, if $x \perp y$ for any x and $y \in \mathbf{P}$, then $\mathbf{P}$ is called a chain of order n, written as $\mathbf{n}$; however, if $x \| y$ for any x and $y \in \mathbf{P}$, then $\mathbf{P}$ is called an antichain of order n, written as $\mathbf{A}_n$.

Let $\mathbf{U} = (X, \leq_{\mathbf{U}})$ and $\mathbf{V} = (Y, \leq_{\mathbf{V}})$ be posets, where $X \cap Y = \varnothing$. Then $\mathbf{U} + \mathbf{V}$ is a poset defined as

(1) $X \cup Y$ is a ground set of $\mathbf{U} + \mathbf{V}$,
(2) For x and $y \in \mathbf{U} + \mathbf{V}$, if $x \leq_{\mathbf{U}} y$ or $x \leq_{\mathbf{V}} y$, then $x \leq_{\mathbf{U}+\mathbf{V}} y$; however, if $x \in \mathbf{U}$ and $y \in \mathbf{V}$, then $x \| y$,

which is called a disjoint sum of $\mathbf{U}$ and $\mathbf{V}$.

Remark 1. For x and y in a poset $\mathbf{P} = (X, \leq_{\mathbf{P}})$ with $x \leq_{\mathbf{P}} y$, removing the relation (x, y) from $\leq_{\mathbf{P}}$ means removing the relation (x, y) and (z, w) for all z, w with $x \leq_{\mathbf{P}} z \leq_{\mathbf{P}} w \leq_{\mathbf{P}} y$.

Example 2. Let $\mathbf{C} = \{x_1 <:_{\mathbf{C}} x_2 <:_{\mathbf{C}} x_3 <:_{\mathbf{C}} x_4\}$ be a chain of order 4 as seen in Figure 1(a). If the relation (x_1, x_2) is removed from $\mathbf{C}$, then we have the poset as Figure 1(b). If the relation (x_2, x_3) is removed, then we have the poset as Figure 1(c). Finally, if the relation (x_1, x_3) is removed, then we have the poset as Figure 1(d).

Now, we define a new class of posets which is called a *simple poset*. A simple poset has a property that a removal of any relation causes to increase its linear discrepancy. This idea is somewhat similar to the concept of ld-irreducibility. The precise definition of a simple poset is as follows.

Definition 3. Let $\mathbf{S} = (X, \leq_{\mathbf{S}})$ be a poset with $\mathrm{ld}(\mathbf{S}) = l$ and $|X| = n$, and $\mathbf{S}'$ a poset obtained by removing any relation of $\mathbf{S}$. If $\mathrm{ld}(\mathbf{S}') \geq \mathrm{ld}(\mathbf{S}) + 1$, then $\mathbf{S}$ is called an (n, l)-*simple* (*or a simple*) *poset.* In particular, an antichain $\mathbf{A}_n$ of order n is $(n, n-1)$-simple.

Let $\mathbf{P}$ be an n-poset of which ground set is $X = \{x_1, \ldots, x_n\}$. Then we can represent it using a $n \times n$ matrix as follows [2].

$$M(\mathbf{P}) = [a_{ij}], \text{ where } a_{ij} = \begin{cases} 1, & \text{if } x_i \leq_{\mathbf{P}} x_j, \\ 0, & \text{otherwise.} \end{cases}$$

Let $M = [m_{ij}]_{n\times n}$ and $N = [n_{ij}]_{n\times n}$ be $n \times n$ matrices consisting of 0 or 1, and $n_{ij} = 1$ if $m_{ij} = 1$ in M. Then M is called a *submatrix* of N, denoted by $M \hookrightarrow N$.

Let $\mathbf{P} = (X, \leq_{\mathbf{P}})$ and $\mathbf{Q} = (X, \leq_{\mathbf{Q}})$ be posets on the common ground set X, and suppose that the partial relation $\leq_{\mathbf{Q}}$ contains the partial relation $\leq_{\mathbf{P}}$. Then $\mathbf{Q}$ is called an extension of $\mathbf{P}$.

Lemma 4. *Let* $\mathbf{P} = (X, \leq_{\mathbf{P}})$ *and* $\mathbf{Q} = (X, \leq_{\mathbf{Q}})$ *be n-posets. Then,* $\mathbf{Q}$ *is an extension of* $\mathbf{P}$ *if* $M(\mathbf{P}) \hookrightarrow M(\mathbf{Q})$.

