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COMPLETE WEIGHT ENUMERATORS OF SOME CLASSES OF LINEAR CODES WITH A FEW WEIGHTS

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Abstract. By choosing defining set properly, several classes of linear codes with a few weights over the finite field $\mathbb{F}_p$ are constructed for an odd prime p, and the complete weight enumerators of these classes of codes are determined.

1. Introduction

Recently, by using a *defining set*, linear codes with a few weights have been extensively constructed and studied due to their applications to secret sharing, authentication codes, association schemes and strongly regular graphs [4], [5], [9]. The paper is to give further new results along this line, and we will construct several classes of codes with a few weights by choosing the defining set accordingly, and then we will determine the complete weight enumerators of these classes of codes.

Let p be an odd prime, and let $\mathbb{F}_q$ be the finite field with $q = p^m$ elements throughout this paper. Assume $m = es$, m, e and $s \geq 2$ are positive integers. A p-ary $[n, k, d]$ linear code $\mathcal{C}$ is defined as a k-dimensional subspace of $\mathbb{F}_p^n$ with minimum Hamming distance d. The code $\mathcal{C}$ is called *optimal* if no $[n, k, d + 1]$ code exists, and is called *almost optimal* if the code $[n, k, d + 1]$ is optimal [6, Chapter 2].

Denote $\mathbb{F}_p = \{w_0, w_1, \ldots, w_{p-1}\}$ the finite field with p elements, where $w_i = i$ for $0 \leq i \leq p - 1$, and denote $\mathbb{F}_p^* = \mathbb{F}_p \backslash \{0\}$. For a codeword $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \in \mathbb{F}_p^n$, define

$$w[\mathbf{c}] = w_0^{k_0} w_1^{k_1} \cdots w_{p-1}^{k_{p-1}},$$

where k_j is the number of components of $\mathbf{c}$ equal to w_j, and $\sum_{j=0}^{p-1} k_j = n$. For any $(k_0, \ldots, k_{p-1})$ with $\sum_{j=0}^{p-1} k_j = n$, define

$$\theta_{k_0 \cdots k_{p-1}} = |\{\mathbf{c} \in \mathcal{C} : w[\mathbf{c}] = w_0^{k_0} w_1^{k_1} \cdots w_{p-1}^{k_{p-1}}\}|.$$

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Then, the complete weight enumerator (CWE) of $\mathcal{C}$ is defined as

$$\mathrm{CWE}(\mathcal{C}) = \sum_{\mathbf{c}\in\mathcal{C}} w[\mathbf{c}] = \sum_{\theta_{k_0\cdots k_{p-1}}\neq 0} \theta_{k_0\cdots k_{p-1}} w_0^{k_0} w_1^{k_1}\cdots w_{p-1}^{k_{p-1}}.$$

Thus, the key to determining CWE($\mathcal{C}$) of a linear code $\mathcal{C}$ is determining those $\theta_{k_0\cdots k_{p-1}}$ such that $\theta_{k_0\cdots k_{p-1}} \neq 0$.

The complete weight enumerator of $\mathcal{C}$ could be helpful in soft decision decoding [2], and it can also be applied to the calculation of the deception probabilities of certain authentication codes [4]. Many new results are dedicated to determining the complete weight enumerators of codes constructed by using defining set [1], [11].

Let Tr_e^m denote the trace function from $\mathbb{F}_q$ onto $\mathbb{F}_{p^e}$, namely, $Tr_e^m(x) = \sum_{k=0}^{m/e-1} x^{p^{ke}}$. Assume f is a function over $\mathbb{F}_q$. For any set $D = \{d_1, d_2, \ldots, d_n\} \subseteq \mathbb{F}_q^*$, we may construct a linear code of length n over $\mathbb{F}_p$ as follows

$$\mathcal{C}_D = \{(Tr_1^m(xf(d_1)), Tr_1^m(xf(d_2)), \ldots, Tr_1^m(xf(d_n))) : x \in \mathbb{F}_q\},$$

and call D the *defining set* of $\mathcal{C}_D$ [3].

By choosing the defining set properly, many classes of codes with a few weights have been obtained [3], [5], [7], [9], [10], [11]. It is interesting that in [11] the complete weight enumerators of $\mathcal{C}_D$ are determined, where D and $\mathcal{C}_D$ are given as follows, respectively,

$$\begin{aligned} D &= \{x \in \mathbb{F}_q^* : Tr_1^m(x) = 0\} = \{d_1, d_2, \ldots, d_n\}, \\ \mathcal{C}_D &= \{(Tr_1^m(xd_1^2), Tr_1^m(xd_2^2), \ldots, Tr_1^m(xd_n^2)) : x \in \mathbb{F}_q\}. \end{aligned} \tag{1}$$

In this paper, we will determine the complete weight enumerators of the code $\mathcal{C}_D$ in (1) by replacing D by D_a, where

$$D_a = \{x \in \mathbb{F}_q^* : Tr_e^m(x) = a\} = \{d_1, d_2, \ldots, d_n\},$$

and a is any element of $\mathbb{F}_{p^e}$ (recall $m = es$). Obviously, the result in [11] can be considered as a special of this paper by taking $e = 1$ and $a = 0$.

Let $\rho \in \mathbb{F}_p$, define

$$n_{(b,a)}(\rho) = |\{x \in \mathbb{F}_q : Tr_e^m(x) = a \text{ and } Tr_1^m(bx^2) = \rho\}|.$$

Then, for any $b \in \mathbb{F}_q^*$, the codeword $\mathbf{c}_b \in \mathcal{C}_D$ obtained by taking x as b in (1) satisfies

$$w[\mathbf{c}_b] = w_0^{k_0} w_1^{k_1}\cdots w_{p-1}^{k_{p-1}}, \text{ where } k_\rho = n_{(b,a)}(\rho). \tag{2}$$

To calculate $k_\rho = n_{(b,a)}(\rho)$ for $0 \leq \rho \leq p-1$, it suffices to get each k_ρ for $\rho > 0$ since $\sum_{\rho=0}^{p-1} k_\rho = \sum_{\rho=0}^{p-1} n_{(b,a)}(\rho) = n$. Thus, we will focus on determining $n_{(b,a)}(\rho)$ for $\rho \neq 0$ in the sequel.

2. Preliminaries

We present some results of group characters, exponential sums, and Gauss sums for later use, please see [8] for the details. An additive character χ of $\mathbb{F}_q$ is a function from $\mathbb{F}_q$ to the set of complex numbers of absolute value 1 such that $\chi(x+y)=\chi(x)\chi(y)$ for all $x,y\in\mathbb{F}_q$. For each $b\in\mathbb{F}_q$, the function

$$\chi^{(b)}(x)=\zeta_p^{Tr(bx)}, \forall\, x\in\mathbb{F}_q \tag{3}$$

defines an additive character of $\mathbb{F}_q$, where $\zeta_p=e^{\frac{2\pi\sqrt{-1}}{p}}$, and every additive character of $\mathbb{F}_q$ can be obtained in this way. When $b=0$, the character $\chi^{(0)}(x)=1$ for all $x\in\mathbb{F}_q$ and is called the *trivial* character of $\mathbb{F}_q$. All other additive character are called *non-trivial*. When $b=1$, the character $\chi^{(1)}$ in (3) is called the canonical additive character of $\mathbb{F}_q$. It is obvious that $\chi^{(b)}(x)=\chi^{(1)}(bx)$.

The orthogonal property of additive characters of $\mathbb{F}_q$ which can be found in Theorem 5.4 in [8] is given by

$$\sum_{x\in\mathbb{F}_q}\chi(x)=\begin{cases} q, & \text{if } \chi \text{ is trivial},\\ 0, & \text{if } \chi \text{ is non-trivial}.\end{cases} \tag{4}$$

Characters of the *multiplicative group* $\mathbb{F}_q^*$ of $\mathbb{F}_q$ are called multiplicative character of $\mathbb{F}_q$. By Theorem 5.8 in [8], for each $j=0,1,\ldots,q-2$, the function ψ_j with

$$\psi_j(g^k)=e^{2\pi\sqrt{-1}jk/(q-1)} \quad \text{for} \quad k=0,1,\ldots,q-2$$

defines a multiplicative character of $\mathbb{F}_q$, where g is a generator of $\mathbb{F}_q^*$. For $j=(q-1)/2$, we have the *quadratic character* $\eta=\psi_{(q-1)/2}$ defined by

$$\eta(g^k)=\begin{cases} -1, & \text{if } 2\nmid k,\\ 1, & \text{if } 2\mid k.\end{cases}$$

In the sequel, we assume that $\eta(0)=0$.

