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A NOTE ON PROOF OF GORDON'S CONJECTURE

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Abstract. In this paper, we show a proof of Gordon's Conjecture by using Qiu's labels and two new labels.

1. Introduction

All 3-manifolds in this paper are assumed to be compact and orientable. All surfaces in 3-manifolds are assumed to be orientable.

Let M be a 3-manifold. If there is a closed surface S which cuts M into two compression bodies V and W with $S = \partial_+ W = \partial_+ V$, then we say M has a Heegaard splitting, denoted by $M = V \cup_S W$; and S is called a Heegaard surface of M. If there are essential disks $B \subset V$ and $D \subset W$ such that $\partial B = \partial D$ (resp. $\partial B \cap \partial D = \emptyset$), then $M = V \cup_S W$ is said to be reducible (resp. weakly reducible); otherwise, $M = V \cup_S W$ is said to be irreducible (resp. strongly irreducible). If there are essential disks $B \subset V$ and $D \subset W$ such that $|B \cap D| = 1$, then $M = V \cup_S W$ is said to be stabilized; otherwise, $M = V \cup_S W$ is said to be unstabilized.

Let M be a 3-manifold, F be a connected closed surface in M, which cuts M into two 3-manifolds M_1 and M_2. Suppose that $M_i = V_i \cup_{S_i} W_i$ is a Heegaard splitting of M_i ($i = 1, 2$). Then, M has a natural Heegaard splitting $M = V \cup_S W$ called the amalgamation of $M_1 = V_1 \cup_{S_1} W_1$ and $M_2 = V_2 \cup_{S_2} W_2$ along F, see [8]. From this construction, we have $g(M) \leq g(M_1)+g(M_2)-g(F)$. So, there is an interesting question as follows:

Question 1.1. *When $M = V \cup_S W$ is unstabilized?*

If $g(F) = 0$, then it is the Gordon's Conjecture ([2]). Bachman ([1]), Qiu ([6]), Qiu and Scharlemann ([7]) give an affirmative answer about this question. But for generally case, it is not true. There are two counterexamples, such that $g(M) < g(M_1) + g(M_2) - g(F)$, see [4] and [9]. In [3], Kabayashi and Qiu

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proved the uniqueness of minimal Heegaard splitting $M = V \cup_S W$ by using sufficiently complicated manifolds, i.e., the amalgamation of $M_1 = V_1 \cup_{S_1} W_1$ and $M_2 = V_2 \cup_{S_2} W_2$ along F. In [5], Lackenby proved the uniqueness of minimal Heegaard splitting $M = V \cup_S W$ by using sufficiently complicated map, i.e., the amalgamation of $M_1 = V_1 \cup_{S_1} W_1$ and $M_2 = V_2 \cup_{S_2} W_2$ along F.

If $g(F) = 0$, then S can be isotoped, such that $F \cap S$ is an essential simple closed curve on S. Hence, $M = V \cup_S W$ is the reducible Heegaard splitting and F is the reducing 2-sphere. So, F cuts V into V^1 and V^2 and cuts W into $W_1^{'}$ and $W_2^{'}$ such that $M_1 = V^1 \cup W_1^{'}$ and $M_2 = V^2 \cup W_2^{'}$. Let $W^i = W_i^{'} \cup_{\partial F = \partial B_i^3} B_i^3$ $(i = 1, 2)$, where B_i^3 is a 3-ball. Then, W^i is a compression body and $M^i = V^i \cup_{S^i} W^i$ is a Heegaard splitting of M^i with $S^i = \partial_+ V^i = \partial_+ W^i$. So, $M = V \cup_S W$ is said to be the connected sum of $M^1 = V^1 \cup_{S^1} W^1$ and $M^2 = V^2 \cup_{S^2} W^2$. In this paper, we show a proof of Gordon's Conjecture by using Qiu's labels in [6] and two new labels as follows:

Theorem 1.2. *The connected sum $M = V \cup_S W$ of $M^1 = V^1 \cup_{S^1} W^1$ and $M^2 = V^2 \cup_{S^2} W^2$ is stabilized if and only if one of $M^1 = V^1 \cup_{S^1} W^1$ and $M^2 = V^2 \cup_{S^2} W^2$ is stabilized.*

2. The proof of Theorem 1.2

Proof. If one of $M^1 = V^1 \cup_{S^1} W^1$ and $M^2 = V^2 \cup_{S^2} W^2$ is stabilized, then by the construction of Heegaard splitting of connected sum, $M = V \cup_S W$ is stabilized. So, we only prove that if $M = V \cup_S W$ is stabilized, then one of $M^1 = V^1 \cup_{S^1} W^1$ and $M^2 = V^2 \cup_{S^2} W^2$ is stabilized.

Since $M = V \cup_S W$ is stabilized, there are two disks $D_V \subset V$ and $D_W \subset W$ such that $|D_V \cap D_W| = 1$. Let $x = D_V \cap D_W$, $F_V = F \cap V$ and $F_W = F \cap W$, where F is the reducing 2-sphere of $M = V \cup_S W$. Then F_V is an essential disk in V and F_W is an essential disk in W.

Proposition 2.1. *If either $D_V \cap F_V = \emptyset$ or $D_W \cap F_W = \emptyset$, then one of $M^1 = V^1 \cap_{S^1} W^1$ and $M^2 = V^2 \cap_{S^2} W^2$ is stabilized.*

Proof. If $D_V \cap F_V = \emptyset$, then D_V is a properly embedded disk in V^1 or V^2. We may assume that D_V lies in V^1. If $D_W \cap F_W = \emptyset$, since $|D_V \cap D_W| = 1$, D_W is a properly embedded disk in W^1. Hence, $M^1 = V^1 \cup_{S^1} W^1$ is stabilized and Proposition 2.1 holds. So, we may assume that $D_W \cap F_W \neq \emptyset$ and $|D_W \cap F_W|$ is minimal. Hence, each component of $D_W \cap F_W$ is an arc on both D_W and F_W. Let $S_i^{'} = S^i \cap S$ $(i = 1, 2)$. Since $|D_W \cap F_W|$ is minimal, each component of $\partial D_W \cap S_i^{'}$ is an essential arc on $S_i^{'}$. Let D_1^W be a subdisk of D_W, which is cut by F_W, such that $|D_V \cap D_1^W| = 1$. Since $D_V \cap F_V = \emptyset$, we can push all components of $\partial D_1^W \cap F_W$ into $S_1^{'}$, after isotopy, still denote it by D_1^W. Then, D_1^W is a properly embedded disk in W^1 and $|D_V \cap D_1^W| = 1$. So, $M^1 = V^1 \cup_{S^1} W^1$ is stabilized and Proposition 2.1 holds.

