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ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR THE GENERALIZED MHD AND HALL-MHD SYSTEMS IN $\mathbb{R}^n$

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Abstract. This paper deals with the asymptotic behavior of solutions to the generalized MHD and Hall-MHD systems. Firstly, the upper bound for the generalized MHD and Hall-MHD systems is investigated in L^2 space. Then, the effect of the Hall term is analyzed. Finally, we optimize the upper bound of decay and obtain their algebraic lower bound for the generalized MHD system by using Fourier splitting method.

1. Introduction

We consider the following incompressible generalized MHD system:

$$u_t + u \cdot \nabla u + \Lambda^{2\alpha} u + \nabla P = B \cdot \nabla B, \tag{1.1}$$

$$B_t + u \cdot \nabla B + \Lambda^{2\beta} B = B \cdot \nabla u, \tag{1.2}$$

$$\operatorname{div} u = \operatorname{div} B = 0, \tag{1.3}$$

here $u = u(x,t) \in \mathbb{R}^n$, $B = B(x,t) \in \mathbb{R}^n$ and $P = P(x,t) \in \mathbb{R}$ represent the unknown velocity field, the magnetic field and the pressure, respectively. We define $\Lambda = (-\Delta)^{\frac{1}{2}}$ in terms of Fourier transform by $\widehat{\Lambda f}(\xi) = |\xi|\hat{f}(\xi)$.

In [25], Wu obtained the existence of weak solutions for any $u_0, b_0 \in L^2(\mathbb{R}^3)$. It was also shown that if $\alpha \geq \frac{1}{2} + \frac{N}{4}$, $\alpha + \beta \geq 1 + \frac{N}{2}$, then the solution $(u,b)(x,t)$ remains smooth for all time (see [25, 26] for details). The special case $\alpha = \beta = \frac{5}{4}$ for 3D can also be found in [29] via a different approach. In [8], it was proved that the system (1.1)-(1.4) is locally well-posed for any given initial data $u_0, b_0 \in H^s$, $s \geq \max\{\frac{n}{2} + 1 - \alpha, 1\}$ (see [19] for classical MHD system $\alpha = \beta = 1$).

The generalized Hall-MHD system reads

$$u_t + u \cdot \nabla u + \Lambda^{2\alpha} u + \nabla P = (\nabla \times B) \times B, \tag{1.4}$$

$$B_t - \nabla \times (u \times B) + \nabla \times ((\nabla \times B) \times B) + \Lambda^{2\beta} B = 0, \tag{1.5}$$

$$\operatorname{div} u = \operatorname{div} B = 0. \tag{1.6}$$

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Chae, Wan and Wu [4] proved the local well-posedness for the case $\alpha = 0$ and $\beta > \frac{1}{2}$. Regularity criteria and global existence were studied in [11, 12, 20, 27, 28]. One can check that the generalized Hall-MHD system reduces to the classical Hall-MHD system (1.1)-(1.3), when $\alpha = \beta = 1$. Recently, there have been extensive mathematical studies for the classical Hall-MHD system [1–3, 5, 21–23]. In [1], the local existence and uniqueness of smooth solutions were shown by Chae and his collaborators. They also established some blow-up criteria in [1, 2]. In [3], they proved that when $(u_0, B_0) \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, the weak solutions satisfy $\|u\|_{L^2} + \|B\|_{L^2} \le C(1+t)^{-\frac{3}{4}}$. Singularity formation was investigated in [5]. In [23], Weng obtained the same space-time decay rates as those of the heat equation. Based on the temporal decay results in [3], he found that one could obtain weighted estimates of the magnetic field B by direct weighted energy estimate, and then by regarding the magnetic convection term as a forcing term in the velocity equations, Weng obtained the weighted estimates for the vorticity, which yields the corresponding estimates for the velocity field. In [22], upper and lower bounds on the decay of higher order derivatives were obtained. In [2], Chae and Lee proved that if $\|u_0\|_{\dot{H}^{\frac{3}{2}}} + \|b_0\|_{\dot{H}^{\frac{3}{2}}}$ or $\|u_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|b_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}}$ is small enough, then the Hall MHD system has a unique global classical solution. $\dot{B}^s_{p,q}$ is the homogeneous Besov space. Later, Wan and Zhou [21] improved the global existence for the Hall-MHD system provided that the initial data $\|u_0\|_{\dot{H}^{\frac{1}{2}+\varepsilon}} + \|b_0\|_{\dot{H}^{\frac{3}{2}}}$ or $\|u_0\|_{\dot{B}^{\frac{3}{q}-1}_{q,1}} + \|b_0\|_{\dot{B}^{\frac{3}{q}-1}_{q,1} \cap \dot{B}^{\frac{3}{q}}_{q,1}}$ is small enough, where $\varepsilon > 0,\ 1 < q < \infty$.

In this paper, we deal with the asymptotic behavior of the solutions to the generalized MHD and Hall-MHD system by using Fourier splitting method. The Fourier splitting method [13] was first applied to the parabolic conservation laws to obtain algebraic energy decay rates. Then, it was used in the study of the classical Navier-Stokes equations [6, 7, 9, 10, 14–18] and the references therein. It is worth to point out that Zhou used a new method to get the famous result in [30].