We define the quadratic Gauss sum $G(\eta,\chi)$ over $\mathbb{F}_q$ by

$$G(\eta,\chi)=\sum_{x\in\mathbb{F}_q^*}\eta(x)\chi(x). \tag{5}$$

In this paper, we denote η_m and χ_m as the quadratic character and the canonical additive character over $\mathbb{F}_{p^m}$, respectively. Let G_m denote $G(\eta_m,\chi_m)$. Then the explicit values of quadratic Gauss sums G_m are given as follows.

Lemma 1 ([8], Theorem 5.15)**.** *Let the symbols be the same as previous, $q=p^m$. Then*

$$G_m=(-1)_m\sqrt{q},$$

where $(-1)_m=(-1)^{(m-1)}\sqrt{-1}^{\frac{(p-1)^2m}{4}}$.

Lemma 2. *Let the symbols be the same as before. Then*

(i) *if s is even, then $\eta_m(y)=1$ for each $y\in\mathbb{F}_{p^e}^*$;*

(ii) *if s is odd, then $\eta_m(y)=\eta_e(y)$ for each $y\in\mathbb{F}_{p^e}^*$.*

Proof. Let g be a generator of $\mathbb{F}_{p^m}^*$. Notice that every $y \in \mathbb{F}_{p^e}^*$ can be expressed as $g^{\frac{p^m-1}{p^e-1}j}$, where $0 \leq j \leq p^e - 2$. Since p is odd, then

$$\frac{p^m - 1}{p^e - 1} = 1 + p^e + \cdots + p^{e(s-1)} \equiv s \mod 2.$$

Hence, every element $y \in \mathbb{F}_{p^e}^*$ is a square in $\mathbb{F}_{p^m}^*$ when s is an even positive integer, and $\eta_m(y) = \eta_e(y)$ for each $y \in \mathbb{F}_{p^m}^*$ when s is odd. This completes the proof. □

Lemma 3 ([8], Theorem 5.33)**.** *Let χ be a non-trivial additive character of $\mathbb{F}_q$, $q = p^m$, and let $f(x) = a_2x^2 + a_1x + a_0 \in \mathbb{F}_q[x]$ with $a_2 \neq 0$. Then*

$$\sum_{x \in \mathbb{F}_q} \chi(f(x)) = \chi\left(a_0 - a_1^2(4a_2)^{-1}\right) \eta_m(a_2) G(\eta_m, \chi).$$

Lemma 4. *For each $c \in \mathbb{F}_{p^e}$, let*

$$M_c = \{x \in \mathbb{F}_q^* : \eta_m(x) = -1 \text{ and } Tr_e^m(x^{-1}) = c\}.$$

Then

$$|M_c| = \begin{cases} \frac{q-p^e}{2p^e}, & \text{if } s \text{ is odd and } c = 0, \\ \frac{q}{2p^e} - \frac{1}{2}(-1)^{\frac{(p-1)e}{2}}(-1)_{m+e}\eta_e(c)p^{\frac{m-e}{2}}, & \text{if } s \text{ is odd and } c \neq 0, \\ \frac{q-p^e}{2p^e} - \frac{1}{2}(-1)_m(p^e-1)p^{\frac{m-2e}{2}}, & \text{if } s \text{ is even and } c = 0, \\ \frac{q}{2p^e} + \frac{1}{2}(-1)_m p^{\frac{m-2e}{2}}, & \text{if } s \text{ is even and } c \neq 0. \end{cases}$$

Proof. By (4), (5) and Lemma 2 for any $c \in \mathbb{F}_{p^e}$, we have

$$\begin{aligned} |M_c| &= \frac{1}{2p^e} \sum_{x \in F_q^*} (1 - \eta_m(x)) \left(\sum_{y \in F_{p^e}} \zeta_p^{Tr_1^e(y(Tr_e^m(x)-c))} \right) \\ &= \frac{1}{2p^e} \sum_{x \in F_q^*} \sum_{y \in F_{p^e}} \zeta_p^{Tr_1^e(y(Tr_e^m(x)-c))} \\ &\quad - \frac{1}{2p^e} \sum_{x \in F_q^*} \eta_m(x) \sum_{y \in F_{p^e}} \zeta_p^{Tr_1^e(y(Tr_e^m(x)-c))} \\ &= \frac{1}{2p^e} \left(\sum_{x \in F_q} \sum_{y \in F_{p^e}} \zeta_p^{Tr_1^e(y(Tr_e^m(x)-c))} - \sum_{y \in F_{p^e}} \zeta_p^{Tr_1^e(-yc)} \right) \\ &\quad - \frac{1}{2p^e} \sum_{x \in F_q^*} \eta_m(x) \sum_{y \in F_{p^e}^*} \zeta_p^{Tr_1^e(y(Tr_e^m(x)-c))} \\ &= \frac{1}{2p^e} \left(q - \sum_{y \in F_{p^e}} \zeta_p^{Tr_1^e(-yc)} \right) \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2p^e} \sum_{y \in F_{p^e}^*} \eta_m(y) \zeta_p^{Tr_1^e(-yc)} \sum_{x \in F_q^*} \eta_m(yx) \zeta_p^{Tr_1^m(yx)} \\
&= \frac{q}{2p^e} - \frac{1}{2p^e} \sum_{y \in F_{p^e}} \zeta_p^{Tr_1^e(-yc)} - \frac{1}{2p^e} G_m \sum_{y \in F_{p^e}^*} \eta_m(y) \zeta_p^{Tr_1^e(-yc)} \\
&= \begin{cases} \frac{q}{2p^e} - \frac{1}{2p^e} \sum_{y \in F_{p^e}} \zeta_p^{Tr_1^e(-yc)} - \frac{1}{2p^e} \eta_e(-c) G_m \sum_{y \in F_{p^e}^*} \eta_e(-yc) \zeta_p^{Tr_1^e(-yc)}, \\ \text{if } s \text{ is odd,} \\ \frac{q}{2p^e} - \frac{1}{2p^e} \sum_{y \in F_{p^e}} \zeta_p^{Tr_1^e(-yc)} - \frac{1}{2p^e} G_m \sum_{y \in F_{p^e}^*} \zeta_p^{Tr_1^e(-yc)}, \\ \text{if } s \text{ is even.} \end{cases} \\
&= \begin{cases} \frac{q}{2p^e} - \frac{1}{2}, & \text{if } s \text{ is odd and } c = 0, \\ \frac{q}{2p^e} - \frac{1}{2p^e} \eta_e(-c) G_m G_e, & \text{if } s \text{ is odd and } c \neq 0, \\ \frac{q}{2p^e} - \frac{1}{2} - \frac{1}{2p^e} G_m (p^e - 1), & \text{if } s \text{ is even and } c = 0, \\ \frac{q}{2p^e} + \frac{1}{2p^e} G_m, & \text{if } s \text{ is even and } c \neq 0. \end{cases}
\end{aligned}
$$

Then, the desired results follow from Lemma 1. □

Remark 5. For each $c \in \mathbb{F}_{p^e}$, let $\overline{M_c} = \{x \in \mathbb{F}_q^* : \eta_m(x) = 1 \text{ and } Tr_e^m(x^{-1}) = c\}$, then $|\overline{M_c}|$ can be determined, since $|M_c| + |\overline{M_c}| = p^{m-e} - 1$ if $c = 0$ and $|M_c| + |\overline{M_c}| = p^{m-e}$ if $c \neq 0$.

Remark 6. If $c = 0$, $m = 2e$ and e is odd, then by Lemma 4 and Remark 5, we have $|M_0| = |\{x \in \mathbb{F}_q^* : \eta_m(x) = -1 \text{ and } Tr_e^m(x^{-1}) = 0\}| = \frac{1}{2}(1 - (-1)_m)(p^e - 1)$, and so

$$
|M_0| = \begin{cases} p^e - 1, & p \equiv 1 \bmod 4, \\ 0, & p \equiv 3 \bmod 4, \end{cases} \qquad |\overline{M_0}| = \begin{cases} 0, & p \equiv 1 \bmod 4, \\ p^e - 1, & p \equiv 3 \bmod 4. \end{cases}
$$

Hence, $\{x \in \mathbb{F}_q^* : Tr_e^m(x^{-1}) = 0\}$ is the set of non-square numbers when $p \equiv 1 \bmod 4$ and square numbers when $p \equiv 3 \bmod 4$ in $\mathbb{F}_q^*$. It follows that $\eta_m(x) = (-1)^{\frac{p+1}{2}}$ for any non-zero element satisfying $Tr_e^m(x^{-1}) = 0$.