If $D_W \cap F_W = \emptyset$, then D_W is a properly embedded disk in W^1 or W^2. We may assume that D_W lies in W^1. If $D_V \cap F_V = \emptyset$, then by the same argument as above, $M^1 = V^1 \cup_{S^1} W^1$ is stabilized and Proposition 2.1 holds. So, we may assume that $D_V \cap F_V \neq \emptyset$ and $|D_V \cap F_V|$ is minimal. Hence, each component of $D_V \cap F_V$ is an arc on both D_V and F_V, and each component of $\partial D_V \cap S_i^{'}$ is an essential arc on $S_i^{'}$ $(i = 1, 2)$. Let D_1^V be a subdisk of D_V, which is cut by F_V, such that $|D_1^V \cap D_W| = 1$. Then, D_1^V is a properly embedded disk in V^1. Hence, $M^1 = V^1 \cup_{S^1} W^1$ is stabilized and Proposition 2.1 holds. □

By Proposition 2.1, we may assume that $D_V \cap F_V \neq \emptyset$, $D_W \cap F_W \neq \emptyset$, both $|D_V \cap F_V|$ and $|D_W \cap F_W|$ are minimal. Hence, each component of $D_V \cap F_V$ is an arc on both D_V and F_V, each component of $D_W \cap F_W$ is an arc on both D_W and F_W, each component of $\partial D_V \cap S_i^{'}$ is essential on $S_i^{'}$, and each component of $\partial D_W \cap S_i^{'}$ is essential on $S_i^{'}$ $(i = 1, 2)$. After isotopy, we may assume that $x = D_V \cap D_W$ lies in $S_1^{'}$. Let $|D_V \cap F_V| = p$ and $|D_W \cap F_W| = n$. Now we show Qiu's labels (see [6]) and two new labels for each arc of $D_V \cap F_V$ on F_V and $D_W \cap F_W$ on F_W as follows:

For each component e of $D_V \cap F_V$ on F_V, e cuts D_V into two disks $V_e^{'}$ and $V_e^{''}$, such that x lies in $\partial V_e^{'}$. Let V_e be a subdisk of D_V, which is cut by F_V, such that ∂V_e contains e and $V_e \subset V_e^{''}$, see Figure 3 in [6]. Then, V_e is a properly embedded disk in V^1 or V^2. If V_e lies in V^1, then we label e with "$+$"; if V_e lies in V^2, then we label e with "$-$". Similarly, for each component e of $D_W \cap F_W$ on F_W, e cuts D_W into two disks $W_e^{'}$ and $W_e^{''}$, such that x lies in $\partial W_e^{'}$. Let W_e^1 be a subdisk of D_W, which is cut by F_W, such that ∂W_e^1 contains e and $W_e^1 \subset W_e^{''}$. Then, W_e^1 is a properly embedded disk in $W_1^{'}$ or $W_2^{'}$. If W_e^1 lies in $W_1^{'}$, then we label e with "$+$"; if W_e^1 lies in $W_2^{'}$, then we label e with "$-$".

Since $|D_V \cap F_V| = p$ and $|D_W \cap F_W| = n$, we label the arcs of $D_V \cap F_V$ on F_V with $\{v_1, \ldots, v_p\}$ and label the arcs of $D_W \cap F_W$ on F_W with $\{w_1^{'}, \ldots, w_n^{'}\}$, such that if $V_{v_i}^{''} \subsetneq V_{v_k}^{''}$ and $W_{w_j^{'}}^{''} \subsetneq W_{w_l^{'}}^{''}$, then $i < k$ and $j < l$. So, each subdisk of D_V which is cut by F_V and does not contain x is denoted by V_{v_i} $(1 \leq i \leq p)$ and each subdisk of D_W which is cut by F_W and does not contain x is denoted by $W_{w_j^{'}}^1$ $(1 \leq j \leq n)$. For convenience, we denote V_{v_i} by V_i and denote $W_{w_j^{'}}^1$ by W_j^1. Let V_x be the subdisk of D_V which is cut by F_V, such that ∂V_x contains x, W_x^1 be the subdisk of D_W which is cut by F_W, such that ∂W_x^1 contains x.

Remark 2.2. Since x lies in $S_1^{'}$, each subdisk of D_V which is cut by F_V and lies in V^1, is either V_i, where the label v_i is "$+$", or V_x; each subdisk of D_V which is cut by F_V and lies in V^2, is V_i, where the label v_i is "$-$"; each subdisk of D_W which is cut by F_W and lies in $W_1^{'}$, is either W_j^1, where the label $w_j^{'}$ is "$+$", or W_x^1; and each subdisk of D_W which is cut by F_W and lies in $W_2^{'}$, is W_j^1, where the label $w_j^{'}$ is "$-$".

For each component $w_j^{'}$ $(1 \leq j \leq n)$ of $D_W \cap F_W$ on F_W, w_j is said to be the dual arc of $w_j^{'}$ on F_V, if $\partial w_j = \partial w_j^{'}$. After isotopy, we may assume that for each component v_i $(1 \le i \le p)$ of $D_V \cap F_V$ on F_V, $|w_j \cap v_i| \le 1$. We may assume that w_j and $w_j^{'}$ have the same labels. For each subdisk W_j^1 $(1 \le j \le n$ or $j = x)$ of D_W which is cut by F_W, we can push each arc $w_k^{'}$ of $\partial W_j^1 \cap F_W$ on F_W into F_V, such that $w_k^{'}$ is replaced by w_k on F_V. After isotopy, we denote it by W_j. Then, W_j is a properly embedded disk in W^1 or W^2.