Proof. Let $X = \{x_1, \ldots, x_n\}$, and let $M(\mathbf{P}) = [p_{ij}]_{n\times n}$ and $M(\mathbf{Q}) = [q_{ij}]_{n\times n}$. Then $M(\mathbf{P})$ and $M(\mathbf{Q})$ are $n\times n$ matrices since $\mathbf{P}$ and $\mathbf{Q}$ are n-posets. Suppose that $M(\mathbf{P}) \hookrightarrow M(\mathbf{Q})$. Then, $q_{ij} = 1$ if $p_{ij} = 1$, i.e., $x_i \leq_{\mathbf{Q}} x_j$ if $x_i \leq_{\mathbf{P}} x_j$. Therefore, $\mathbf{Q}$ is an extension of $\mathbf{P}$. $\square$

Let $\mathbf{P} = (X, \leq_{\mathbf{P}})$ be a poset with a ground set $X = \{x_1, \ldots, x_n\}$, and f a natural labeling from X to $[n] = \{1, 2, \ldots, n\}$ with preserving the order $\leq_{\mathbf{P}}$, where such f is called just a labeling of $\mathbf{P}$. Since a labeling f is a one-to-one correspondence, there is $f^{-1} : [n] \to X$ such that $f \circ f^{-1} = 1_{[n]}$ and $f^{-1} \circ f = 1_X$. Constructing a matrix with P and f, we obtain a matrix $U^f_{\mathbf{P}} = [u_{ij}]_{n\times n}$ satisfying that

$$u_{ij} = \begin{cases} 1, & \text{if } f^{-1}(i) \leq_{\mathbf{P}} f^{-1}(j), \\ 0, & \text{otherwise.} \end{cases}$$

Then $u_{ij} = 0$ for $j < i$ since f is a labeling. Hence, $U^f_{\mathbf{P}}$ is an upper triangular matrix.

For a labeling f of a poset $\mathbf{P}$, if $\mathrm{ld}(\mathbf{P}) = \max\{|f(x) - f(y)| : x, y \in \mathbf{P} \text{ with } x\|y\}\}$, then f is called an *optimal labeling* of $\mathbf{P}$. In $U^f_{\mathbf{P}}$, the maximum difference $|f(x) - f(y)|$ for x, $y \in \mathbf{P}$ with $x\|y$, called the *tightness* of f on $\mathbf{P}$, is equal to the maximum difference $|i - j|$ for all $i, j \in [n]$ with $a_{ij} = 0$ in $U^f_{\mathbf{P}}$ since the indices of a_{ij} are the value of a labeling f of $\mathbf{P}$ represented by $U^f_{\mathbf{P}}$ [2]. This implies that the tightness of f is the maximum distance from the diagonal line of $U^f_{\mathbf{P}}$ to the entry which is 0. Hence, we have

$$\mathrm{ld}(\mathbf{P}) = \min\{\max\{|i - j| : a_{ij} = 0 \text{ in } U^f_{\mathbf{P}}\} : f \text{ is a labeling of } \mathbf{P}\}.$$

We have the following lemma about a simple poset S and its matrix representation with an optimal labeling f of S, which is shown in [2].

Lemma 5 ([2])**.** *Let* $\mathbf{S} = (X, \leq_{\mathbf{S}})$ *be a poset with* $|X| = n$ *and* $\mathrm{ld}(\mathbf{S}) = l$. *Let f be an optimal labeling on* $\mathbf{S}$. *Then the followings are equivalent.*

(i) $\mathbf{S}$ *is* (n,l)*-simple.*
(ii) $U^f_{\mathbf{S}} = [a_{ij}]$, *where*

$$a_{ij} = \begin{cases} 1 & \text{if } j-i>l \text{ or } i=j, \\ 0 & \text{otherwise.} \end{cases}$$

We write such $U^f_{\mathbf{S}}$ *as* $\nabla(n,l)$.

3. Some properties of simple posets

If two posets have the same matrix representations, then we can consider that two posets are identical up to isomorphism. This can be summarized as follows.