Remark 7. For each $c \in \mathbb{F}_{p^e}$ and $a \in \mathbb{F}_q^*$, let $M_{(c,a)} = \{x \in \mathbb{F}_q^* : \eta_m(x) = -1 \text{ and } Tr_e^m(ax^{-1}) = c\}$. From Lemma 4, we have

$$
|M_{(c,a)}| = \begin{cases} |M_c|, & \text{if } a \text{ is square number,} \\ |\overline{M_c}|, & \text{if } a \text{ is non-square number.} \end{cases}
$$

3. Complete weight enumerators of $\mathcal{C}_{D_a}$

It is well known that [8]

$$
N_a = |\{x \in \mathbb{F}_q : Tr_e^m(x) = a, a \in \mathbb{F}_{p^e}\}| = p^{m-e}. \tag{6}
$$

Thus, the length n_a of the code $\mathcal{C}_{D_a}$ $(a \in \mathbb{F}_{p^e})$ satisfies

$$n_a = \begin{cases} p^{m-e} - 1, & \text{if } a = 0, \\ p^{m-e}, & \text{if } a \neq 0. \end{cases} \tag{7}$$

By (4) and Lemma 3, for $b \in \mathbb{F}_q^*$, we have

$$\begin{aligned}
n_{(b,a)}(\rho) &= p^{-e-1} \sum_{x\in\mathbb{F}_q} \left(\sum_{y\in F_p} \zeta_p^{y(Tr_1^m(bx^2)-\rho)} \right) \left(\sum_{z\in F_{p^e}} \zeta_p^{Tr_1^e(z(Tr_e^m(x)-a))} \right) \\
&= p^{-e-1} \sum_{x\in\mathbb{F}_q} \sum_{z\in F_{p^e}} \zeta_p^{Tr_1^e(z(Tr_e^m(x)-a))} \\
&\quad + p^{-e-1} \sum_{y\in F_p^*} \sum_{z\in F_{p^e}} \zeta_p^{Tr_1^e(-za)-y\rho} \sum_{x\in\mathbb{F}_q} \zeta_p^{Tr_1^m(byx^2+zx)} \\
&= p^{m-e-1} + p^{-e-1} \sum_{y\in F_p^*} \sum_{z\in F_{p^e}} \zeta_p^{Tr_1^e(-za)-y\rho} \sum_{x\in\mathbb{F}_q} \zeta_p^{Tr_1^m(byx^2+zx)} \\
&= p^{m-e-1} + p^{-e-1} \sum_{y\in F_p^*} \sum_{z\in F_{p^e}} \zeta_p^{Tr_1^e(-za)-y\rho} \zeta_p^{Tr_1^m(-z^2(4by)^{-1})} \eta_m(by) G_m \\
&= p^{m-e-1} + p^{-e-1} \eta_m(b) G_m \\
&\quad \times \sum_{y\in F_p^*} \eta_m(y) \zeta_p^{-y\rho} \sum_{z\in F_{p^e}} \zeta_p^{Tr_1^e(-4^{-1}y^{-1}Tr_e^m(b^{-1})z^2-az)} \\
&= \begin{cases} p^{m-e-1} + p^{-e-1}\eta_m(b)G_m \sum_{y\in F_p^*} \eta_m(y)\zeta_p^{-y\rho} \sum_{z\in F_{p^e}} \zeta_p^{Tr_1^e(-az)}, \\ \text{if } Tr_e^m(b^{-1}) = 0, \\ p^{m-e-1} + p^{-e-1}\eta_m(b)G_m G_e \\ \times \sum_{y\in F_p^*} \eta_m(y) \zeta_p^{y(Tr_1^e(a^2(Tr_e^m(b^{-1}))^{-1})-\rho)} \eta_e(-y^{-1}Tr_e^m(b^{-1})), \\ \text{if } Tr_e^m(b^{-1}) \neq 0. \end{cases}
\end{aligned} \tag{8}$$

In the sequel, we will divide our analysis into two cases according to whether $a = 0$ or not.

3.1. $a = 0$.

From (8), Lemmas 1 and 2, we get

$$n_{(b,0)}(\rho) = \begin{cases} p^{m-e-1} + p^{-1}\eta_m(b)G_m \sum_{y\in F_p^*} \eta_m(y)\zeta_p^{-y\rho}, \\ \text{if } Tr_e^m(b^{-1}) = 0, \\ p^{m-e-1} + p^{-e-1}\eta_m(b)\eta_e(-Tr_e^m(b^{-1}))G_m G_e \sum_{y\in F_p^*} \eta_m(y)\eta_e(y^{-1})\zeta_p^{-y\rho}, \\ \text{if } Tr_e^m(b^{-1}) \neq 0. \end{cases}$$

$$
= \begin{cases}
p^{m-e-1} + p^{-1}\eta_m(b)\eta_1(-\rho)G_m \sum_{y\in F_p^*} \eta_1(-y\rho)\zeta_p^{-y\rho}, \\
\text{if } m \text{ is odd and } Tr_e^m(b^{-1}) = 0, \\
p^{m-e-1} + p^{-e-1}\eta_m(b)\eta_e(-Tr_e^m(b^{-1}))G_m G_e \sum_{y\in F_p^*} \zeta_p^{-y\rho}, \\
\text{if } m \text{ is odd and } Tr_e^m(b^{-1}) \neq 0, \\
p^{m-e-1} + p^{-1}\eta_m(b)G_m \sum_{y\in F_p^*} \zeta_p^{-y\rho}, \\
\text{if } m \text{ is even and } Tr_e^m(b^{-1}) = 0, \\
p^{m-e-1} + p^{-e-1}\eta_m(b)\eta_e(-Tr_e^m(b^{-1}))G_m G_e \sum_{y\in F_p^*} \eta_e(y^{-1})\zeta_p^{-y\rho}, \\
\text{if } m \text{ is even and } Tr_e^m(b^{-1}) \neq 0.
\end{cases}
$$

$$
= \begin{cases}
p^{m-e-1} + p^{-1}\eta_m(b)\eta_1(-\rho)G_m G_1, \\
\text{if } m \text{ is odd and } Tr_e^m(b^{-1}) = 0, \\
p^{m-e-1} - p^{-e-1}\eta_m(b)\eta_e(-Tr_e^m(b^{-1}))G_m G_e, \\
\text{if } m \text{ is odd and } Tr_e^m(b^{-1}) \neq 0, \\
p^{m-e-1} - p^{-1}\eta_m(b)G_m, \\
\text{if } m \text{ is even and } Tr_e^m(b^{-1}) = 0, \\
p^{m-e-1} + p^{-e-1}\eta_m(b)\eta_e(Tr_e^m(b^{-1}))\eta_1(\rho)G_m G_e \sum_{y\in F_p^*} \eta_1(-y\rho)\zeta_p^{-y\rho}, \\
\text{if } m \text{ is even, } e \text{ is odd and } Tr_e^m(b^{-1}) \neq 0, \\
p^{m-e-1} + p^{-e-1}\eta_m(b)\eta_e(Tr_e^m(b^{-1}))G_m G_e \sum_{y\in F_p^*} \zeta_p^{-y\rho}, \\
\text{if } m \text{ is even, } e \text{ is even and } Tr_e^m(b^{-1}) \neq 0.
\end{cases}
$$

$$
= \begin{cases}
p^{m-e-1} + p^{-1}\eta_m(b)\eta_1(-\rho)G_m G_1, \\
\text{if } m \text{ is odd and } Tr_e^m(b^{-1}) = 0, \\
p^{m-e-1} - p^{-e-1}\eta_m(b)\eta_e(-Tr_e^m(b^{-1}))G_m G_e, \\
\text{if } m \text{ is odd and } Tr_e^m(b^{-1}) \neq 0, \\
p^{m-e-1} - p^{-1}\eta_m(b)G_m, \\
\text{if } m \text{ is even and } Tr_e^m(b^{-1}) = 0, \\
p^{m-e-1} + p^{-e-1}\eta_m(b)\eta_e(Tr_e^m(b^{-1}))\eta_1(\rho)G_m G_e G_1, \\
\text{if } m \text{ is even, } e \text{ is odd and } Tr_e^m(b^{-1}) \neq 0, \\
p^{m-e-1} - p^{-e-1}\eta_m(b)\eta_e(Tr_e^m(b^{-1}))G_m G_e, \\
\text{if } m \text{ is even, } e \text{ is even and } Tr_e^m(b^{-1}) \neq 0.
\end{cases}
$$