Throughout this paper, C denotes a generic positive constant (generally large), it may be different from line to line. Our main results are stated as follows. The upper bound for the weak solution of generalized MHD is established in L^2 space.

Theorem 1.1. *Assume v_u and v_B are the solutions to the generalized heat equation $v_t + \Lambda^{2\alpha} v = 0$ with the initial data $u_0 \in L^2(\mathbb{R}^n)$ and $B_0 \in L^2(\mathbb{R}^n)$, and*

$$\|v_u\|^2_{L^2} \le C(1+t)^{-\theta_1}, \quad \|v_B\|^2_{L^2} \le C(1+t)^{-\theta_2} \tag{1.7}$$

for some $\theta_1, \theta_2 > 0$. Then, for $n \ge 2$ and $\alpha, \beta \in (0, \frac{n+2}{4}]$, there exists a weak solution $(u, B)(x,t)$ for the GMHD system, such that

$$\|u\|^2_{L^2} + \|B\|^2_{L^2} \le C(1+t)^{-\theta_0}, \quad t \ge 0 \tag{1.8}$$

with $\theta_0 = \min\{\theta_1, \theta_2, \frac{n+2}{2\alpha}, \frac{n+2}{2\beta}\}$.

The similar results can also be established for generalized Hall-MHD system.

Theorem 1.2. *Assume v_u and v_B are the solutions to the generalized heat equation $v_t + \Lambda^{2\alpha} v = 0$ with the initial data $u_0 \in L^2(\mathbb{R}^n)$ and $B_0 \in L^2(\mathbb{R}^n)$, and*

$$\|v_u\|_{L^2}^2 \le C(1+t)^{-\theta_1}, \quad \|v_B\|_{L^2}^2 \le C(1+t)^{-\theta_2} \tag{1.9}$$

for some $\theta_1, \theta_2 > 0$. Then, for $n \ge 3$ and $\alpha, \beta \in (0, \frac{n+2}{4}]$, there exists a weak solution $(u, B)(x,t)$ for the Hall-GMHD system, such that

$$\|u\|_{L^2}^2 + \|B\|_{L^2}^2 \le C(1+t)^{-\theta_0}, \quad t \ge 0 \tag{1.10}$$

with $\theta_0 = \min\{\theta_1, \theta_2, \frac{n+2}{2\alpha}, \frac{n+2}{2\beta}\}$.

Remark 1.1. Similar result was established in [22] for the classical Hall-MHD system. So Theorem 1.2 can be seem as a generalization. Some results for the higher derivatives in different Sobolev spaces were obtained in [27].

Remark 1.2. It seems that the Hall term doesn't affect the decay result. Actually, if we set the fluid velocity $u \equiv 0$, then the system reduce to

$$B_t + \nabla \times ((\nabla \times B) \times B) + \Lambda^{2\beta} B = 0. \tag{1.11}$$

We can get the following decay result for (1.11):

Theorem 1.3. *Assume v_B is the solution to the generalized heat equation $v_t + \Lambda^{2\alpha} v = 0$ with the same initial data $B_0 \in L^2(\mathbb{R}^n)$, and*

$$\|v_B\|_{L^2}^2 \le C(1+t)^{-\theta_2} \tag{1.12}$$

for some $\theta_2 > 0$. Then, for $n \ge 3$ and $\alpha, \beta \in (0, \frac{n+4}{4}]$, there exists a weak solution $B(x,t)$ for (1.11), such that

$$\|B\|_{L^2}^2 \le C(1+t)^{-\theta_0}, \quad t \ge 0 \tag{1.13}$$

with $\theta_0 = \min\{\theta_2, \frac{n+4}{2\beta}\}$.

It seems that the decay result for (1.11) is better than that for generalized MHD and Hall-MHD systems.

Then, we optimize the upper bound for the strong solutions of the generalized MHD system and obtain their algebraic lower bound. Before going to present the main result, we introduce the notation $R_\mu^\epsilon = \{u : |\hat{u}(\xi)| \ge \mu \text{ for } |\xi| \le \epsilon\}$ as that in [15].

Theorem 1.4. *Assume $\alpha = \beta \in (0, \frac{n+2}{4}]$, $n \ge 2$, v_u and v_B are the solutions to the generalized heat equation $v_t + \Lambda^{2\alpha} v = 0$ with the same initial data u_0, $B_0 \in H^1(\mathbb{R}^n) \cap R_\mu^\epsilon$ for some $\mu, \epsilon > 0$, and*

$$\|v_u\|_{L^2}^2 + \|v_B\|_{L^2}^2 \le M(1+t)^{-\frac{n}{2\alpha}}, \tag{1.14}$$

where M are positive constants. Then, we have

$$C_1(1+t)^{-\frac{n}{2\alpha}} \le \|u\|_{L^2}^2 + \|B\|_{L^2}^2 \le C_2(1+t)^{-\frac{n}{2\alpha}}, \tag{1.15}$$

where C_1, C_2 are positive constants.