Lemma 6. *For two posets* $\mathbf{P}$ *and* $\mathbf{Q}$ *with* $|\mathbf{P}| = |\mathbf{Q}|$, *if there exist two labelings* f *of* $\mathbf{P}$ *and* g *of* $\mathbf{Q}$ *such that* $U^f_{\mathbf{P}} = U^g_{\mathbf{Q}}$, *then* $\mathbf{P}$ *and* $\mathbf{Q}$ *are identical up to isomorphism.*

Proof. Let f and g be labelings of $\mathbf{P}$ and $\mathbf{Q}$, respectively, and suppose that $U^f_{\mathbf{P}} = U^g_{\mathbf{Q}}$. Define $h = g^{-1} \circ f$. Then h is a well-define map from $\mathbf{P}$ to $\mathbf{Q}$, and clearly one-to-one correspondence. Let x, $y \in \mathbf{P}$ with $x \le_{\mathbf{P}} y$. Then $f(x) \le f(y)$ so that the $(f(x), f(y))$-entry of $U^f_{\mathbf{P}}$ is 1. Since $U^f_{\mathbf{P}} = U^g_{\mathbf{Q}}$, the $(f(x), f(y))$-entry of $U^g_{\mathbf{Q}}$ is 1 so that $g^{-1}(f(x)) \le_{\mathbf{Q}} g^{-1}(f(y))$, i.e., $h(x) \le_{\mathbf{Q}} h(y)$. Hence, h is order-preserving, which implies that h is an isomorphism. Therefore, $\mathbf{P}$ and $\mathbf{Q}$ are identical up to isomorphism. □

For two positive integers n and l, let $\mathbf{S}$ be an (n,l)-simple poset. Then $U^f_{\mathbf{S}} = \nabla(n,l)$ for an optimal labeling f on $\mathbf{S}$ by Lemma 5. From Lemma 6, we can easily obtain the following important theorem.

Theorem 7. *For a positive integers* n *and* l, *there exists a unique* (n,l)*-simple poset up to isomorphism.*

Considering the uniqueness of a simple poset up to isomorphism, we can write a simple poset as follows.

Notation 8. For two positive integers n and l, we write the (n,l)-simple posets as $S_l(n)$.

The number of elements in a maximum chain of $\mathbf{P}$ is called a height of $\mathbf{P}$, denoted by $ht(\mathbf{P})$, and the number of elements in a maximum antichain of $\mathbf{P}$ is called a width of $\mathbf{P}$, denoted by $width(\mathbf{P})$. The following lemmas give us the information on the width and the height of a given simple poset with ld l and its cardinality n.

Lemma 9. *For two positive integers* n *and* l, *let* $\mathbf{S}$ *be* (n,l)*-simple, i.e.,* $\mathbf{S} = S_l(n)$. *Then* $width(\mathbf{S}) = l+1$.

Proof. Let $\mathbf{S} = (X, \leq_{\mathbf{S}})$ be a simple poset with a ground set $X = \{x_1, \ldots, x_n\}$ and $\mathrm{ld}(\mathbf{S}) = l$, and f an optimal labeling of $\mathbf{S}$ such that $f(x_i) = i$. Then $U^f_{\mathbf{S}} = \nabla(n, l)$ so that $x_i \| x_j$ for $i \in [n-l-1]$ and $j \in [i+1, i+l]$. However, $x_i \leq_{\mathbf{S}} x_{i+l+1}$ for $i \in [n-l-1]$ since $\mathrm{ld}(\mathbf{S}) = l$. Hence, for $i \in [n-l]$, a set $\{x_i, x_{i+1}, \ldots, x_{i+l}\}$ is an antichain, which is maximal so that $width(\mathbf{S}) \geq l+1$. Moreover, $\mathrm{ld}(\mathbf{S}) \geq width(\mathbf{S}) - 1$, i.e., $width(\mathbf{S}) \leq l+1$. Therefore, $width(\mathbf{S}) = l+1$. □

Lemma 10. *For two positive integers n and l, let $\mathbf{S}$ be (n,l)-simple, i.e., $\mathbf{S} = S_l(n)$. Then $ht(\mathbf{S}) = \left\lceil \frac{n}{l+1} \right\rceil$.*

Proof. For a positive integer t and a set $X = \{x_1, \ldots, x_n\}$, let $\mathbf{S} = (X, \leq_{\mathbf{S}})$ be the (n,l)-simple poset of height t, and f an optimal labeling of $\mathbf{S}$ such that $f(x_i) = i$. Since $\mathrm{ld}(\mathbf{S}) = l$, we have

$$x_i \leq_{\mathbf{S}} x_{i+l+1} \text{ for } i \in [n-l-1]. \tag{1}$$

Since $\mathbf{S}$ is simple, we have $U^f_{\mathbf{S}} = \nabla(n, l)$ so that

$$x_i \| x_j \text{ for } i \in [n-1] \text{ and } j \text{ with } 0 < j - i \leq l. \tag{2}$$