$$
(9) \quad = \begin{cases} p^{m-e-1} + (-1)^{\frac{p-1}{2}}(-1)_{m+1}\eta_m(b)\eta_1(\rho)p^{\frac{m-1}{2}}, \\ \text{if } m \text{ is odd and } Tr_e^m(b^{-1}) = 0, \\ p^{m-e-1} - (-1)^{\frac{p-1}{2}}(-1)_{m+e}\eta_m(b)\eta_e(Tr_e^m(b^{-1}))p^{\frac{m-e-2}{2}}, \\ \text{if } m \text{ is odd and } Tr_e^m(b^{-1}) \neq 0, \\ p^{m-e-1} - (-1)_m\eta_m(b)p^{\frac{m-2}{2}}, \\ \text{if } m \text{ is even and } Tr_e^m(b^{-1}) = 0, \\ p^{m-e-1} + (-1)_{m+e+1}\eta_m(b)\eta_e(Tr_e^m(b^{-1}))\eta_1(\rho)p^{\frac{m-e-1}{2}}, \\ \text{if } m \text{ is even, } e \text{ is odd and } Tr_e^m(b^{-1}) \neq 0, \\ p^{m-e-1} - (-1)_{m+e}\eta_m(b)\eta_e(Tr_e^m(b^{-1}))p^{\frac{m-e-2}{2}}, \\ \text{if } m \text{ is even, } e \text{ is even and } Tr_e^m(b^{-1}) \neq 0. \end{cases}
$$

Lemma 8. *Let $k \in \{-1, 1\}$, denote*

$$T_k = \{x \in \mathbb{F}_q^* : \eta_m(x)\eta_e(Tr_e^m(x^{-1})) = k \text{ and } Tr_e^m(x^{-1}) \neq 0\}.$$

Then

$$
|T_k| = \begin{cases} \frac{1}{2}(p^e - 1)(p^{m-e} + (-1)^{\frac{(p-1)e}{2}}(-1)_{m+e}kp^{\frac{m-e}{2}}), & \text{if } s \text{ is odd}, \\ \frac{1}{2}(p^e - 1)p^{m-e}, & \text{if } s \text{ is even}. \end{cases}
$$

Proof. Notice that

$$|T_1| = |\{x \in \mathbb{F}_q^* : Tr_e^m(x^{-1}) \neq 0\}| - |T_{-1}| = (p^e - 1)p^{m-e} - |T_{-1}|.$$

Thus, we only focus on $|T_{-1}|$.

For $i, j \in \{-1, 1\}$, define

$$N_{i,j} = \{x \in \mathbb{F}_q^* : \eta_m(x) = i,\ \eta_e(Tr_e^m(x^{-1})) = j \text{ and } Tr_e^m(x^{-1}) \neq 0\}.$$

Let $y = Tr_e^m(x^{-1})$, from Lemma 4 and Remark 5, we have

$$
\begin{aligned}
|N_{i,j}| &= |\{x \in \mathbb{F}_q^* : \eta_m(x) = i,\ \eta_e(y) = j \text{ and } y \neq 0\}| \\
&= |\{x \in \mathbb{F}_q^* : \eta_m(x) = i \text{ and } Tr_e^m(x^{-1}) = y \neq 0\}||\{y \in \mathbb{F}_{p^e}^* : \eta_e(y) = j\}| \\
&= \begin{cases} \frac{1}{4}(p^{m-e} + (-1)^{\frac{(p-1)e}{2}}(-1)_{m+e}ijp^{\frac{m-e}{2}})(p^e - 1), & \text{if } s \text{ is odd}, \\ \frac{1}{4}(p^{m-e} - (-1)_m ip^{\frac{m-2e}{2}})(p^e - 1), & \text{if } s \text{ is even}. \end{cases}
\end{aligned}
$$

Then,

$$
\begin{aligned}
|T_{-1}| &= |N_{-1,1}| + |N_{1,-1}| \\
&= \begin{cases} \frac{1}{2}(p^e - 1)(p^{m-e} - (-1)^{\frac{(p-1)e}{2}}(-1)_{m+e}p^{\frac{m-e}{2}}), & \text{if } s \text{ is odd}, \\ \frac{1}{2}(p^e - 1)p^{m-e}, & \text{if } s \text{ is even}. \end{cases}
\end{aligned}
$$

This completes the proof. □

Our main result for $a = 0$ is:

TABLE 1. The corresponding code $\mathcal{C}_{D_0}$ in Theorem 9, when e is odd and s is odd.

$n_{(b,0)}(0)$	$n_{(b,0)}(\rho)(\rho \neq 0)$	Frequency
$p^{m-e-1}-1$	$p^{m-e-1}-\eta_1(\rho)p^{\frac{m-1}{2}}$	$\frac{1}{2}(p^{m-e}-1)$
$p^{m-e-1}-1$	$p^{m-e-1}+\eta_1(\rho)p^{\frac{m-1}{2}}$	$\frac{1}{2}(p^{m-e}-1)$
$p^{m-e-1}+(p-1)p^{\frac{m-e-2}{2}}-1$	$p^{m-e-1}-p^{\frac{m-e-2}{2}}$	$\frac{1}{2}(p^e-1)(p^{m-e}+p^{\frac{m-e}{2}})$
$p^{m-e-1}-(p-1)p^{\frac{m-e-2}{2}}-1$	$p^{m-e-1}+p^{\frac{m-e-2}{2}}$	$\frac{1}{2}(p^e-1)(p^{m-e}-p^{\frac{m-e}{2}})$
$p^{m-e}-1$	0	1

TABLE 2. The corresponding code $\mathcal{C}_{D_0}$ in Theorem 9, when e is odd and $s(>2)$ is even.

$n_{(b,0)}(0)$	$n_{(b,0)}(\rho)(\rho \neq 0)$	Frequency
$p^{m-e-1}-1$	$p^{m-e-1}-\eta_1(\rho)p^{\frac{m-e-1}{2}}$	$\frac{1}{2}(p^e-1)p^{m-e}$
$p^{m-e-1}-1$	$p^{m-e-1}+\eta_1(\rho)p^{\frac{m-e-1}{2}}$	$\frac{1}{2}(p^e-1)p^{m-e}$
$p^{m-e-1}+(p-1)p^{\frac{m-2}{2}}-1$	$p^{m-e-1}-p^{\frac{m-2}{2}}$	$\frac{1}{2}(p^{\frac{m}{2}}-1)(p^{\frac{m-2e}{2}}+1)$
$p^{m-e-1}-(p-1)p^{\frac{m-2}{2}}-1$	$p^{m-e-1}+p^{\frac{m-2}{2}}$	$\frac{1}{2}(p^{\frac{m}{2}}+1)(p^{\frac{m-2e}{2}}-1)$
$p^{m-e}-1$	0	1

Theorem 9. *The code $\mathcal{C}_{D_0}$ is a $[p^{m-e}-1, m]$ linear code except for $s=2$, which is a $[p^e-1, e]$ linear code. Their complete weight enumerators are described as follow*:

(i) *If e is odd and s is odd, then the complete weight enumerator of the code $\mathcal{C}_{D_0}$ is described as in Table 1*;

(ii) *If e is odd and s is even, then the complete weight enumerator of the code $\mathcal{C}_{D_0}$ is described as in Table 2 when $s>2$ and as in Table 3 when $s=2$*;

(iii) *If e is even and s is odd, then the complete weight enumerator of the code $\mathcal{C}_{D_0}$ is described as in Table 4*;

(iv) *If e is even and s is even, then the complete weight enumerator of the code $\mathcal{C}_{D_0}$ is described as in Table 5 when $s>2$ and as in Table 6 when $s=2$.*

Proof. (i) If e is odd and s is odd, then m is odd. To obtain the complete weight enumerators in this case, we give a partition of $\mathbb{F}_q^*$ as follows

$$\mathbb{F}_q^* = M_0 \cup \overline{M_0} \cup T_{-1} \cup T_1.$$

Note that from (9) the elements in the same part of the partition correspond to the codewords with the same coordinate symbol distribution, and the converse is also right. Thus, it suffices to calculate $n_{(b,0)}(\rho)$ with $0 \leq \rho \leq p-1$ for the elements b in each part of the partition, and then we determine the size of each part of the partition.