Remark 1.3. Unfortunately, due to the Hall term, we can't get the similar result for generalized Hall-MHD system. We hope some lower bound results can be obtained for the generalized Hall-MHD system in the future. We are looking forward to some results for $\alpha, \beta > \frac{5}{4}$ of both generalized MHD and Hall-MHD systems.

Now, we list some notations that will be used in our paper. Use $\|u\|_{L^p}$ to denote the $L^p(\mathbb{R}^n)$ norm. Use $\hat{f}$ to denote the Fourier transform of f.

The proof of our main results will be shown in Section 2.

2. Proof of the main results

2.1. Proof of Theorem 1.1

Multiplying on (1.1) and (1.2) by u and B, integration by parts, we get the following energy equality

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|B\|_{L^2}^2) + \|\Lambda^\alpha u\|_{L^2}^2 + \|\Lambda^\beta B\|_{L^2}^2 = 0. \tag{2.1}$$

By Plancherel's theorem $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$ and using Fourier splitting method, we get

$$\begin{aligned} \|\Lambda^\alpha u\|_{L^2}^2 = \|\widehat{\Lambda^\alpha u}\|_{L^2}^2 &= \int_{\mathbb{R}^n} |\xi|^{2\alpha}|\widehat{u}|^2 d\xi \ge |r(t)|^{2\alpha}\int_{|\xi|\ge r(t)} |\widehat{u}|^2 d\xi \\ &\ge |r(t)|^{2\alpha}\int_{\mathbb{R}^n} |\widehat{u}|^2 d\xi - |r(t)|^{2\alpha}\int_{|\xi|\le r(t)} |\widehat{u}|^2 d\xi \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \|\Lambda^\beta B\|_{L^2}^2 = \int_{\mathbb{R}^n} |\xi|^{2\beta}|\widehat{B}|^2 d\xi &\ge |s(t)|^{2\beta}\int_{|\xi|\ge s(t)} |\widehat{B}|^2 d\xi \\ &\ge |s(t)|^{2\beta}\int_{\mathbb{R}^n} |\widehat{B}|^2 d\xi - |s(t)|^{2\beta}\int_{|\xi|\le s(t)} |\widehat{B}|^2 d\xi, \end{aligned} \tag{2.3}$$

here $r(t)$ and $s(t)$ will be chosen later. Combining (2.2) and (2.3) to (2.1), we get

$$\begin{aligned} &\frac{d}{dt}(\|\hat{u}\|_{L^2}^2(t) + \|\widehat{B}\|_{L^2}^2(t)) + 2r(t)^{2\alpha}\|\hat{u}\|_{L^2}^2 + 2s(t)^{2\beta}\|\hat{B}\|_{L^2}^2 \\ &\le 2r(t)^{2\alpha}\int_{|\xi|\le r(t)} |\hat{u}|^2 d\xi + 2s(t)^{2\beta}\int_{|\xi|\le s(t)} |\hat{B}|^2 d\xi. \end{aligned} \tag{2.4}$$

As the assumption, $v_u(x,t)$ is the solution of $v_t + \Lambda^{2\alpha} v = 0$ with the initial data $u_0(x)$ and $v_B(x,t)$ is the solution of $v_t + \Lambda^{2\beta} v = 0$ with the initial data

$B_0(x)$. By direct computation, we have $\hat{v}_u(\xi,t) = e^{-|\xi|^{2\alpha}t}\hat{u}_0(\xi)$ and $\hat{v}_B(\xi,t) = e^{-|\xi|^{2\beta}t}\hat{B}_0(\xi)$.

From the generalized MHD equations, we get

$$\hat{u}(\xi,t) = \hat{v}_u(\xi,t) - \int_0^t e^{-|\xi|^{2\alpha}(t-s)}(1-\tfrac{\xi\otimes\xi}{\xi^2})\{\xi(\widehat{u\otimes u})(\xi,s) - \xi(\widehat{B\otimes B})(\xi,s)\}ds$$

and

$$\hat{B}(\xi,t) = \hat{v}_B(\xi,t) - \int_0^t e^{-|\xi|^{2\beta}(t-s)}\{\xi(\widehat{u\otimes B})(\xi,s) - \xi(\widehat{B\otimes u})(\xi,s)\}ds.$$

Then, we get

$$|\hat{u}(\xi,t)| \le |\hat{v}_u(\xi,t)| + |\xi|\int_0^t \|u\|_{L^2}^2 + \|B\|_{L^2}^2 d\tau$$

and

$$|\hat{B}(\xi,t)| \le |\hat{v}_B(\xi,t)| + |\xi|\int_0^t \|u\|_{L^2}^2 + \|B\|_{L^2}^2 d\tau.$$