Since $ht(\mathbf{S}) = t$, there is a maximal chain C of order t in $\mathbf{S}$ consisting of $x_i <:_{\mathbf{S}} x_{i+(l+1)} <:_{\mathbf{S}} \cdots <:_{\mathbf{S}} x_{i+(t-1)(l+1)}$ from (1) and (2). Since $i + (t-1)(l+1) \leq n$ and t is a positive integer, we have

$$ht(\mathbf{S}) = t \leq \left\lfloor \frac{n-i}{l+1} \right\rfloor + 1 \leq \left\lfloor \frac{n-1}{l+1} \right\rfloor + 1 \leq \left\lceil \frac{n}{l+1} \right\rceil.$$

Note that $width(\mathbf{S}) = l+1$ from Lemma 9. Hence, we have $ht(\mathbf{S}) \geq \left\lceil \frac{n}{l+1} \right\rceil$. Therefore, $ht(\mathbf{S}) = \left\lceil \frac{n}{l+1} \right\rceil$. □

A simple poset is defined as a matrix representation. Now, we represent a simple poset as a covering graph.

Let $\mathbf{S} = (X, \leq_{\mathbf{S}})$ be a simple poset with n elements and $\mathrm{ld}(\mathbf{S}) = l$. Since $width(\mathbf{S}) = l+1$ and $ht(\mathbf{S}) = \left\lceil \frac{n}{l+1} \right\rceil$, we can arrange all elements of $\mathbf{S}$ in the $ht(\mathbf{S}) \times width(\mathbf{S})$-grid, where each intersection of lines can be an element of $\mathbf{S}$, as follows.

Theorem 11. *For a positive integer n with $n \geq 3$, let l be a positive integer with $l \leq n-1$, and let $w = l+1$, and $h = \left\lceil \frac{n}{l+1} \right\rceil$. Let X be a subset of $[w] \times [h]$ defined as follows*:

$$\begin{aligned} X = \{&(1,1), (2,1), \ldots, (w,1), \\ &(1,2), (2,2), \ldots, (w,2), \\ &\ldots, \\ &(1,h), \ldots, (n-(h-1)w, h)\} \subseteq [w] \times [h]. \end{aligned}$$

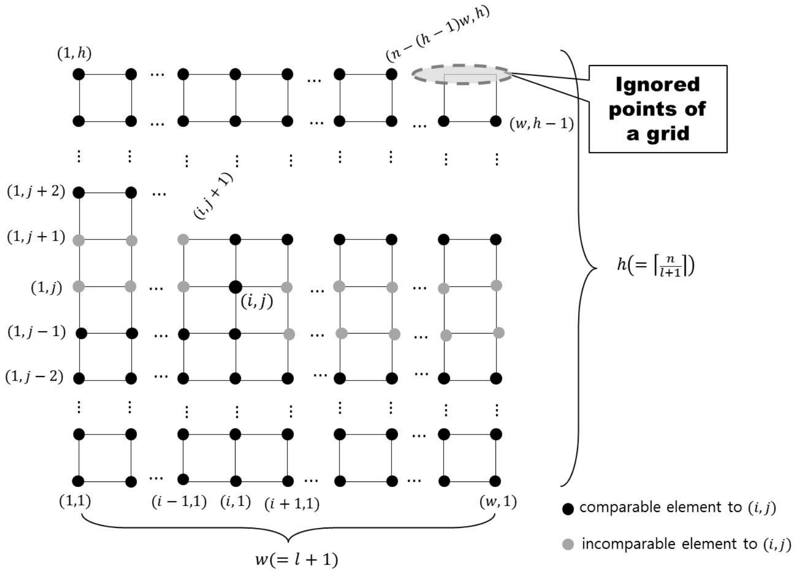

FIGURE 2. Comparable elements and incomparable elements to (i, j)

Now, we define a relation $\leq_{\mathbf{S}}$ *on* X *as*

$$(i, j) \leq_{\mathbf{S}} (s, t) \text{ if and only if } (s - i) + (t - j)w > l$$

for (i, j) *and* $(s, t) \in X$*. Then* $\mathbf{S} = (X, \leq_{\mathbf{S}})$ *is the* (n, l)*-simple poset, i.e.,* $\mathbf{S} = S_l(n)$*.*