For $\rho \neq 0$, we get by using (9) that

$$n_{(b,0)}(\rho) = p^{m-e-1} - (-1)^{\frac{p-1}{2}}(-1)_{m+1}\eta_1(\rho)p^{\frac{m-1}{2}} \text{ and}$$
$$n_{(b,0)}(\rho) = p^{m-e-1} + (-1)^{\frac{p-1}{2}}(-1)_{m+e}p^{\frac{m-e-2}{2}}$$

for $b \in M_0$ and $b \in T_{-1}$, respectively.

Similarly, for the cases $b \in \overline{M_0}$ and $b \in T_1$, we may get $n_{(b,0)}(\rho)$ by (9). Since $n_{(b,0)}(0) = n_0 - \sum_{\rho=1}^{p-1} n_{(b,0)}(\rho)$ (see (7)), we can also get $n_{(b,0)}(0)$ in all the cases.

Finally, Lemmas 4 and 8 yield $|M_0|, |\overline{M_0}|, |T_{-1}|$ and $|T_1|$, that is, the number of the codewords with the same coordinate symbols distribution corresponding to each part of the partition.

Since, for any $b \in \mathbb{F}_q^*$, the codeword $\mathbf{c}_b \in \mathcal{C}_{D_0}$ obtained by taking x as b in (1) is non-zero, we obtain $\dim(\mathcal{C}_{D_0}) = m$ over $\mathbb{F}_p$, that is, $\mathcal{C}_{D_0}$ is a $[p^{m-e}-1, m]$ linear code.

We list the complete weight enumerator in Table 1. Note that each row in Table 1 stands for a kind of coordinate symbols distribution in the code $\mathcal{C}_{D_0}$, and the last column stands for the number of the codewords with each kind of coordinate symbols distribution. Other tables in the paper will preserve similar meaning as Table 1.

(ii) For the case e is odd and s is even, we can obtain the complete weight enumerators listed in Table 2 by using m being even and similar arguments as in (i). It needs to point out that the case $s = 2$ should be separated from Table 2 because of the dimension variation of $\mathcal{C}_{D_0}$.

If $s = 2$, then $m = se = 2e$. For $\rho \neq 0$, we may get from Remark 6 and (9) that

$$n_{(b,0)}(\rho) = p^{e-1} + (-1)_{2e}(-1)^{\frac{p-1}{2}}p^{e-1} = 0 \quad \text{whenever } Tr_e^{2e}(b^{-1}) = 0.$$

That is, $wt(\mathbf{c}_b) = 0$ whenever $Tr_e^{2e}(b^{-1}) = 0$. Thus, there are p^e elements b in $\mathbb{F}_q$ such that $wt(\mathbf{c}_b) = 0$ ($wt(\cdot)$ stands for Hamming weight), thus, the code $\mathcal{C}_{D_0}$ only has $p^{2e}/p^e = p^e$ different codewords and is a $[p^e - 1, e]$ code. We get the complete weight enumerators with respect to $m = se = 2e$ in Table 3 which can be obtained by modifying Table 2. Note from Table 3 that all the non-zero codewords have the same Hamming weight, and thus the code $\mathcal{C}_{D_0}$ with respect to $m = 2e$ is constant-weight.

(iii) If e is even and s is odd, then m is even. We may obtain the complete weight enumerators in Table 4 by using similar arguments as in (i).

(iv) If e is even and s is even, then m is even. We get the results in Tables 5 and 6 for $s > 2$ and $s = 2$, respectively, by using similar arguments as in (ii).

This completes the proof. □

Example 10. Let $(p, m, e, s) = (3, 3, 1, 3)$ and $a = 0$. Then by Theorem 9 and Table 1, the code $\mathcal{C}_{D_0}$ has parameters $[8, 3, 4]$ and complete weight enumerator

$$w_0^8 + 4w_0^2w_2^6 + 4w_0^2w_1^6 + 12w_0^4w_1^2w_2^2 + 6w_1^4w_2^4$$

TABLE 3. The corresponding code $\mathcal{C}_{D_0}$ in Theorem 9, when e is odd and $s = 2$.

$n_{(b,0)}(0)$	$n_{(b,0)}(\rho)(\rho \neq 0)$	Frequency
$p^{e-1} - 1$	$p^{e-1} - \eta_1(\rho)p^{\frac{e-1}{2}}$	$\frac{1}{2}(p^e - 1)$
$p^{e-1} - 1$	$p^{e-1} + \eta_1(\rho)p^{\frac{e-1}{2}}$	$\frac{1}{2}(p^e - 1)$
$p^e - 1$	0	1

TABLE 4. The corresponding code $\mathcal{C}_{D_0}$ in Theorem 9, when e is even and s is odd.

$n_{(b,0)}(0)$	$n_{(b,0)}(\rho)(\rho \neq 0)$	Frequency
$p^{m-e-1} + (p-1)p^{\frac{m-2}{2}} - 1$	$p^{m-e-1} - p^{\frac{m-2}{2}}$	$\frac{1}{2}(p^{m-e} - 1)$
$p^{m-e-1} - (p-1)p^{\frac{m-2}{2}} - 1$	$p^{m-e-1} + p^{\frac{m-2}{2}}$	$\frac{1}{2}(p^{m-e} - 1)$
$p^{m-e-1} + (p-1)p^{\frac{m-e-2}{2}} - 1$	$p^{m-e-1} - p^{\frac{m-e-2}{2}}$	$\frac{1}{2}(p^e - 1)(p^{m-e} + p^{\frac{m-e}{2}})$
$p^{m-e-1} - (p-1)p^{\frac{m-e-2}{2}} - 1$	$p^{m-e-1} + p^{\frac{m-e-2}{2}}$	$\frac{1}{2}(p^e - 1)(p^{m-e} - p^{\frac{m-e}{2}})$
$p^{m-e} - 1$	0	1

TABLE 5. The corresponding code $\mathcal{C}_{D_0}$ in Theorem 9, when e is even and $s(> 2)$ is even.

$n_{(b,0)}(0)$	$n_{(b,0)}(\rho)(\rho \neq 0)$	Frequency
$p^{m-e-1} + (p-1)p^{\frac{m-2}{2}} - 1$	$p^{m-e-1} - p^{\frac{m-2}{2}}$	$\frac{1}{2}(p^{\frac{m}{2}} - 1)(p^{\frac{m-2e}{2}} + 1)$
$p^{m-e-1} - (p-1)p^{\frac{m-2}{2}} - 1$	$p^{m-e-1} + p^{\frac{m-2}{2}}$	$\frac{1}{2}(p^{\frac{m}{2}} + 1)(p^{\frac{m-2e}{2}} - 1)$
$p^{m-e-1} + (p-1)p^{\frac{m-e-2}{2}} - 1$	$p^{m-e-1} - p^{\frac{m-e-2}{2}}$	$\frac{1}{2}(p^e - 1)p^{m-e}$
$p^{m-e-1} - (p-1)p^{\frac{m-e-2}{2}} - 1$	$p^{m-e-1} + p^{\frac{m-e-2}{2}}$	$\frac{1}{2}(p^e - 1)p^{m-e}$
$p^{m-e} - 1$	0	1

TABLE 6. The corresponding code $\mathcal{C}_{D_0}$ in Theorem 9, when e is even and $s = 2$.

$n_{(b,0)}(0)$	$n_{(b,0)}(\rho)(\rho \neq 0)$	Frequency
$p^{m-e-1} + (p-1)p^{\frac{m-e-2}{2}} - 1$	$p^{m-e-1} - p^{\frac{m-e-2}{2}}$	$\frac{1}{2}(p^e - 1)$
$p^{m-e-1} - (p-1)p^{\frac{m-e-2}{2}} - 1$	$p^{m-e-1} + p^{\frac{m-e-2}{2}}$	$\frac{1}{2}(p^e - 1)$
$p^{m-e} - 1$	0	1

and is almost optimal. This code can be punctured into a shorter code (see [10, Corollary 2.3]) with parameters $[4, 3, 2]$ which is optimal.