So, for each arc v_i $(1 \le i \le p)$ of $D_V \cap F_V$ on F_V and each dual arc w_j $(1 \le j \le n)$ of $D_W \cap F_W$ on F_V, $|v_i \cap w_j| \le 1$. Let $I(v_i) = \{r \,|\, v_r \subset \partial V_i$ and $v_r \ne v_i\}$, $I(w_j) = \{r \,|\, w_r \subset \partial W_j$ and $w_r \ne w_j\}$, $I(v) = \{r \,|\, v_r \subset \partial V_x\}$ and $I(w) = \{r \,|\, w_r \subset \partial W_x\}$. Then, there are some properties for $I(v_i)$, $I(w_j)$, $I(v)$, $I(w)$, V_i, V_x, W_j and W_x as follows:

Proposition 2.3 ([6]). (1) *If* $r \in I(v_i)$, *then* $r < i$;

(2) *if* $r \in I(w_j)$, *then* $r < j$;

(3) *the label* v_i *is* "$+$" *if and only if the label* v_r *is* "$-$" *for each* $r \in I(v_i)$;

(4) *the label* w_j *is* "$+$" *if and only if the label* w_r *is* "$-$" *for each* $r \in I(w_j)$;

(5) *if* $r \in I(v)$, *then the label* v_r *is* "$-$";

(6) *if* $r \in I(w)$, *then the label* w_r *is* "$-$";

(7) $p \in I(v)$, $n \in I(w)$;

(8) *there are four sets of pairwise disjoint properly embedded disks* $\{V_i \,|\, 1 \le i \le p$ *and the label* v_i *is* "$+$"$\} \cup \{V_x\}$ *in* V^1, $\{V_i \,|\, 1 \le i \le p$ *and the label* v_i *is* "$-$"$\}$ *in* V^2, $\{W_j \,|\, 1 \le j \le n$ *and the label* w_j *is* "$+$"$\} \cup \{W_x\}$ *in* W^1, *and* $\{W_j \,|\, 1 \le j \le n$ *and the label* w_j *is* "$-$"$\}$ *in* W^2, *satisfying the following conditions*:

(i) $V_i \cap F_V = v_i \cup_{r \in I(v_i)} v_r$, $W_j \cap F_V = w_j \cup_{r \in I(w_j)} w_r$, $V_x \cap F_V = \cup_{r \in I(v)} v_r$, $W_x \cap F_V = \cup_{r \in I(w)} w_r$;

(ii) *if* V_i *lies in* V^1 *and* W_j *lies in* W^1, *then* $V_i \cap W_j = V_i \cap W_j \cap F_V$, $V_i \cap W_x = V_i \cap W_x \cap F_V$, $V_x \cap W_j = V_x \cap W_j \cap F_V$, *and* $V_x \cap W_x = \{x\} \cup (V_x \cap W_x \cap F_V)$;

(iii) *if* V_i *lies in* V^2 *and* W_j *lies in* W^2, *then* $V_i \cap W_j = V_i \cap W_j \cap F_V$.

Since F_V cuts V into V^1 and V^2, let F_V^k $(k = 1, 2)$ be a copy of F_V, such that F_V^k lies in S^k, v_i^k be a copy of v_i on F_V^k and w_j^k be a copy of w_j on F_V^k $(1 \le i \le p; 1 \le j \le n)$. We may assume that v_i^k and v_i have the same label, and w_j^k and w_j have the same label. For convenience, $v_i^1 = v_i^2$ means that both v_i^1 and v_i^2 are the copies of v_i, and $w_j^1 = w_j^2$ means that both w_j^1 and w_j^2 are the copies of w_j.

Outline of the proof of Theorem 2. By using Qiu's labels and two new labels, we band sum disks of $\{V_i \,|\, 1 \le i \le p$ and the label v_i is "$+$"$\} \cup \{V_x\}$ in V^1 along some arcs obtained from $\{w_1^1, w_2^1, \ldots, w_n^1\}$ on S^1, band sum disks of $\{W_j \,|\, 1 \le j \le n$ and the label w_j is "$+$"$\} \cup \{W_x\}$ in W^1 along some arcs obtained from $\{v_1^1, v_2^1, \ldots, v_p^1\}$ on S^1, band sum disks of $\{V_i \,|\, 1 \le i \le p$ and the label v_i is "$-$"$\}$ in V^2 along some arcs obtained from $\{w_1^2, w_2^2, \ldots, w_n^2\}$ on S^2, and band sum disks of $\{W_j \,|\, 1 \le j \le n$ and the label w_j is "$-$"$\}$ in W^2 along some arcs obtained from $\{v_1^2, v_2^2, \ldots, v_p^2\}$ on S^2. Finally, either there are two disks $D_{V^1} \subset V^1$ and $D_{W^1} \subset W^1$ with $|D_{V^1} \cap D_{W^1}| = 1$ or there are two disks

$D_{V^2} \subset V^2$ and $D_{W^2} \subset W^2$ with $|D_{V^2} \cap D_{W^2}| = 1$. So, one of $M^1 = V^1 \cup_{S^1} W^1$ and $M^2 = V^2 \cup_{S^2} W^2$ is stabilized.

Proposition 2.4. *Either there are two disks $D_{V^1} \subset V^1$ and $D_{W^1} \subset W^1$ with $|D_{V^1} \cap D_{W^1}| = 1$, where D_{V^1} is obtained by banding sum disks of $\{V_i \,|\, 1 \leq i \leq p$ and the label v_i is "$+$"$\} \cup \{V_x\}$ in V^1 along some arcs obtained from $\{w_1^1, w_2^1, \ldots, w_n^1\}$ on S^1, and D_{W^1} is obtained by banding sum disks of $\{W_j \,|\, 1 \leq j \leq n$ and the label w_j is "$+$"$\} \cup \{W_x\}$ in W^1 along some arcs obtained from $\{v_1^1, v_2^1, \ldots, v_p^1\}$ on S^1, or there are two disks $D_{V^2} \subset V^2$ and $D_{W^2} \subset W^2$ with $|D_{V^2} \cap D_{W^2}| = 1$, where D_{V^2} is obtained by banding sum disks of $\{V_i \,|\, 1 \leq i \leq p$ and the label v_i is "$-$"$\}$ in V^2 along some arcs obtained from $\{w_1^2, w_2^2, \ldots, w_n^2\}$ on S^2, and D_{W^2} is obtained by banding sum disks of $\{W_j \,|\, 1 \leq j \leq n$ and the label w_j is "$-$"$\}$ in W^2 along some arcs obtained from $\{v_1^2, v_2^2, \ldots, v_p^2\}$ on S^2.*