Proof. Clearly, $\leq_{\mathbf{S}}$ is a partial order so that $S = (X, \leq_{\mathbf{S}})$ is a poset. Note that $(s, t)\|(i, j)$ for (i, j) and (s, t) in X with $|(s-i)+(t-j)w| \leq l$ by the definition of the partial order $\leq_{\mathbf{S}}$. Hence, for an element (i, j) of X with $i+(j-1)w > l$, the number of incomparable elements to (i, j) is $2l$ so that $\mathrm{ld}(\mathbf{S}) \geq l$. (See Figure 2.)

Now, we define a map $f : X \to [n]$ as

$$f((i, j)) = i + (j - 1)w \quad \text{for all } (i, j) \in X.$$

Then f is clearly a labeling of $\mathbf{S}$. Let $a_1 = (i_1, j_1)$ and $a_2 = (i_2, j_2)$ be elements in X with $a_1\|a_2$. Then $(i_2 - i_1) + (j_2 - j_1)w \leq l$ so that $\mathrm{ld}(\mathbf{S}) \leq T_f(\mathbf{S}) \leq l$. Since $\mathrm{ld}(S) \leq l$, we have $\mathrm{ld}(\mathbf{S}) = l$, and f is optimal.

Let $b_1 = (m_1, n_1)$ and $b_2 = (m_2, n_2)$ be elements with $b_1 \leq_{\mathbf{S}} b_2$. Then $(m_2 - m_1) + (n_2 - n_1)w > l$ and $m_2 + (n_2 - 1)w > l$. Hence,

$$|\{x \in X \mid x\|_{\mathbf{S}} b_2\}| = 2l.$$

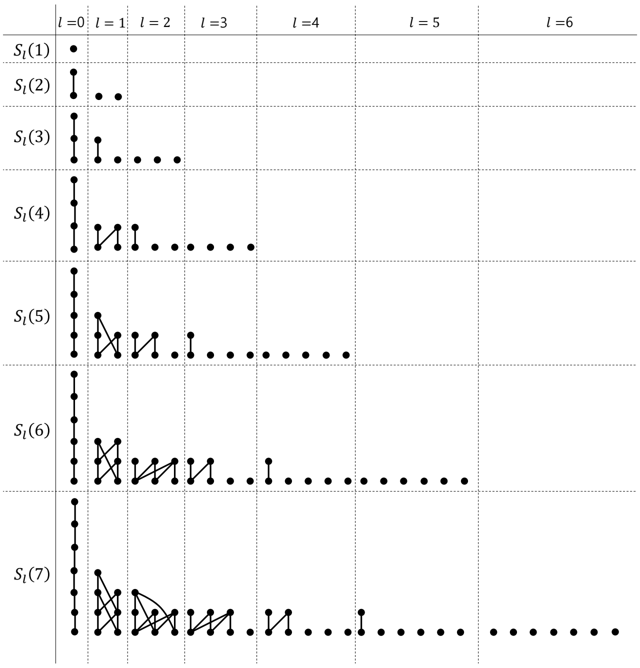

FIGURE 3. Hesse diagrams of some simple posets $S_l(n)$

Let $\mathbf{S}'$ be a poset obtained from removing relations $b_1 \leq_{\mathbf{S}} b_2$ and all relations between b_1 and b_2. Then, since $\mathbf{S}$ is an extension of $\mathbf{S}'$, we have $\mathrm{ld}(\mathbf{S}') \geq \mathrm{ld}(\mathbf{S})$. Note that

$$|\{x \in X \mid x\|_{\mathbf{S}'}b_2\}| = |\{x \in X \mid x\|_{\mathbf{S}}b_2\}| + 1 = 2l + 1.$$

Hence, $\mathrm{ld}(S) \geq l + 1 \geq \mathrm{ld}(\mathbf{S})$.

Therefore, $\mathbf{S}$ is a simple poset with $\mathrm{ld}(\mathbf{S}) = l$. $\square$

Theorem 11 gives us how to construct a simple poset $\mathbf{S} = (X, \leq_{\mathbf{S}})$ with $|X| = n$ and $\mathrm{ld}(\mathbf{S}) = l$. Especially, Figure 2 shows the covering relations of an upset of an element (i, j) in $X = [h] \times [w]$. Also, Figure 3 gives examples of (n, l)-simple posets constructed from Theorem 11.