Therefore, it follows from (2.4) that

$$\begin{aligned}
&\frac{d}{dt}(\|\hat{u}\|_{L^2}^2 + \|\hat{B}\|_{L^2}^2)(t) + 2r(t)^{2\alpha}\|\hat{u}\|_{L^2}^2 + 2s(t)^{2\beta}\|\hat{B}\|_{L^2}^2 \\
&\le Cr(t)^{2\alpha}\left[\|\hat{v}_u\|_{L^2}^2 + r(t)^{2+n}\left(\int_0^t \|u\|_{L^2}^2 + \|B\|_{L^2}^2 d\tau\right)^2\right] \\
&\quad + Cs(t)^{2\beta}\left[\|\hat{v}_B\|_{L^2}^2 + s(t)^{2+n}\left(\int_0^t \|u\|_{L^2}^2 + \|B\|_{L^2}^2 d\tau\right)^2\right].
\end{aligned} \tag{2.5}$$

Let $r(t)^{2\alpha} = s(t)^{2\beta} = \frac{1}{2(t+e)\ln(t+e)}$. It yields

$$\begin{aligned}
&\frac{d}{dt}[\ln(t+e)(\|\hat{u}\|_{L^2}^2 + \|\hat{B}\|_{L^2}^2)(t)] \\
&\le C(t+e)^{-1-\theta_1} + C(t+e)^{-1-\theta_2} + C(t+e)^{-1-\frac{n+2}{2\alpha}}\left(\int_0^t \|u\|_{L^2}^2 + \|B\|_{L^2}^2 d\tau\right)^2 \\
&\quad + C(t+e)^{-1-\frac{n+2}{2\beta}}\left(\int_0^t \|u\|_{L^2}^2 + \|B\|_{L^2}^2 d\tau\right)^2.
\end{aligned} \tag{2.6}$$

By the energy estimate (2.1), one can get that $\|u\|_{L^2}^2 + \|B\|_{L^2}^2 \le C$. This is enough for the case $\alpha, \beta \in (0, \frac{n+2}{4})$. However, for the special case $\max\{\alpha,\beta\} = \frac{n+2}{4}$, from (2.6), one can't get some decay property. So, we need the following claim.

We claim that

$$\|u(s)\|_{L^2}^2 + \|B(s)\|_{L^2}^2 \le C(1+s)^{-\varrho}$$

for some $\varrho > 0$ when $\max\{\alpha, \beta\} = \frac{n+2}{4}$. In order to prove the claim, we need to show that

$$\ln(t+e)^2(\|u\|_{L^2}^2 + \|B\|_{L^2}^2) \le C.$$

Let $r(t)^{2\alpha} = s(t)^{2\beta} = \frac{1}{(t+e)\ln(t+e)}$ in (2.5). We get

$$\begin{aligned}
&\frac{d}{dt}[\ln(t+e)^2(\|\hat{u}\|_{L^2}^2 + \|\hat{B}\|_{L^2}^2)] \\
&\le C\frac{\ln(t+e)}{t+e}\big\{(t+e)^{-\theta_1} + (t+e)^{-\theta_2} \\
&\quad + (\ln(t+e)(t+e))^{-\frac{n+2}{2\alpha}}(\int_0^t \|u\|_{L^2}^2 + \|B\|_{L^2}^2 d\tau)^2 \\
&\quad + (\ln(t+e)(t+e))^{-\frac{n+2}{2\beta}}(\int_0^t \|u\|_{L^2}^2 + \|B\|_{L^2}^2 d\tau)^2\big\}.
\end{aligned}$$

Note that $\|u\|_{L^2}^2 + \|B\|_{L^2}^2$ is bounded, we have $\ln(t+e)^2(\|u\|_{L^2}^2 + \|B\|_{L^2}^2) \le C$. It follows that

$$\int_0^s (\|u(\tau)\|_{L^2}^2 + \|B(\tau)\|_{L^2}^2)d\tau \le C_3(s+1)\ln(s+e)^{-2}.$$

Then, by the same argument as that in [24], we complete this claim.

Suppose that $\|u(s)\|_{L^2}^2 + \|B(s)\|_{L^2}^2 \le C(1+s)^{-\varrho}$ with $\varrho > 0$ for $\max\{\alpha, \beta\} = \frac{n+2}{4}$ and $\varrho \ge 0$ for $\alpha, \beta \in (0, \frac{n+2}{4})$. From (2.6), we get

$$\begin{aligned}
&\ln(t+e)(\|\hat{u}\|_{L^2}^2 + \|\hat{B}\|_{L^2}^2)(t) \\
&\le C(t+e)^{-\theta_1} + C(t+e)^{-\theta_2} + C(t+e)^{-\frac{n+2}{2\alpha}+2-2\varrho} + C(t+e)^{-\frac{n+2}{2\beta}+2-2\varrho},
\end{aligned}$$

which implies that

$$\|u\|_{L^2}^2 + \|B\|_{L^2}^2 \le C(1+t)^{-\tilde{\varrho}},$$

with

$$\tilde{\varrho} = \min\{\theta_1, \theta_2, \frac{n+2}{2\alpha} - 2 + 2\varrho, \frac{n+2}{2\beta} - 2 + 2\varrho\}.$$

When $\max\{\alpha, \beta\} = \frac{n+2}{4}$, if we start with $\varrho = 0$, we would get $\tilde{\varrho} = 0$. This is why we need the claim above.