Example 11. Let $(p, m, e, s) = (3, 6, 3, 2)$ and $a = 0$. Then by Theorem 9 and Table 3, the code $\mathcal{C}_{D_0}$ has parameters $[26, 3, 18]$ and complete weight enumerator

$$w_0^{26} + 13w_0^8 w_1^6 w_2^{12} + 13w_0^8 w_1^{12} w_2^6.$$

This code is optimal.

Example 12. Let $(p, m, e, s) = (3, 4, 2, 2)$ and $a = 0$. Then by Theorem 9 and Table 6, the code $\mathcal{C}_{D_0}$ has parameters $[8, 2, 4]$ and complete weight enumerator

$$w_0^8 + 4w_0^4 w_1^2 w_2^2 + 4w_1^4 w_2^4.$$

This code can be punctured into a shorter code (see [10, Corollary 2.3]) with parameters $[4, 2, 2]$ which is almost optimal.

3.2. $a \neq 0$.

Still from (8), Lemmas 1 and 2, we have

$$n_{(b,a)}(\rho) = \begin{cases} p^{m-e-1}, \text{ if } Tr_e^m(b^{-1}) = 0, \\ p^{m-e-1} + p^{-e-1}\eta_m(b)\eta_e(-Tr_e^m(b^{-1})) \\ G_m G_e \sum_{y \in F_p^*} \zeta_p^{y(Tr_1^e(a^2(Tr_e^m(b^{-1}))^{-1}) - \rho)} \eta_1(y), \\ \text{if } e \text{ is odd, } s \text{ is even and } Tr_e^m(b^{-1}) \neq 0, \\ p^{m-e-1} + p^{-e-1}\eta_m(b)\eta_e(-Tr_e^m(b^{-1})) \\ G_m G_e \sum_{y \in F_p^*} \zeta_p^{y(Tr_1^e(a^2(Tr_e^m(b^{-1}))^{-1}) - \rho)}, \\ \text{if } e \text{ is even or } s \text{ is odd and } Tr_e^m(b^{-1}) \neq 0. \end{cases}$$

$$= \begin{cases} p^{m-e-1}, \text{ if } Tr_e^m(b^{-1}) = 0, \\ p^{m-e-1}, \text{ if } e \text{ is odd, } s \text{ is even, } Tr_e^m(b^{-1}) \neq 0 \text{ and } Tr_1^e(a^2(Tr_e^m(b^{-1}))^{-1}) = \rho, \\ p^{m-e-1} + p^{-e-1}\eta_m(b)\eta_e(-Tr_e^m(b^{-1})) \\ \eta_1(Tr_1^e(a^2(Tr_e^m(b^{-1}))^{-1}) - \rho) G_m G_e G_1, \\ \text{if } e \text{ is odd, } s \text{ is even, } Tr_e^m(b^{-1}) \neq 0 \text{ and } Tr_1^e(a^2(Tr_e^m(b^{-1}))^{-1}) \neq \rho, \\ p^{m-e-1} + p^{-e-1}(p-1)\eta_m(b)\eta_e(-Tr_e^m(b^{-1})) G_m G_e, \\ \text{if } e \text{ is even or } s \text{ is odd, } Tr_e^m(b^{-1}) \neq 0 \text{ and } Tr_1^e(a^2(Tr_e^m(b^{-1}))^{-1}) = \rho, \\ p^{m-e-1} - p^{-e-1}\eta_m(b)\eta_e(-Tr_e^m(b^{-1})) G_m G_e, \\ \text{if } e \text{ is even or } s \text{ is odd } Tr_e^m(b^{-1}) \neq 0 \text{ and } Tr_1^e(a^2(Tr_e^m(b^{-1}))^{-1}) \neq \rho. \end{cases}$$

$$(10) \qquad = \begin{cases} p^{m-e-1}, \text{ if } Tr_e^m(b^{-1}) = 0, \\ p^{m-e-1}, \text{ if } e \text{ is odd, } s \text{ is even, } Tr_e^m(b^{-1}) \neq 0 \text{ and } Tr_1^e(a^2(Tr_e^m(b^{-1}))^{-1}) = \rho, \\ p^{m-e-1} + (-1)^{\frac{p-1}{2}}(-1)_{m+e+1} \\ \eta_m(b)\eta_e(Tr_e^m(b^{-1}))\eta_1(Tr_1^e(a^2(Tr_e^m(b^{-1}))^{-1}) - \rho) p^{\frac{m-e-1}{2}}, \\ \text{if } e \text{ is odd, } s \text{ is even, } Tr_e^m(b^{-1}) \neq 0 \text{ and } Tr_1^e(a^2(Tr_e^m(b^{-1}))^{-1}) \neq \rho, \\ p^{m-e-1} + (-1)^{\frac{(p-1)e}{2}}(-1)_{m+e}\eta_m(b)\eta_e(Tr_e^m(b^{-1}))(p-1)p^{\frac{m-e-2}{2}}, \\ \text{if } e \text{ is even or } s \text{ is odd, } Tr_e^m(b^{-1}) \neq 0 \text{ and } Tr_1^e(a^2(Tr_e^m(b^{-1}))^{-1}) = \rho, \\ p^{m-e-1} - (-1)^{\frac{(p-1)e}{2}}(-1)_{m+e}\eta_m(b)\eta_e(Tr_e^m(b^{-1}))p^{\frac{m-e-2}{2}}, \\ \text{if } e \text{ is even or } s \text{ is odd, } Tr_e^m(b^{-1}) \neq 0 \text{ and } Tr_1^e(a^2(Tr_e^m(b^{-1}))^{-1}) \neq \rho. \end{cases}$$

Lemma 13. *Let $a \in \mathbb{F}_{p^e}^*$, $\gamma \in \mathbb{F}_p$ and $k \in \{-1, 1\}$, denote*

$$S_{k,\gamma} = \{x \in \mathbb{F}_q^* : \eta_m(x)\eta_e(Tr_e^m(x^{-1})) = k \text{ and } Tr_1^e(a^2(Tr_e^m(x^{-1}))^{-1}) = \gamma\}.$$

Then, for e being odd and s being odd, we have

$$S_{k,\gamma} = \begin{cases} \frac{1}{2}p^{e-1}(p^{m-e} + (-1)^{\frac{p-1}{2}}(-1)_{m+e}kp^{\frac{m-e}{2}}), & \text{if } \gamma \neq 0, \\ \frac{1}{2}(p^{e-1}-1)(p^{m-e} + (-1)^{\frac{p-1}{2}}(-1)_{m+e}kp^{\frac{m-e}{2}}), & \text{if } \gamma = 0, \end{cases}$$

for e being odd and s being even, we have

$$S_{k,\gamma} = \begin{cases} \frac{1}{2}(p^{m-1} - (-1)^{\frac{p-1}{2}}(-1)_{m+e+1}k\eta_1(\gamma)p^{\frac{m-e-1}{2}}), & \text{if } \gamma \neq 0, \\ \frac{1}{2}(p^{m-1} - p^{m-e}), & \text{if } \gamma = 0, \end{cases}$$

for e being even and s being odd, we have

$$S_{k,\gamma} = \begin{cases} \frac{1}{2}p^{e-1}(p^{m-e} - kp^{\frac{m-e}{2}}), & \text{if } \gamma \neq 0, \\ \frac{1}{2}(p^{e-1}-1)(p^{m-e} - kp^{\frac{m-e}{2}}), & \text{if } \gamma = 0, \end{cases}$$

and for e being even and s being even, we have

$$S_{k,\gamma} = \begin{cases} \frac{1}{2}(p^{m-1} - (-1)_e kp^{\frac{m-e-2}{2}}), & \text{if } \gamma \neq 0, \\ \frac{1}{2}(p^{m-1} - p^{m-e} + (-1)_e k(p-1)p^{\frac{m-e-2}{2}}), & \text{if } \gamma = 0. \end{cases}$$

Proof. Let $y = Tr_e^m(x^{-1})$. We only consider $\gamma \neq 0$ in the sequel. Observe that

$$\begin{aligned} |S_{1,\gamma}| &= |\{x \in \mathbb{F}_q^* : Tr_1^e(a^2y^{-1}) = \gamma \neq 0\}| - |S_{-1,\gamma}| \\ &= |\{x \in \mathbb{F}_q^* : Tr_e^m(x^{-1}) = y \neq 0\}||\{y \in \mathbb{F}_{p^e}^* : Tr_1^e(a^2y^{-1}) = \gamma \neq 0\}| \\ &\quad - |S_{-1,\gamma}| \\ &= p^{m-1} - |S_{-1,\gamma}|. \end{aligned}$$

Thus, we only focus on $|S_{-1,\gamma}|$.