Proof. We consider all arcs $\{v_1, v_2, \ldots, v_p\}$ of $D_V \cap F_V$ on F_V in sequence. If we consider all dual arcs $\{w_1, w_2, \ldots, w_n\}$ of $D_W \cap F_W$ on F_V in sequence, then the argument is the same. So, we may assume that $p \leq n$. First, we consider v_1^1 on F_V^1. Let m^1 be the minimal label among all arcs of $\{w_j^1 \,|\, 1 \leq j \leq n\}$ on F_V^1 with $|w_{m^1}^1 \cap v_1^1| = 1$. If $m^1 = \emptyset$, then for each arc w_j^1 $(1 \leq j \leq n)$, $w_j^1 \cap v_1^1 = \emptyset$. If $m^1 \neq \emptyset$, then $|v_1^1 \cap w_{m^1}^1| = 1$ $(1 \leq m^1 \leq n)$. Since $v_i^1 = v_i^2$ $(1 \leq i \leq p)$ and $w_j^1 = w_j^2$ $(1 \leq j \leq n)$, $|v_i^1 \cap w_j^1| = |v_i^2 \cap w_j^2|$. So, m^1 is the minimal label among all arcs of $\{w_j^2 |\ 1 \leq j \leq n\}$ on F_V^2 with $|w_{m^1}^2 \cap v_1^2| = 1$. We may assume that the label v_1 on F_V is "$+$". If the label v_1 on F_V is "$-$", then the argument is the same.

If $m^1 = \emptyset$, then for each arc w_j^k $(1 \leq j \leq n; k = 1, 2)$, $w_j^k \cap v_1^k = \emptyset$. Since the label v_1 on F_V is "$+$", the label v_1^k $(k = 1, 2)$ on F_V^k is "$+$". We label v_1^1 on F_V^1 with "$\times$" and label v_1^2 on F_V^2 with "$\circ$". The label "$\times$" on v_1^1 means that we delete the arc v_1^1 on F_V^1, and the label "$\circ$" on v_1^2 means that we retain the arc v_1^2 on F_V^2. For each arc v_i^2 $(2 \leq i \leq p)$ and w_j^2 $(1 \leq j \leq n)$, since $v_i^2 \cap v_1^2 = \emptyset$ and $w_j^2 \cap v_1^2 = \emptyset$, there is no influence on v_1^2 when we consider v_i^2 and w_j^2. Hence, the label "$\circ$" on v_1^2 means that we retain the arc v_1^2 on F_V^2. For convenience, for each arc v_i^k $(2 \leq i \leq p)$ and w_j^k $(1 \leq j \leq n)$ on S^k $(k = 1, 2)$, we denote them by $v_{i_1}^k$ and $w_{j_1}^k$. We may assume that $v_{i_1}^k$ and v_i^k have the same label, and $w_{j_1}^k$ and w_j^k have the same label. For each disk V_i $(2 \leq i \leq p$ or $i = x)$ and W_j $(1 \leq j \leq n$ or $j = x)$, we denote them by V_{i_1} and W_{j_1}. Since v_1^2 is retained, we also denote it by $v_{1_1}^2$. But in the future banding sum process, we do not consider $v_{1_1}^2$.

If $m^1 \neq \emptyset$, then $|v_1^1 \cap w_{m^1}^1| = |v_1^2 \cap w_{m^1}^2| = 1$.

Lemma 2.5. *If the label w_{m^1} on F_V is "$+$", then $M^1 = V^1 \cup_{S^1} W^1$ is stabilized.*

Proof. Since the label v_1 on F_V is "$+$", V_1 is a properly embedded disk in V^1. For each $r \in I(v_1)$, by (1) in Proposition 2.3, $r < 1$. So, $I(v_1) = \emptyset$. By (i) of (8) in Proposition 2.3, $V_1 \cap F_V^1 = v_1^1 \cup_{r \in I(v_1)} v_r^1 = v_1^1$. Since the label w_{m^1} on

F_V is "$+$", W_{m^1} is a properly embedded disk in W^1. For each $r \in I(w_{m^1})$, by (2) in Proposition 2.3, $r < m^1$. By the minimality of m^1, $w_r^1 \cap v_1^1 = \emptyset$. By (8) in Proposition 2.3, $|V_1 \cap W_{m^1}| = |V_1 \cap W_{m^1} \cap F_V^1| = |v_1^1 \cap (w_{m^1}^1 \cup_{r \in I(w_{m^1})} w_r^1)| = |v_1^1 \cap w_{m^1}^1| = 1$. So, $M^1 = V^1 \cup_{S^1} W^1$ is stabilized. □

By Lemma 2.5, Proposition 2.4 holds. So, we may assume that the label w_{m^1} on F_V is "$-$". Then, W_{m^1} is a properly embedded disk in W^2. Now we label v_1^1 and $w_{m^1}^1$ on F_V^1, and label v_1^2 and $w_{m^1}^2$ on F_V^2, respectively:

(I_1) Label v_1^1 and $w_{m^1}^1$ on F_V^1.