Now, starting with the new exponent, and after finitely many iterations, we get if $\theta \le 1$, then $\|u\|_{L^2}^2 + \|B\|_{L^2}^2 \le C(1+t)^{-\theta}$. If $\theta > 1$, then we have $\tilde{\varrho} = 1 + \varepsilon$ with $\varepsilon > 0$. It follows

$$\int_0^s \|u(\tau)\|_{L^2}^2 + \|B(\tau)\|_{L^2}^2 d\tau \le C,$$

here C is without respect to the time s. By (2.6), we have

$$\begin{aligned}
&\ln(t+e)(\|\hat{u}\|_{L^2}^2 + \|\hat{B}\|_{L^2}^2)(t) \\
&\le C(t+e)^{-\theta_1} + C(t+e)^{-\theta_2} + C(t+e)^{-\frac{n+2}{2\alpha}} + C(t+e)^{-\frac{n+2}{2\beta}}.
\end{aligned}$$

Which implies that

$$\|u\|_{L^2}^2 + \|B\|_{L^2}^2 \le (1+t)^{-\theta_0} \quad \text{for} \quad \theta_0 = \min\left\{\theta_1, \theta_2, \frac{n+2}{2\alpha}, \frac{n+2}{2\beta}\right\}.$$

This complete the proof of Theorem 1.

2.2. Proof of Theorems 1.2 and 1.3

One can rewrite (1.4)-(1.6) as

$$\begin{aligned} &u_t + u\cdot\nabla u + \Lambda^{2\alpha}u + \nabla(P + \frac{|B|^2}{2}) = B\cdot\nabla B,\\ &B_t + u\cdot\nabla B + \nabla\times((\nabla\times B)\times B) = B\cdot\nabla u - \Lambda^{2\beta}B,\\ &\operatorname{div} u = \operatorname{div} B = 0. \end{aligned}$$

By the same argument, we get the following inequality for the generalized Hall-GMHD system.

$$\begin{aligned} &\frac{d}{dt}(\|\hat{u}\|_{L^2}^2(t) + \|\hat{B}\|_{L^2}^2(t)) + 2r(t)^{2\alpha}\|\hat{u}\|_{L^2}^2 + 2s(t)^{2\beta}\|\hat{B}\|_{L^2}^2\\ &\le 2r(t)^{2\alpha}\int_{|\xi|\le r(t)}|\hat{u}|^2 d\xi + 2s(t)^{2\beta}\int_{|\xi|\le s(t)}|\hat{B}|^2 d\xi. \end{aligned} \tag{2.7}$$

As the assumption, $v_u(x,t)$ is the solution of $v_t + \Lambda^{2\alpha}v = 0$ with the initial data $u_0(x)$ and $v_B(x,t)$ is the solution of $v_t + \Lambda^{2\beta}v = 0$ with the initial data $B_0(x)$. By direct computation, we have $\hat{v}_u(\xi,t) = e^{-|\xi|^{2\alpha}t}\hat{u}_0(\xi)$ and $\hat{v}_B(\xi,t) = e^{-|\xi|^{2\beta}t}\hat{B}_0(\xi)$.

From the generalized Hall-MHD equations, we get

$$\hat{u}(\xi,t) = \hat{v}_u(\xi,t) - \int_0^t e^{-|\xi|^{2\alpha}(t-s)}(1 - \tfrac{\xi\otimes\xi}{\xi^2})\{\xi(\widehat{u\otimes u})(\xi,s) - \xi(\widehat{B\otimes B})(\xi,s)\}ds$$

and

$$\hat{B}(\xi,t) = \hat{v}_B(\xi,t) - \int_0^t e^{-|\xi|^{2\beta}(t-s)}\{\xi(\widehat{u\otimes B})(\xi,s) - \xi(\widehat{B\otimes u})(\xi,s) - \xi\times(\xi\cdot\widehat{B\otimes B})\}ds),$$

which imply that

$$|\hat{u}|(\xi,t) \le |\hat{v}_u(\xi,t)| + |\xi|\int_0^t \|u\|_{L^2}^2 + \|B\|_{L^2}^2 d\tau$$

and

$$|\hat{B}|(\xi,t) \le |\hat{v}_B(\xi,t)| + |\xi|\int_0^t \|u\|_{L^2}^2 + \|B\|_{L^2}^2 dt + |\xi|^2\int_0^t \|B\|_{L^2}^2 d\tau.$$