For $i, j \in \{-1, 1\}$, define

$$N_{i,j,\gamma} = \{x \in \mathbb{F}_q^* : \eta_m(x) = i,\ \eta_e(Tr_e^m(x^{-1})) = j \text{ and } Tr_1^e(a^2(Tr_e^m(x^{-1}))^{-1}) = \gamma \neq 0\}.$$

From Lemma 4 and Remark 5, we have

$$\begin{aligned} |N_{i,j,\gamma}| &= |\{x \in \mathbb{F}_q^* : \eta_m(x) = i,\ \eta_e(y) = j \text{ and } Tr_1^e(a^2y^{-1}) = \gamma \neq 0\}| \\ &= |\{x \in \mathbb{F}_q^* : \eta_m(x) = i \text{ and } Tr_e^m(x^{-1}) = y \neq 0\}| \\ &\quad |\{y \in \mathbb{F}_{p^e}^* : \eta_e(y) = j \text{ and } Tr_1^e(a^2y^{-1}) = \gamma \neq 0\}| \\ &= |\{x \in \mathbb{F}_q^* : \eta_m(x) = i \text{ and } Tr_e^m(x^{-1}) = y \neq 0\}| \\ &\quad |\{y \in \mathbb{F}_{p^e}^* : \eta_e(y) = j \text{ and } Tr_1^e(y^{-1}) = \gamma \neq 0\}| \\ &= \begin{cases} \frac{1}{4}(p^{m-e} + (-1)^{\frac{p-1}{2}}(-1)_{m+e}ijp^{\frac{m-e}{2}})(p^{e-1} + (-1)^{\frac{p-1}{2}}(-1)_{e+1}j\eta_1(\gamma)p^{\frac{e-1}{2}}), \\ \text{if } e \text{ is odd and } s \text{ is odd}, \\ \frac{1}{4}(p^{m-e} - (-1)_m ip^{\frac{m-2e}{2}})(p^{e-1} + (-1)^{\frac{p-1}{2}}(-1)_{e+1}j\eta_1(\gamma)p^{\frac{e-1}{2}}), \\ \text{if } e \text{ is odd and } s \text{ is even}, \\ \frac{1}{4}(p^{m-e} - ijp^{\frac{m-e}{2}})(p^{e-1} - (-1)_e jp^{\frac{e-2}{2}}), \\ \text{if } e \text{ is even and } s \text{ is odd}, \\ \frac{1}{4}(p^{m-e} + ip^{\frac{m-2e}{2}})(p^{e-1} - (-1)_e jp^{\frac{e-2}{2}}), \\ \text{if } e \text{ is even and } s \text{ is even}. \end{cases} \end{aligned}$$

Then,

$$|S_{-1,\gamma}| = |N_{-1,1,\gamma}| + |N_{1,-1,\gamma}|$$
$$= \begin{cases} \frac{1}{2}p^{e-1}(p^{m-e} - (-1)^{\frac{p-1}{2}}(-1)_{m+e}p^{\frac{m-e}{2}}), \\ \text{if } e \text{ is odd and } s \text{ is odd,} \\ \frac{1}{2}(p^{m-1} + (-1)^{\frac{p-1}{2}}(-1)_{m+e+1}\eta_1(\gamma)p^{\frac{m-e-1}{2}}), \\ \text{if } e \text{ is odd and } s \text{ is even,} \\ \frac{1}{2}p^{e-1}(p^{m-e} + p^{\frac{m-e}{2}}), \\ \text{if } e \text{ is even and } s \text{ is odd,} \\ \frac{1}{2}(p^{m-1} + (-1)_e p^{\frac{m-e-2}{2}}), \\ \text{if } e \text{ is even and } s \text{ is even.} \end{cases}$$

By the same method, we can also compute $S_{k,\gamma}$ when $\gamma = 0$. This completes the proof. □

TABLE 7. The corresponding code $\mathcal{C}_{D_a}$ in Theorem 14, when s is odd.

$n_{(b,a\neq 0)}(0)$	$n_{(b,a\neq 0)}(\rho)(\rho \neq 0)$	Frequency
p^{m-e-1}	p^{m-e-1}	$p^{m-e} - 1$
p^{m-e}	0	1
$p^{m-e-1} - (p-1)p^{\frac{m-e-2}{2}}$	$p^{m-e-1} + p^{\frac{m-e-2}{2}}$	$\frac{1}{2}(p^{e-1} - 1)(p^{m-e} - p^{\frac{m-e}{2}})$
$p^{m-e-1} + (p-1)p^{\frac{m-e-2}{2}}$	$p^{m-e-1} - p^{\frac{m-e-2}{2}}$	$\frac{1}{2}(p^{e-1} - 1)(p^{m-e} + p^{\frac{m-e}{2}})$
$n_{(b,a\neq 0)}(\rho)(\rho = \gamma)$	$n_{(b,a\neq 0)}(\rho)(\rho \neq \gamma)$	Frequency
$p^{m-e-1} - (p-1)p^{\frac{m-e-2}{2}}$	$p^{m-e-1} + p^{\frac{m-e-2}{2}}$	$\frac{1}{2}p^{e-1}(p^{m-e} - p^{\frac{m-e}{2}})$
$p^{m-e-1} + (p-1)p^{\frac{m-e-2}{2}}$	$p^{m-e-1} - p^{\frac{m-e-2}{2}}$	$\frac{1}{2}p^{e-1}(p^{m-e} + p^{\frac{m-e}{2}})$

where γ runs though $\mathbb{F}_p^*$.

Our main result for $a \neq 0$ is:

Theorem 14. *Let $\mathcal{C}_{D_a}$ be defined as previous and $a \neq 0$, Then, the code $\mathcal{C}_{D_a}$ is a $[p^{m-e}, m]$ linear code and its the complete weight enumerator is described as follow*:

(i) *If s is odd, then the complete weight enumerator of the code $\mathcal{C}_{D_a}$ is described as in Table 7*;

(ii) *If s is even and e is odd, then the complete weight enumerator of the code $\mathcal{C}_{D_a}$ is described as in Table 8*;

(iii) *If s is even and e is even, then the complete weight enumerator of the code $\mathcal{C}_{D_a}$ is described as in Table 9.*

Proof. (i) Assume s is odd. Then, we will state that the complete weight enumerators are described as in Table 7 no matter whether e is odd or even.

TABLE 8. The corresponding code $\mathcal{C}_{D_a}$ in Theorem 14, when s is even and e is odd.

$n_{(b,a\neq 0)}(0)$	$n_{(b,a\neq 0)}(\rho)(\rho \neq 0)$	Frequency
p^{m-e-1}	p^{m-e-1}	$p^{m-e}-1$
p^{m-e}	0	1
p^{m-e-1}	$p^{m-e-1}-\eta_1(\rho)p^{\frac{m-e-1}{2}}$	$\frac{1}{2}(p^{m-1}-p^{m-e})$
p^{m-e-1}	$p^{m-e-1}+\eta_1(\rho)p^{\frac{m-e-1}{2}}$	$\frac{1}{2}(p^{m-1}-p^{m-e})$
$n_{(b,a\neq 0)}(\rho)(\rho=\gamma)$	$n_{(b,a\neq 0)}(\rho)(\rho\neq\gamma)$	Frequency
p^{m-e-1}	$p^{m-e-1}-\eta_1(\gamma-\rho)p^{\frac{m-e-1}{2}}$	$\frac{1}{2}(p^{m-1}+\eta_1(\gamma)p^{\frac{m-e-1}{2}})$
p^{m-e-1}	$p^{m-e-1}+\eta_1(\gamma-\rho)p^{\frac{m-e-1}{2}}$	$\frac{1}{2}(p^{m-1}-\eta_1(\gamma)p^{\frac{m-e-1}{2}})$

where γ runs though $\mathbb{F}_p^*$.

TABLE 9. The corresponding code $\mathcal{C}_{D_a}$ in Theorem 14, when s is even and e is even.