By (8) in Proposition 2.3, $|V_1 \cap w_{m^1}^1| = |(V_1 \cap F_V^1) \cap w_{m^1}^1| = |v_1^1 \cap w_{m^1}^1| = 1$. If there is a disk V_l of $\{V_i \mid \text{the label } v_i \text{ is "}+\text{" and } 2 \le i \le p\} \cup \{V_x\}$ in V^1 with $\partial V_l \cap w_{m^1}^1 \neq \emptyset$, then we band sum V_l and k copies of V_1 along $w_{m^1}^1$ in some order, where $|\partial V_l \cap w_{m^1}^1| = k$. After banding sum and isotopy, we obtain a properly embedded disk in V^1 and denote it by V_{l_1}. So, $V_{l_1} \cap V_1 = \emptyset$ and $\partial V_{l_1} \cap (w_{m^1}^1 \cup v_1^1) = \emptyset$. If there is a disk V_l of $\{V_i \mid \text{the label } v_i \text{ is "}+\text{" and } 2 \le i \le p\} \cup \{V_x\}$ in V^1 with $\partial V_l \cap w_{m^1}^1 = \emptyset$, then we do nothing and denote it by V_{l_1}. After isotopy, we obtain a collection of mutually disjoint disks $\{V_{i_1} \mid \text{the label } v_i \text{ is "}+\text{" and } 2 \le i \le p\} \cup \{V_{x_1}\}$ in V^1. For each disk W_l of $\{W_j \mid \text{the label } w_j \text{ is "}+\text{", } 1 \le j \le n \text{ and } j \neq m^1\} \cup \{W_x\}$ in W^1, we do nothing and denote it by W_{l_1}. So, we obtain a collection of mutually disjoint disks $\{W_{j_1} \mid \text{the label } w_j \text{ is "}+\text{", } 1 \le j \le n \text{ and } j \neq m^1\} \cup \{W_{x_1}\}$ in W^1.

This procedure can be viewed as for each arc v_i^1 $(2 \le i \le p)$, if $|v_i^1 \cap w_{m^1}^1| = 1$, then we band sum v_i^1 and a copy ∂V_1^i of ∂V_1 along $w_{m^1}^1$, where V_1^i is a copy of V_1 and $\partial V_1^i \cap F_V^1$ lies between v_1^1 and v_i^1. After banding sum and isotopy, we obtain a new arc and denote it by $v_{i_1}^1$. Before banding sum, if there is an arc v_k^1 $(k \neq 1, i)$ with $|v_k^1 \cap w_{m^1}^1| = 1$, such that v_k^1 lies between v_1^1 and v_i^1, then v_k^1 lies between $\partial V_1^i \cap F_V^1$ and v_i^1. Let ∂V_1^k be a copy of ∂V_1, where V_1^k is a copy of V_1, such that $\partial V_1^k \cap F_V^1$ lies between $\partial V_1^i \cap F_V^1$ and v_k^1. Then, we band sum v_k^1 and ∂V_1^k along $w_{m^1}^1$. After banding sum and isotopy, we obtain a new arc and denote it by $v_{k_1}^1$, such that $v_{k_1}^1 \cap v_{i_1}^1 = \emptyset$, see Figure 1. If $v_i^1 \cap w_{m^1}^1 = \emptyset$, then we do nothing and denote it by $v_{i_1}^1$.

After banding sum and isotopy, we obtain a collection of mutually disjoint arcs $\{v_{i_1}^1 \mid 2 \le i \le p\}$ on S^1. Also, for each arc w_j^1 $(1 \le j \le n$ and $j \neq m^1)$ before banding sum, we do nothing and denote it by $w_{j_1}^1$ after banding sum. So, there is a collection of mutually disjoint arcs $\{w_{j_1}^1 \mid 1 \le j \le n \text{ and } j \neq m^1\}$ on S^1. Hence, v_i^1 and w_j^1 $(2 \le i \le p; 1 \le j \le n$ and $j \neq m^1)$ represent the arcs before banding sum, $v_{i_1}^1$ and $w_{j_1}^1$ represent the arcs after banding sum. We may assume that v_i^1 and $v_{i_1}^1$ $(2 \le i \le p)$ have the same label, and w_j^1 and $w_{j_1}^1$ $(1 \le j \le n$ and $j \neq m^1)$ have the same label.

For each arc $v_{i_1}^1$ $(2 \le i \le p)$ and $w_{j_1}^1$ $(1 \le j \le n$ and $j \neq m^1)$, $v_{i_1}^1 \cap (v_1^1 \cup w_{m^1}^1) = \emptyset$, $|v_{i_1}^1 \cap w_{j_1}^1| \le 1$ and $|v_{i_1}^1 \cap F_V^1| \le 2$. Since the label v_1^1 on F_V^1 is "$+$" and the label $w_{m^1}^1$ on F_V^1 is "$-$", we label v_1^1 on F_V^1 with "$\times$" and label $w_{m^1}^1$ on

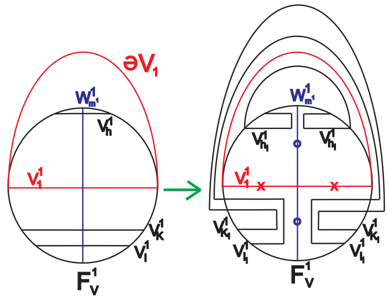

FIGURE 1. Band sum subdisk V_i and V_1 in V^1 along $w^1_{m^1}$ $(2 \le i \le p)$

F^1_V with "$\circ$", see Figure 1. The label "$\times$" on v^1_1 means that we delete the arc v^1_1 on F^1_V, the label "$\circ$" on $w^1_{m^1}$ means that we retain the arc $w^1_{m^1}$ on F^1_V. For each arc $v^1_{i_1}$ $(2 \le i \le p)$ and $w^1_{j_1}$ $(1 \le j \le n$ and $j \ne m^1)$, since $v^1_{i_1} \cap w^1_{m^1} = \emptyset$ and $w^1_{j_1} \cap w^1_{m^1} = \emptyset$, there is no influence on $w^1_{m^1}$ when we consider $v^1_{i_1}$ and $w^1_{j_1}$. Hence, the label "$\circ$" on $w^1_{m^1}$ means that we retain the arc $w^1_{m^1}$. So, we also denote it by $w^1_{m^1_1}$, but in the future banding sum process, we do not need to consider it.