Therefore, it follows from (2.7) that

(2.8)

$$\frac{d}{dt}(\|\hat{u}\|_{L^2}^2 + \|\hat{B}\|_{L^2}^2)(t) + 2r(t)^{2\alpha}\|\hat{u}\|_{L^2}^2 + 2s(t)^{2\beta}\|\hat{B}\|_{L^2}^2$$

$$\leq Cr(t)^{2\alpha}\left[\|\hat{v}_u\|_{L^2}^2+r(t)^{2+n}\left(\int_0^t\|u\|_{L^2}^2+\|B\|_{L^2}^2d\tau\right)^2\right]$$
$$+Cs(t)^{2\beta}\left[\|\hat{v}_B\|_{L^2}^2+s(t)^{2+n}\left(\int_0^t\|u\|_{L^2}^2+\|B\|_{L^2}^2d\tau\right)^2+s(t)^{4+n}\left(\int_0^t\|B\|_{L^2}^2d\tau\right)^2\right].$$

Let $r(t)^{2\alpha}=s(t)^{2\beta}=\frac{1}{2(t+e)\ln(t+e)}$. Then $s(t)^{2+n}>s(t)^{4+n}$. It yields

$$\frac{d}{dt}[\ln(t+e)(\|\hat{u}\|_{L^2}^2+\|\hat{B}\|_{L^2}^2)(t)]$$
$$\leq C(t+e)^{-1-\theta_1}+C(t+e)^{-1-\theta_2}+C(t+e)^{-1-\frac{n+2}{2\alpha}}\left(\int_0^t\|u\|_{L^2}^2+\|B\|_{L^2}^2d\tau\right)^2$$
$$+C(t+e)^{-1-\frac{n+2}{2\beta}}\left(\int_0^t\|u\|_{L^2}^2+\|B\|_{L^2}^2d\tau\right)^2.$$

By the same argument as the generalized MHD system, we complete the proof of Theorem 1.2.

Now, we will show the proof of Theorem 1.3. For the system (1.11)

$$B_t+\nabla\times((\nabla\times B)\times B)+\Lambda^{2\beta}B=0,$$

let $u\equiv 0$, (2.8) can be reduced to

$$\|\hat{B}\|_{L^2}^2(t)+2s(t)^{2\beta}\|\hat{B}\|_{L^2}^2\leq s(t)^{4+n}\left(\int_0^t\|B\|_{L^2}^2dt\right)^2.$$

Let $s(t)^{2\beta}=\frac{1}{2(t+e)\ln(t+e)}$. We have

$$\frac{d}{dt}[\ln(t+e)\|\hat{B}\|_{L^2}^2(t)]\leq C(t+e)^{-1-\theta_2}+C(t+e)^{-1-\frac{n+4}{2\beta}}\left(\int_0^t\|B\|_{L^2}^2dt\right)^2.$$

By the same discussion, we complete the proof of Theorem 1.3.

2.3. Proof of Theorem 1.4

In order to prove Theorem 1.4, we need the following lemma.

Lemma 2.1. *Choosing T_1 large enough and fixed (will be chosen later). Let h_1 and h_2 be the solution to the generalized heat equation*

$$h_t+\Lambda^{2\alpha}h=0$$

with the initial data $h_1(x,0)=u(x,T_1)$ and $h_2=B(x,T_1)$, respectively. For $t>T_1$, we have

$$C(\delta)(1+t)^{-\frac{n}{2\alpha}}\leq\|h_1\|_{L^2}^2+\|h_2\|_{L^2}^2\leq C_1(1+t)^{-\frac{n}{2\alpha}},$$

here $C(\delta)=\frac{\delta^2\pi^{\frac{n}{2}}}{e^2\Gamma(\frac{n}{2}+1)}$ and $\delta=\frac{1}{2}e^{-1}\mu-C|\xi|$.

Proof. For $|\xi| \le T_1^{-\frac{1}{2\alpha}}$ such that $T_1 \ge \max\{\epsilon^{-2\alpha}, 1\}$, we can directly compute

$$
\begin{aligned}
&|\hat{u}(\xi, T_1)| \\
= &\left| e^{-|\xi|^{2\alpha} T_1} \hat{u}_0 - \int_0^{T_1} e^{-|\xi|^{2\alpha}(T_1 - s)} (\widehat{u \cdot \nabla u} - \widehat{B \cdot \nabla B} + \widehat{\nabla p})(\xi, s) ds \right| \\
\ge &\left| e^{-|\xi|^{2\alpha} T_1} \hat{u}_0 \right| - \left| \int_0^{T_1} e^{-|\xi|^{2\alpha}(T_1 - s)} (\sum_{j=1}^n i\xi_j \widehat{u_j u} - \sum_{j=1}^n i\xi_j \widehat{B_j B} + i\xi \sum_{i,j=1}^n \frac{\xi_i \xi_j}{|\xi|^2} \widehat{u_i u_j})(\xi, s) ds \right| \\
\ge &\left| e^{-|\xi|^{2\alpha} T_1} \hat{u}_0 \right| - |\xi| \left| \int_0^{T_1} \|\widehat{u_j u}\|_{L^\infty} + \|\widehat{B_j B}\|_{L^\infty} + \|\widehat{u_i u_j}\|_{L^\infty} ds \right| \\
\ge &\left| e^{-|\xi|^{2\alpha} T_1} \hat{u}_0 \right| - |\xi| \left| \int_0^{T_1} \|u\|_{L^2}^2 + \|B\|_{L^2}^2 + \|u\|_{L^2}^2 ds \right| \\
\ge &\left| e^{-|\xi|^{2\alpha} T_1} \hat{u}_0 \right| - C|\xi| \left| \int_0^{T_1} (1+t)^{-\frac{n}{2\alpha}} ds \right| \\
\ge &\, e^{-1}\mu - C|\xi| |(1 + T_1)^{1 - \frac{3}{2\alpha}} - 1|.
\end{aligned}
$$