4. A characterization of an n-poset with ld $n - k$

Let $\mathbf{P}$ be an n-poset with $\mathrm{ld}(\mathbf{P}) = l$. The following theorem provides the reason why $\nabla(n, l)$ is a submatrix of $U_{\mathbf{P}}^f$ for an optimal labeling f of $\mathbf{P}$, and it can be proved easily using a matrix representation of a poset. The theorem can be seen in [2], however, we give a detail proof as follows.

Theorem 12. *The linear discrepancy of an n-poset $\mathbf{P} = (X, \leq_{\mathbf{P}})$ is l if and only if $\mathbf{P}$ is not any extension of an $(n, l-1)$-simple poset but it contains an extension of an (n, l)-simple poset.*

Proof. ('if' part) Suppose not, i.e., $\mathrm{ld}(\mathbf{P}) \neq l$. If $\mathrm{ld}(\mathbf{P}) < l$, then there is an optimal labeling f such that $T_f(\mathbf{P}) < l$, and the optimal isotone matrix $U^f_{\mathbf{P}}$ of $\mathbf{P}$ with respect to f has an element $a_{ij} = 0$ such that $|i-j| < l$, and $a_{ij} = 1$ for $i \in [n]$ and $j \geq i + l$. This implies that $U^f_{\mathbf{P}}$ has $\nabla(n, l-1)$ as a submatrix so that $\mathbf{P}$ is an extension of $S_{l-1}(n)$. This is a contradiction.

If $\mathrm{ld}(\mathbf{P}) > l$, then there is $i \in [n]$ such that $a_{i,\, i+l+1} = 0$ so that $\nabla(n, l)$ is not a submatrix of $U^f_{\mathbf{P}}$. Hence, P is not an extension of $S_l(n)$, which is also a contradiction. Therefore, if $\mathbf{P}$ is not an extension of $S_{l-1}(n)$ but an extension of $S_l(n)$, then $\mathrm{ld}(\mathbf{P}) = l$.

('only if' part) Suppose $\mathrm{ld}(\mathbf{P}) = l$. Then $U^f_{\mathbf{P}}$ has an element $a_{i\ i+l} = 0$ and no element $a_{ij} = 0$ for $j \geq i + l + 1$ and $i \in [n]$, i.e., $U^f_{\mathbf{P}}$ has $\nabla(n, l)$ and no $\nabla(n, l-1)$ as a submatrix. Therefore, $\mathbf{P}$ is an extension of $S_l(n)$ and not an extension of $S_{l-1}(n)$. $\square$

Let $\mathbf{A}_n$ be an antichain of order n. An antichain $\mathbf{A}_n$ is a $(n, n-1)$-simple and $(n-1)$-ld-irreducible poset. So, we have a question whether there are any posets which are simple and ld-irreducible simultaneously. Fortunately, we can easily characterize such posets, as follows.

Theorem 13. *Let $\mathbf{P}$ be an l-ld-irreducible and (n, l)-simple poset. Then $\mathbf{P}$ is an antichain of order n.*

Proof. Let $\mathbf{P}$ be an l-ld-irreducible and (n, l)-simple poset, and suppose not, i.e., $\mathbf{P}$ is not an antichain. Since $\mathbf{P}$ is (n, l)-simple, we have $width(\mathbf{P}) = l + 1$ from Lemma 9. Hence, $\mathbf{P}$ has an antichain $\mathbf{A}_{l+1}$ of order $(l+1)$ as a subposet. However, since $\mathbf{A}_{l+1}$ and P are l-ld-irreducible, it is clear that P does not have any more elements except those of $\mathbf{A}_{l+1}$, i.e., $\mathbf{P} = \mathbf{A}_{l+1}$. This is a contradiction. Therefore, $\mathbf{P}$ is an antichain. $\square$

In 2004, S.-L. Ng gave a characterization of almost antichain posets which are posets of ld $n-2$. The following theorem is the characterization.