$n_{(b,a\neq 0)}(0)$	$n_{(b,a\neq 0)}(\rho)(\rho \neq 0)$	Frequency
p^{m-e-1}	p^{m-e-1}	$p^{m-e}-1$
p^{m-e}	0	1
$p^{m-e-1}-(p-1)p^{\frac{m-e-2}{2}}$	$p^{m-e-1}+p^{\frac{m-e-2}{2}}$	$\frac{1}{2}(p^{m-1}-p^{m-e}+(p-1)p^{\frac{m-e-2}{2}})$
$p^{m-e-1}+(p-1)p^{\frac{m-e-2}{2}}$	$p^{m-e-1}-p^{\frac{m-e-2}{2}}$	$\frac{1}{2}(p^{m-1}-p^{m-e}-(p-1)p^{\frac{m-e-2}{2}})$
$n_{(b,a\neq 0)}(\rho)(\rho=\gamma)$	$n_{(b,a\neq 0)}(\rho)(\rho\neq\gamma)$	Frequency
$p^{m-e-1}-(p-1)p^{\frac{m-e-2}{2}}$	$p^{m-e-1}+p^{\frac{m-e-2}{2}}$	$\frac{1}{2}(p^{m-1}-p^{\frac{m-e-2}{2}})$
$p^{m-e-1}+(p-1)p^{\frac{m-e-2}{2}}$	$p^{m-e-1}-p^{\frac{m-e-2}{2}}$	$\frac{1}{2}(p^{m-1}+p^{\frac{m-e-2}{2}})$

where γ runs though $\mathbb{F}_p^*$.

We only consider the case e being odd, and for e being even, similar arguments can be carried out.

When both s and e are odd, due to (10) we get a partition of the set $\mathbb{F}_q^*$ as follows:

$$\begin{aligned}\mathbb{F}_q^* &= \{b \in \mathbb{F}_q^* : Tr_e^m(b^{-1}) = 0\} \cup \{b \in \mathbb{F}_q^* : Tr_e^m(b^{-1}) \neq 0\}\\ &= \{b \in \mathbb{F}_q^* : Tr_e^m(b^{-1}) = 0\} \cup (\cup_{k,\gamma} S_{k,\gamma}),\end{aligned}$$

where k, γ and $S_{k,\gamma}$ are described as in Lemma 13.

Observe that from (10) the elements in the same part of the partition of $\mathbb{F}_q^*$ correspond to the codewords with the same coordinate symbol distribution, and the converse is also right. Thus, it suffices to calculate $n_{(b,a)}(\rho)$ with $0 \leq \rho \leq p-1$ for the elements b in each part of the partition, and then we determine the size of each part of the partition.

As a first step, since the codeword $\mathbf{c}_b \in \mathcal{C}_{D_a}$ is non-zero for any $b \in \mathbb{F}_q^*$ by (10), we obtain $\dim(\mathcal{C}_{D_a}) = m$. that is, $\mathcal{C}_{D_a}$ is a $[p^{m-e}, m]$ linear code.

Then, consider the part of the partition $\{b \in \mathbb{F}_q^* : Tr_e^m(b^{-1}) = 0\}$. For each element b of this part, it holds $n_{(b,a)}(\rho) = p^{m-e-1}$ for any $\rho \neq 0$ by (10). Furthermore, the number of codewords with the above property is equal to

$$|\{b \in \mathbb{F}_q^* : Tr_e^m(b^{-1}) = 0\}| = p^{m-e} - 1.$$

Now, let us consider $S_{k,\gamma}$, if $\gamma = 0$ and $b \in S_{k,\gamma} = S_{k,0}$, then we have $n_{(b,a)}(\rho) = p^{m-e-1} - (-1)^{\frac{p-1}{2}}(-1)_{m+e}kp^{\frac{m-e-2}{2}}$ for any $\rho \neq 0$ by (10). If $\gamma\rho \neq 0$ and $\rho \neq \gamma$, we have $n_{(b,a)}(\rho) = p^{m-e-1} - (-1)^{\frac{p-1}{2}}(-1)_{m+e}kp^{\frac{m-e-2}{2}}$ by (10), and if $\gamma\rho \neq 0$ and $\rho = \gamma$, we have $n_{(b,a)}(\rho) = p^{m-e-1} + (-1)^{\frac{p-1}{2}}(-1)_{m+e}k(p-1)p^{\frac{m-e-2}{2}}$ by (10).

Since $n_{(b,a)}(0) = p^{m-e} - \sum_{\rho=1}^{p-1} n_{(b,a)}(\rho)$, we get $n_{(b,a)}(0)$ in the above cases.

Finally, by using Lemma 13, the size $|S_{k,\gamma}|$ of each part $S_{k,\gamma}$ can be obtained, and we list the complete weight enumerators of the code $\mathcal{C}_{D_a}$ in Table 7.

Similar arguments as in (i) yield the results in (ii) and (iii), which are listed in Tables 8 and 9, respectively, and the details are omitted.

This completes the proof. □

Remark 15. It is observed that, when $a \neq 0$, the complete weight enumerator of the code $\mathcal{C}_{D_a}$ are independent of the choice of a.

Example 16. Let $(p, m, e, s) = (3, 3, 1, 3)$ and $a \neq 0$. Then by Theorem 14 and Table 7, the code $\mathcal{C}_{D_a}$ has parameters $[9, 3, 5]$ and complete weight enumerators

$$w_0^9 + 8w_0^3w_1^3w_2^3 + 3w_0^4w_1^1w_2^4 + 6w_0^2w_1^5w_2^2 + 3w_0^4w_1^4w_2^1 + 6w_0^2w_1^2w_2^5.$$

This code is almost optimal.

Example 17. Let $(p, m, e, s) = (3, 6, 3, 2)$ and $a \neq 0$. Then by Theorem 14 and Table 8, the code $\mathcal{C}_{D_a}$ has parameters $[27, 6, 15]$ and complete weight enumerator

$$\begin{aligned} &w_0^{27} + 26w_0^9w_1^9w_2^9 + 108w_0^9w_1^6w_2^{12} + 108w_0^9w_1^{12}w_2^6 + 123w_0^6w_1^9w_2^{12} \\ &+ 120w_0^{12}w_1^9w_2^6 + 120w_0^{12}w_1^6w_2^9 + 123w_0^6w_1^{12}w_2^9. \end{aligned}$$

This code is optimal.

Example 18. Let $(p, m, e, s) = (3, 4, 2, 2)$ and $a \neq 0$. Then by Theorem 14 and Table 9, the code $\mathcal{C}_{D_a}$ has parameters $[9, 4, 4]$ and complete weight enumerator

$$\begin{aligned} &w_0^9 + 8w_0^3w_1^3w_2^3 + 10w_0^1w_1^4w_2^4 + 8w_0^5w_1^2w_2^2 + 13w_0^4w_1^1w_2^4 \\ &+ 14w_0^2w_1^5w_2^2 + 13w_0^4w_1^4w_2^1 + 14w_0^2w_1^2w_2^5. \end{aligned}$$

This code is almost optimal.

Remark 19. The defining set in this paper can be generally defined by $D_a(\alpha, \beta) = \{x \in \mathbb{F}_q^* : Tr_e^m(\alpha x + \beta) = a\}$, where $a \in \mathbb{F}_{p^e}$, $\alpha \in \mathbb{F}_q^*$ and $\beta \in \mathbb{F}_q$.

If $(\alpha, \beta) = (\alpha, 0)$, then by using similar proof as in this paper we can obtain that the code $\mathcal{C}_{D_a(\alpha,0)}$ has the same complete weight enumerators as $\mathcal{C}_{D_a}$. To get the result, we need the following modifications: the parameter b^{-1} is replaced by $\alpha^2 b^{-1}$ in (8), (9) and (10), and x^{-1} is replaced by $\alpha^2 x^{-1}$ in Lemmas 4, 8 and 13 due to Remark 7.

If $(\alpha, \beta) = (\alpha, \beta \neq 0)$, then we may rewrite the defining set as $\{x \in \mathbb{F}_q^* : Tr_e^m(\alpha x) = a - Tr_e^m(\beta)\}$, and then the complete weight enumerators of the code $\mathcal{C}_{D_a(\alpha,\beta)}$ can be determined from $\mathcal{C}_{D_a(\alpha,0)}$. The details are omitted.

4. Conclusion

In this paper, we constructed several classes of linear codes with a few weights by choosing the defining set properly, and then we determined the complete weight enumerators of these classes of codes.

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