For $|\xi| \le T_1^{-\frac{1}{2\alpha}}$, and T_1 large enough, we can obtain

$$
|\hat{u}(\xi, T_1)| \ge \frac{1}{2} e^{-1}\mu - C|\xi| = \delta.
$$

Then

$$
\|h_1\|_{L^2}^2 \ge \int_{|\xi| \le T_1^{-\frac{1}{2\alpha}}} e^{-2|\xi|^{2\alpha} t} |\hat{u}(\xi, T_1)|^2 d\xi \ge \delta^2 t^{-\frac{n}{2\alpha}} \int_{|y| \le \frac{\sqrt{t}}{\sqrt{T_1}}} e^{-2|y|^2} dy.
$$

For $t > T_1$, we have

$$
\|h_1\|_{L^2}^2 \ge \delta^2 t^{-\frac{n}{2\alpha}} \int_{|y| \le 1} e^{-2|y|^2} dy \ge C(\delta)(1+t)^{-\frac{n}{2\alpha}},
$$

here $C(\delta) = \frac{\delta^2 \pi^{\frac{n}{2}}}{e^2 \Gamma(\frac{n}{2}+1)}$.

Now we give the upper bound for $\|h_1\|_{L^2}$. Due to the fact that

$$
\begin{aligned}
|\hat{h_1}(\xi, t)| &= |e^{-|\xi|^{2\alpha} t} \hat{u}(\xi, T_1)| \\
&\le |e^{-|\xi|^{2\alpha}(t + T_1)} \hat{u}_0| + \left| e^{-|\xi|^{2\alpha} t} \int_0^{T_1} e^{-|\xi|^{2\alpha}(T_1 - s)} (\widehat{u \cdot \nabla u} + \widehat{B \cdot \nabla B} + \widehat{\nabla p})(\xi, s) ds \right| \\
&\le |e^{-|\xi|^{2\alpha}(t + T_1)} \hat{u}_0| + |C e^{-|\xi|^{2\alpha} t} \xi|,
\end{aligned}
$$

we have

$$
\begin{aligned}
\|h_1(\cdot, t)\|_{L^2} &= \|\hat{h}(\xi, t)\|_{L^2} \\
&\le C\|\hat{h}(t + T_1)\|_{L^2} + C\||\xi| e^{-|\xi|^{2\alpha} t}\|_{L^2} \\
&\le C(1+t)^{-\frac{n}{4\alpha}}.
\end{aligned}
$$

The estimate for h_2 is almost same. So, we omit the details. $\square$

Now, we give the proof of the main result. Set $U_1(x,t) = u(x,t+T_1)$, $U_2(x,t) = B(x,t+T_1)$ and $V_1(x,t) = U_1(x,t) - h_1(x,t)$, $V_2(x,t) = U_2(x,t) - h_2(x,t)$. Multiply both sides of the equation of V_1, V_2 by V_1 V_2, and integrate over $\mathbb{R}^n$, after suitable integration by parts, we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|V_1(t)\|_{L^2}^2 + \|V_2(t)\|_{L^2}^2) + 2\|\Lambda^\alpha V_1\|_{L^2}^2 + 2\|\Lambda^\alpha V_2\|_{L^2}^2 \\
&= 2\int_{\mathbb{R}^n} -U_1\cdot\nabla U_1\cdot V_1 + U_2\cdot\nabla U_2\cdot V_1 - U_1\cdot\nabla U_2\cdot V_2 + U_2\cdot\nabla U_1\cdot V_2 dx \\
&\le 2(\|\nabla h_1\|_{L^\infty} + \|\nabla h_2\|_{L^\infty})(\|U_1\|_{L^2}^2 + \|U_2\|_{L^2}^2).
\end{aligned}$$

Using the Parseval's equality, we have

$$\begin{aligned}
&\frac{d}{dt}(\|\hat{V}_1(t)\|_{L^2}^2 + \|\hat{V}_2(t)\|_{L^2}^2) + 2\|\widehat{\Lambda^\alpha V}_1\|_{L^2}^2 + 2\|\widehat{\Lambda^\alpha V}_2\|_{L^2}^2 \\
&= 2\int_{\mathbb{R}^n} -U_1\cdot\nabla U_1\cdot V_1 + U_2\cdot\nabla U_2\cdot V_1 - U_1\cdot\nabla U_2\cdot V_2 + U_2\cdot\nabla U_1\cdot V_2 dx \\
&\le 2(\|\nabla h_1\|_{L^\infty} + \|\nabla h_2\|_{L^\infty})(\|U_1\|_{L^2}^2 + \|U_2\|_{L^2}^2).
\end{aligned}$$