Theorem 14 ([5]). *Let $\mathbf{P} = (X, \leq_{\mathbf{P}})$ be a poset with $|X| = n \geq 3$. Then $\mathrm{ld}(\mathbf{P}) = n-2$ if and only if $\mathbf{P}$ is a disjoint union of one or more of $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$, but $\mathbf{P} \neq \mathbf{P}_2$ or $\mathbf{P}_3$ or $\mathbf{1} + \cdots + \mathbf{1}$, where*

$$\begin{aligned} \mathbf{P}_1 = \mathbf{U} \cup \mathbf{V}, \quad & \textit{where } \mathbf{U} = \{u\},\ \mathbf{V} = \{v_1, \ldots, v_h\},\ h \leq n-1, \\ & \qquad\qquad u \perp v_1 \textit{ and } x \perp y \textit{ only if } x = u,\ y \in \mathbf{V}; \\ \mathbf{P}_2 = \mathbf{2}&; \\ \mathbf{P}_3 = \mathbf{3}&; \\ \mathbf{P}_4 = \mathbf{1}&. \end{aligned}$$

For a poset $\mathbf{P}$, if $\mathrm{ld}(\mathbf{P}) = n-2$, then P is an extension of a simple poset of ld $(n-2)$, and not an extension of a simple poset of ld $(n-3)$, vice versa, from Theorem 12. Hence, Theorem 14 can be rewritten using simple posets, as follows.

Theorem 15. *Let* $\mathbf{P} = (X, \leq_{\mathbf{P}})$ *be a poset with* $|X| = n \geq 3$*. Then* $\mathrm{ld}(\mathbf{P}) = n-2$ *if and only if* $\mathbf{P}$ *is an extension of* $\mathbf{S}_1$*, but not an extension of* $\mathbf{S}_2$*, where*

$$\mathbf{S}_1 = \mathbf{2} + \mathbf{A}_{n-2};$$

$\mathbf{S}_2 =$  $+ A_{n-4}$ *if* $n \geq 4$,

$\mathbf{S}_2 = \mathbf{3}$ *if* $n = 3$.

In fact, $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ in Theorem 14 are clearly extensions of $\mathbf{S}_1$ in Theorem 15 but not extensions of $\mathbf{S}_2$ in Theorem 15. Hence, it is not difficult to characterize a poset of ld $(n-2)$ with respect to the linear discrepancy using simple posets. With the same manner to Theorem 15, we can easily provide a characterization of n-posets of ld $(n-3)$, as follows.

Theorem 16. *Let* $\mathbf{P} = (X, \leq_{\mathbf{P}})$ *be a poset with* $|X| = n \geq 4$*. Then* $\mathrm{ld}(\mathbf{P}) = n-3$ *if and only if* $\mathbf{P}$ *is an extension of* $\mathbf{S}_2$*, but not an extension of* $\mathbf{S}_3$*, where*

$\mathbf{S}_2 =$  $+ \mathbf{A}_{n-4}$;

$\mathbf{S}_3 =$  $+ \mathbf{A}_{n-6}$ *if* $n \geq 6$,

$\mathbf{S}_3 =$  *if* $n = 5$,

$\mathbf{S}_3 = \mathbf{4}$ *if* $n = 4$.

For a positive integer l, to characterize a poset of ld l with simple posets is very simple and clear, however, one of the most difficult problems to characterize a poset using simple posets is to determine whether a poset is an extension of a simple poset. If the cardinality of a poset is relatively small, we can determine whether a poset is an extension of a simple poset. Otherwise, we may have some trouble to determine. We give another method to characterize a poset with respect to the linear discrepancy, which uses both simple posets and ld-irreducible posets, as follows.

Theorem 17. *For positive integers* n *and* l*, an* n*-poset* $\mathbf{P}$ *has the linear discrepancy* l *if and only if* $\mathbf{P}$ *is an extension of* $S_l(n)$*, and* $\mathbf{P}$ *has an* l*-ld-irreducible poset as its subposet.*

Proof. If $\mathrm{ld}(\mathbf{P}) = l$, then it is clear that $\mathbf{P}$ has an l-ld-irreducible poset as its subposet, and $\mathbf{P}$ is an extension of $S_l(n)$. Conversely, if an n-poset $\mathbf{P}$ is an extension of $S_l(n)$, then $\mathrm{ld}(\mathbf{P}) \leq l$, and if $\mathbf{P}$ has an l-ld-irreducible poset as its subposet, then $\mathrm{ld}(\mathbf{P}) \geq l$. Hence, we easily obtain that $\mathrm{ld}(\mathbf{P}) = l$. $\square$

Theorem 17 is an approach with view of both a removal of a relation of a poset and a removal of an element of a poset. This approach may sometimes