By (8) in Proposition 2.3 and the argument as above, we have:

Lemma 2.6. *There are two sets of pairwise disjoint properly embedded disks* $\{V_{i_1} \mid$ *the label* v_i *is* "$+$" *and* $2 \le i \le p\} \cup \{V_{x_1}\}$ *in* V^1, *and* $\{W_{j_1} \mid$ *the label* w_j *is* "$+$", $1 \le j \le n$ *and* $j \ne m^1\} \cup \{W_{x_1}\}$ *in* W^1, *satisfying the following conditions*:

(1) $V_{i_1} \cap F^1_V = (v^1_{i_1} \cap F^1_V) \cup_{r \in I(v_i)} (v^1_{r_1} \cap F^1_V)$, $W_{j_1} \cap F^1_V = w^1_{j_1} \cup_{r \in I(w_j)} w^1_{r_1}$, $V_{x_1} \cap F^1_V = \cup_{r \in I(v)} (v^1_{r_1} \cap F^1_V)$, $W_{x_1} \cap F^1_V = \cup_{r \in I(w)} w^1_{r_1}$;

(2) $V_{i_1} \cap W_{j_1} = V_{i_1} \cap W_{j_1} \cap F^1_V$, $V_{i_1} \cap W_{x_1} = V_{i_1} \cap W_{x_1} \cap F^1_V$, $V_{x_1} \cap W_{j_1} = V_{x_1} \cap W_{j_1} \cap F^1_V$, $V_{x_1} \cap W_{x_1} = \{x\} \cup (V_{x_1} \cap W_{x_1} \cap F^1_V)$.

Remark 2.7. For each $2 \le i \le p$, $1 \le j \le n$ and $j \ne m^1$, if $|v^1_i \cap w^1_{m^1}| = 1$ and $|w^1_j \cap (v^1_1 \cup v^1_i)| = 1$ before banding sum, then $|v^1_{i_1} \cap w^1_{j_1}| = 1$ after banding sum; if $|v^1_i \cap w^1_{m^1}| = 1$ and $|w^1_j \cap (v^1_1 \cup v^1_i)| = 0$ or 2 before banding sum, then $v^1_{i_1} \cap w^1_{j_1} = \emptyset$ after banding sum and isotopy; if $v^1_i \cap w^1_{m^1} = \emptyset$ before banding sum, then $|v^1_{i_1} \cap w^1_{j_1}| = |v^1_i \cap w^1_j|$, see Figure 2. After banding sum and isotopy, $|v^1_{i_1} \cap w^1_{j_1}| \le 1$.

(I_2) Label v^2_1 and $w^2_{m^1}$ on F^2_V.

Since the label w_{m^1} on F_V is "$-$", W_{m^1} is a properly embedded disk in W^2. For each $r \in I(w_{m^1})$, by (2) in Proposition 2.3, $r < m^1$. By the minimality of m^1, $v^2_1 \cap w^2_r = \emptyset$. By (8) in Proposition 2.3, $|W_{m^1} \cap v^2_1| = |(W_{m^1} \cap F^2_V) \cap v^2_1| = |(w^2_{m^1} \cup_{r \in I(w_{m^1})} w^2_r) \cap v^2_1| = |w^2_{m^1} \cap v^2_1| = 1$. If there is a disk W_l of $\{W_j \mid$ the label w_j is "$-$"; $1 \le j \le n$ and $j \ne m^1\}$ in W^2 with $\partial W_l \cap v^2_1 \ne \emptyset$, then we band

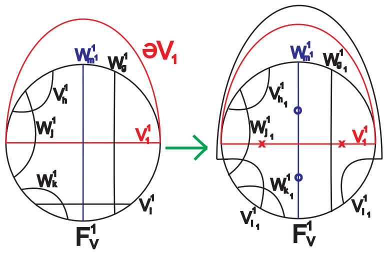

FIGURE 2. $v_i^1 \cap w_j^1$ and $v_{i_1}^1 \cap w_{j_1}^1$ $(2 \leq i \leq p; 1 \leq j \leq n$ and $j \neq m^1)$

sum W_l and k copies of W_{m^1} along v_1^2 in some order, where $|\partial W_l \cap v_1^2| = k$. After banding sum and isotopy, we obtain a properly embedded disk in W^2 and denote it by W_{l_1}. So, $W_{l_1} \cap W_{m^1} = \emptyset$ and $\partial W_{l_1} \cap (w_{m^1}^2 \cup v_1^2) = \emptyset$. If there is a disk W_l of $\{W_j \,|\, \text{the label } w_j \text{ is } \text{“}-\text{”};\ 1 \leq j \leq n \text{ and } j \neq m^1\}$ in W^2 with $\partial W_l \cap v_1^2 = \emptyset$, then we do nothing and denote it by W_{l_1}. After isotopy, we obtain a collection of mutually disjoint disks $\{W_{j_1} \,|\, \text{the label } w_j \text{ is } \text{“}-\text{”};\ 1 \leq j \leq n \text{ and } j \neq m^1\}$ in W^2. For each disk V_l of $\{V_i \,|\, \text{the label } v_i \text{ is } \text{“}-\text{”} \text{ and } 2 \leq i \leq p\}$ in V^2, we do nothing and denote it by V_{l_1}. So, we obtain a collection of mutually disjoint disks $\{V_{i_1} \,|\, \text{the label } v_i \text{ is } \text{“}-\text{”} \text{ and } 2 \leq i \leq p\}$ in V^2.

This procedure can be viewed as for each arc w_j^2 $(1 \leq j \leq n$ and $j \neq m^1)$, if $|w_j^2 \cap v_1^2| = 1$, then we band sum w_j^2 and a copy $\partial W_{m^1}^j$ of ∂W_{m^1} along v_1^2, where $W_{m^1}^j$ is a copy of W_{m^1} and one component of $\partial W_{m^1}^j \cap F_V^2$ which is a copy of $w_{m^1}^2$ lies between $w_{m^1}^2$ and w_j^2. After banding sum and isotopy, we obtain a new arc and denote it by $w_{j_1}^2$. Before banding sum, if there is an arc w_k^2 $(k \neq m^1, j)$ with $|w_k^2 \cap v_1^2| = 1$, such that w_k^2 lies between $w_{m^1}^2$ and w_j^2, then, w_k^2 lies between one component of $\partial W_{m^1}^j \cap F_V^2$ which is a copy of $w_{m^1}^2$ and w_j^2. Let $\partial W_{m^1}^k$ be a copy of ∂W_{m^1}, where $W_{m^1}^k$ is a copy of W_{m^1}, such that one component of $\partial W_{m^1}^k \cap F_V^2$ which is a copy of $w_{m^1}^2$ lies between one component of $\partial W_{m^1}^j \cap F_V^2$ which is a copy of $w_{m^1}^2$ and w_k^2. Then, we band sum w_k^2 and $\partial W_{m^1}^k$ along v_1^2. After banding sum and isotopy, we obtain a new arc and denote it by $w_{k_1}^2$, such that $w_{k_1}^2 \cap w_{j_1}^2 = \emptyset$, see Figure 3. If $w_j^2 \cap v_1^2 = \emptyset$, then we do nothing and denote it by $w_{j_1}^2$.