By the same argument as in (2.2), we known that

$$\begin{aligned}
&\|\widehat{\Lambda^\alpha V}_1\|_{L^2}^2 + \|\widehat{\Lambda^\alpha V}_2\|_{L^2}^2 \\
&= \int_{\mathbb{R}^n} |\xi|^{2\alpha}(|\widehat{V_1}|^2 + |\widehat{V_2}|^2)d\xi \ge |r(t)|^{2\alpha}\int_{|\xi|\ge r(t)}(|\widehat{V_1}|^2 + |\widehat{V_2}|^2)d\xi \\
&\ge |r(t)|^{2\alpha}\int_{\mathbb{R}^n}(|\widehat{V_1}|^2 + |\widehat{V_2}|^2)d\xi - |r(t)|^{2\alpha}\int_{|\xi|\le r(t)}(|\widehat{V_1}|^2 + |\widehat{V_2}|^2)d\xi.
\end{aligned}$$

Let $r(t) = (\frac{k}{1+t})^{\frac{1}{2\alpha}}$. It follows that

$$\begin{aligned}
&\frac{d}{dt}(\|\hat{V}_1(t)\|_{L^2}^2 + \|\hat{V}_2(t)\|_{L^2}^2) + \frac{k}{1+t}(\|\widehat{V}_1\|_{L^2}^2 + \|\widehat{V}_2\|_{L^2}^2) \\
&\le \frac{k}{1+t}\int_{|\xi|\le r(t)} |\hat{V}_1(t)|^2 + |\hat{V}_2(t)|^2 d\xi + 2(\|\nabla h_1\|_{L^\infty} + \|\nabla h_2\|_{L^\infty})(\|U_1\|_{L^2}^2 + \|U_2\|_{L^2}^2).
\end{aligned} \tag{2.9}$$

On the other hand, we have

$$\hat{V}_1(\xi,t) = \int_0^t e^{-|\xi|^{2\alpha}(t-s)}\hat{H}(\xi,s)ds,$$

here

$$\hat{H}(\xi,t) = -\widehat{U_1\cdot\nabla U_1} + \widehat{U_2\cdot\nabla U_2} - \widehat{\nabla P},$$

which follows

$$|\hat{H}(\xi,t)| \le C|\xi|(\|U_1\|_{L^2}^2 + \|U_2\|_{L^2}^2).$$

Thanks to the above inequality, we have

$$\begin{aligned}|\hat{V}_1(\xi,t)| &\leq C\int_0^t |\xi|(\|U_1\|_{L^2}^2+\|U_2\|_{L^2}^2)ds\\ &\leq C|\xi|\int_{T_1}^{t+T_1}(\|u\|_{L^2}^2+\|B\|_{L^2}^2)ds\\ &\leq C|\xi|(1+T_1)^{-\frac{n}{2\alpha}+1}.\end{aligned}$$

Inserting above inequality into the right hand side of (2.9), we can obtain

$$\begin{aligned}&\frac{d}{dt}[(1+t)^k(\|\hat{V}_1(t)]\|_{L^2}^2+\|\hat{V}_2(t)]\|_{L^2}^2)\\ &\leq C(1+t)^{k-1-\frac{n+2}{2\alpha}}(1+T_1)^{-\frac{n}{\alpha}+2}\\ &\quad+2(\|\nabla h_1\|_{L^\infty}+\|\nabla h_2\|_{L^\infty})(\|U_1\|_{L^2}^2+\|U_2\|_{L^2}^2).\end{aligned}\tag{2.10}$$

Before completing the proof, we need to show

$$\begin{aligned}\|\nabla h_1\|_{L^\infty} &\leq \|\widehat{\nabla h}_1\|_{L^1}\leq \int_{\mathbb{R}^n}|\xi||\hat{h}_1(\frac{t-1}{2})|e^{-|\xi|^{2\alpha}\frac{t+1}{2}}d\xi\\ &\leq C\|\hat{h}_1(\frac{t-1}{2})\|_{L^2}(1+t)^{-\frac{n+2}{4\alpha}}\leq C(1+t)^{-\frac{n+1}{2\alpha}}.\end{aligned}\tag{2.11}$$

Similarly, we have

$$\|\nabla h_2\|_{L^\infty}\leq C(1+t)^{-\frac{n+1}{2\alpha}}.\tag{2.12}$$

Combining (2.11) and (2.12) into (2.10) and choosing T_1 large enough, we have

$$\|V_1(\cdot,t)\|_{L^2}^2+\|V_2(\cdot,t)\|_{L^2}^2\leq \frac{C(\delta)}{4}(1+t)^{-\frac{n}{2\alpha}} \text{ as } t\to\infty.$$

Then, we can deduce

$$\|U_1(\cdot,t)\|_{L^2}^2+\|U_2(\cdot,t)\|_{L^2}^2\geq \frac{C(\delta)}{4}(1+t)^{-\frac{n}{2\alpha}} \text{ as } t\to\infty.$$

This completes the proof of Theorem 1.4.

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