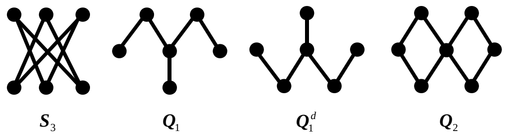

FIGURE 4. 3-ld-irreducible posets

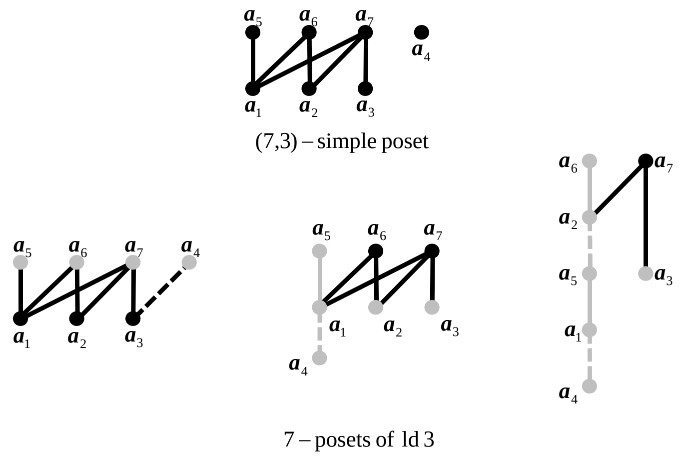

FIGURE 5. The $(7,3)$-simple poset and some 7-posets of ld 3

helps us to characterize a poset with respect to linear discrepancy. For examples, this approach is a great help to characterize a poset of ld 3. Since 3-ld-irreducible posets are given in [3, 4], we are able to give the following corollary of Theorem 17 to characterize posets of ld 3 immediately.

Corollary 18. *An n-poset has linear discrepancy equal to* 3 *if and only if it is an extension of an* $(n, 3)$*-simple poset* $S_3(n)$*, and it contains one of the following*:

(i) $\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}$;
(ii) *any poset obtained from* $\mathbf{1}+\mathbf{5}$ *or* $\mathbf{2}+\mathbf{3}$ *by the removal of a* (*possibly empty*) *subset of over relations*;
(iii) $\mathbf{S}_3$, $\mathbf{Q}_1$, $\mathbf{Q}_1^d$, *or* $\mathbf{Q}_2$; *or*
(iv) *any member of the families* $\mathscr{I}_3^2$ *and* $\mathscr{I}_3^3$,

where $\mathscr{I}_3^2$ *and* $\mathscr{I}_3^3$ *are Howard's infinite families described in* [3]*, and* $\mathbf{S}_3$, $\mathbf{Q}_1$, $\mathbf{Q}_1^d$*, or* $\mathbf{Q}_2$ *are shown in Figure 4.*

For $n = 4, 5, 6$, and 7, the $(n, 3)$-simple posets in Corollary 18 can be illustrated as the figures below the mark $l = 3$ in Figure 3. The following example shows some 7-posets of ld 3 induced from Corollary 18.

Example 19. Figure 5 shows some 7-posets of ld 3 induced from the $(7, 3)$-simple poset. In the figure, the gray solid or dashed line and gray points represent 3-ld-irreducible posets, and dashed lines imply the relations added to $(7, 3)$-simple poset.

References

[1] G.-B. Chae, M. Cheong, and S.-M. Kim, *Irreducible posets of linear discrepancy 1 and 2*, Far East J. Math. Sci. (FJMS) **22** (2006), no. 2, 217–226.

[2] ———, *Simple posets with respect to linear discrepancies*, Adv. Stud. Contemp. Math. (Kyungshang) **23** (2013), no. 1, 173–182.

[3] D. M. Howard, G.-B. Chae, M. Cheong, and S.-M. Kim, *Irreducible width 2 posets of linear discrepancy 3*, Order **25** (2008), no. 2, 105–119.

[4] D. M. Howard, M. T. Keller, and S. J. Young, *A characterization of partially ordered sets with linear discrepancy equal to 2*, Order **24** (2007), no. 3, 139–153.

[5] S.-L. Ng, *A characterisation of posets that are nearly antichains*, Order **21** (2004), no. 1, 1–5 (2005).

[6] P. J. Tanenbaum, A. N. Trenk, and P. C. Fishburn, *Linear discrepancy and weak discrepancy of partially ordered sets*, Order **18** (2001), no. 3, 201–225.

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