After banding sum and isotopy, we obtain a collection of mutually disjoint arcs $\{w_{j_1}^2 \,|\, 1 \leq j \leq n \text{ and } j \neq m^1\}$ on S^2. For each arc v_i^2 $(2 \leq i \leq p)$ before banding sum, we do nothing and denote it by $v_{i_1}^2$ after banding sum. Then, there is a collection of mutually disjoint arcs $\{v_{i_1}^2 \,|\, 2 \leq i \leq p\}$ on S^2. Hence, v_i^2 and w_j^2 $(2 \leq i \leq p; 1 \leq j \leq n$ and $j \neq m^1)$ represent the arcs before banding sum, $v_{i_1}^2$ and $w_{j_1}^2$ represent the arcs after banding sum. So, we may assume

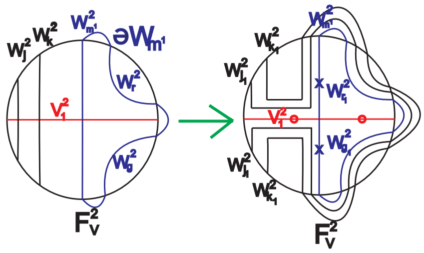

FIGURE 3. Band sum subdisk W_j and W_{m^1} in W^2 along v_1^2 $(1 \leq j \leq n$ and $j \neq m^1)$

that v_i^2 and $v_{i_1}^2$ $(2 \leq i \leq p)$ have the same label, and w_j^2 and $w_{j_1}^2$ $(1 \leq j \leq n$ and $j \neq m^1)$ have the same label.

For each arc $w_{j_1}^2$ $(1 \leq j \leq n$ and $j \neq m^1)$ on S^2, $w_{j_1}^2 \cap (v_1^2 \cup w_{m^1}^2) = \emptyset$. Since the label $w_{m^1}^2$ on F_V^2 is "$-$" and the label v_1^2 on F_V^2 is "$+$", we label $w_{m^1}^2$ on F_V^2 with "$\times$" and label v_1^2 on F_V^2 with "$\circ$", see Figure 3. The label "$\times$" on $w_{m^1}^2$ means that we delete the arc $w_{m^1}^2$ on F_V^2, the label "$\circ$" on v_1^2 means that we retain the arc v_1^2 on F_V^2. For each arc $v_{i_1}^2$ $(2 \leq i \leq p)$ and $w_{j_1}^2$ $(1 \leq j \leq n$ and $j \neq m^1)$, since $v_{i_1}^2 \cap v_1^2 = \emptyset$ and $w_{j_1}^2 \cap v_1^2 = \emptyset$, there is no influence on v_1^2 when we consider $v_{i_1}^2$ and $w_{j_1}^2$. Hence, the label "$\circ$" on v_1^2 means that we retain the arc v_1^2. So, we also denote it by $v_{1_1}^2$, but in the future banding sum process, we do not need to consider it.

By (8) in Proposition 2.3 and the argument as above, we have:

Lemma 2.8. *There are two sets of pairwise disjoint properly embedded disks* $\{V_{i_1} \mid$ *the label* v_i *is* "$-$" *and* $2 \leq i \leq p\}$ *in* V^2, *and* $\{W_{j_1} \mid$ *the label* w_j *is* "$-$"; $1 \leq j \leq n$ *and* $j \neq m^1\}$ *in* W^2, *satisfying the following conditions*:

(1) $V_{i_1} \cap F_V^2 = v_{i_1}^2 \cup_{r \in I(v_i)} v_{r_1}^2$, $W_{j_1} \cap F_V^2 = (w_{j_1}^2 \cap F_V^2) \cup_{r \in I(w_j)} (w_{r_1}^2 \cap F_V^2)$;

(2) $V_{i_1} \cap W_{j_1} = V_{i_1} \cap W_{j_1} \cap F_V^2$.

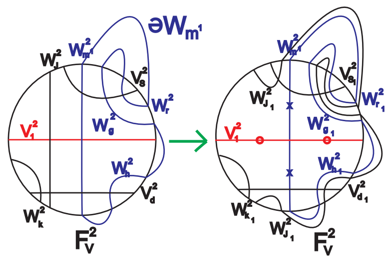

FIGURE 4. $v_i^2 \cap w_j^2$ and $v_{i_1}^2 \cap w_{j_1}^2$ $(2 \leq i \leq p; 1 \leq j \leq n$ and $j \neq m^1)$

Remark 2.9. By the argument as above, for each arc $v^2_{i_1}$ $(2 \leq i \leq p)$ and $w^2_{j_1}$ $(1 \leq j \leq n$ and $j \neq m^1)$ on S^2, if $w^2_j \cap v^2_1 = \emptyset$, then $w^2_{j_1}$ lies in F^2_V and $|w^2_{j_1} \cap v^2_{i_1}| \leq 1$; if $|w^2_j \cap v^2_1| = 1$ and $v^2_i \cap (\cup_{r \in I(w_{m^1})} w^2_r) = \emptyset$, then $|w^2_{j_1} \cap v^2_{i_1}| \leq 1$; if $|w^2_j \cap v^2_1| = 1$ and $v^2_i \cap (\cup_{r \in I(w_{m^1})} w^2_r) \neq \emptyset$, then $|w^2_{j_1} \cap v^2_{i_1}| \geq 1$. Particularly, if $|w^2_{j_1} \cap v^2_{i_1}| \geq 2$, then j is not the minimal label among all arcs of $\{w^2_{l_1} \,|\, 1 \leq l \leq n$ and $l \neq m^1\}$ on S^2 with $w^2_{j_1} \cap v^2_{i_1} \neq \emptyset$. Specifically, if $w^2_j \cap v^2_1 = \emptyset$, then $|w^2_{j_1} \cap v^2_{i_1}| = |w^2_j \cap v^2_i|$; if $|w^2_j \cap v^2_1| = 1$, $|v^2_i \cap (w^2_j \cup w^2_{m^1})| = 1$ and $|v^2_i \cap (\cup_{r \in I(w_{m^1})} w^2_r)| = k$, then $|w^2_{j_1} \cap v^2_{i_1}| = k+1$; if $|w^2_j \cap v^2_1| = 1$, $|v^2_i \cap (w^2_j \cup w^2_{m^1})| = 0$ or 2 and $|v^2_i \cap (\cup_{r \in I(w_{m^1})} w^2_r)| = k$, then $|w^2_{j_1} \cap v^2_{i_1}| = k$ after isotopy, see Figure 4.

Remark 2.10. By (I_1), (I_2), if $m^1 = \emptyset$ and the label v_1 on F_V is "$+$", then we label v^1_1 on F^1_V with "$\times$" and label v^2_1 on F^2_V with "$\circ$"; if $m^1 = \emptyset$ and the label v_1 on F_V is "$-$", then we label v^1_1 on F^1_V with "$\circ$" and label v^2_1 on F^2_V with "$\times$"; if $m^1 \neq \emptyset$, the label v_1 on F_V is "$+$", by Lemma 2.5, we may assume that the label w_{m^1} on F_V is "$-$", then after banding sum, we label v^1_1 on F^1_V with "$\times$", label $w^1_{m^1}$ on F^1_V with "$\circ$", label v^2_1 on F^2_V with "$\circ$", and label $w^2_{m^1}$ on F^2_V with "$\times$"; if $m^1 \neq \emptyset$, the label v_1 on F_V is "$-$", by Lemma 2.5, we may assume that the label w_{m^1} on F_V is "$+$", then after banding sum, we label v^1_1 on F^1_V with "$\circ$", label $w^1_{m^1}$ on F^1_V with "$\times$", label v^2_1 on F^2_V with "$\times$", and label $w^2_{m^1}$ on F^2_V with "$\circ$".

Let $\triangle v^k_{i_1} = \{j|\ v^k_{i_1} \cap w^k_{j_1} \neq \emptyset\}$ $(2 \leq i \leq p; 1 \leq j \leq n$ and $j \neq m^1; k = 1, 2)$. Then, we have:

Lemma 2.11. *For $2 \leq i \leq p$, if $m^1 = \emptyset$ and the label v_1 on F_V is "$+$", then $\triangle v^1_{i_1} = \triangle v^2_{i_1}$; if $m^1 \neq \emptyset$, the label v_1 on F_V is "$+$" and the label w_{m^1} on F_V is "$-$", then $\triangle v^1_{i_1} \subseteq \triangle v^2_{i_1}$, and if $j \in \triangle v^2_{i_1} - \triangle v^1_{i_1}$, then j is not the minimal label among all arcs of $\{w^2_{l_1} \,|\, 1 \leq l \leq n$ and $l \neq m^1\}$ on S^2 with $v^2_{i_1} \cap w^2_{j_1} \neq \emptyset$.*

Proof. Before banding sum, $v^1_i = v^2_i$ $(1 \leq i \leq p)$ and $w^1_j = w^2_j$ $(1 \leq j \leq n)$. If $m^1 = \emptyset$, then we do not need to band sum. So, $v^k_i = v^k_{i_1}$ and $w^k_j = w^k_{j_1}$ $(2 \leq i \leq p; 1 \leq j \leq n; k = 1, 2)$. Since $|v^1_i \cap w^1_j| = |v^2_i \cap w^2_j|$, $|v^1_{i_1} \cap w^1_{j_1}| = |v^2_{i_1} \cap w^2_{j_1}|$ $(2 \leq i \leq p; 1 \leq j \leq n)$. Hence, $\triangle v^1_{i_1} = \triangle v^2_{i_1}$ $(2 \leq i \leq p)$. So, we may assume that $m^1 \neq \emptyset$. There are two cases:

Case 1 in Lemma 2.11. *$v^1_i \cap w^1_{m^1} = \emptyset$ for some $2 \leq i \leq p$.*

Since $v^1_i = v^2_i$ and $w^1_{m^1} = w^2_{m^1}$, $v^2_i \cap w^2_{m^1} = \emptyset$. By (I_2), $v^2_i = v^2_{i_1}$. Since $v^1_i \cap w^1_{m^1} = \emptyset$, by (I_1), $v^1_i = v^1_{i_1}$. By Remark 2.7, for each $1 \leq l \leq n$ and $l \neq m^1$, $|v^1_{i_1} \cap w^1_{l_1}| \leq 1$. Hence, for each $j \in \triangle v^1_{i_1}$, $|w^1_{j_1} \cap v^1_{i_1}| = 1$. By (I_1), since $w^1_j = w^1_{j_1}$ and $v^1_i = v^1_{i_1}$, $|v^1_i \cap w^1_j| = 1$. So, $|v^2_i \cap w^2_j| = 1$. If $w^2_j \cap v^2_1 = \emptyset$, then by (I_2), $w^2_j = w^2_{j_1}$. Since $v^2_i = v^2_{i_1}$, $|v^2_{i_1} \cap w^2_{j_1}| = 1$. Hence, $j \in \triangle v^2_{i_1}$. So, $\triangle v^1_{i_1} \subseteq \triangle v^2_{i_1}$. If $|w^2_j \cap v^2_1| = 1$, since $|v^2_i \cap (w^2_{m^1} \cup w^2_j)| = 1$, by Remark 2.9, $|w^2_{j_1} \cap v^2_{i_1}| = k+1$, where $k = |v^2_i \cap (\cup_{r \in I(w_{m^1})} w^2_r)|$. Hence, $j \in \triangle v^2_{i_1}$. So, $\triangle v^1_{i_1} \subseteq \triangle v^2_{i_1}$.

For each $j \notin \triangle v^1_{i_1}$, $v^1_{i_1} \cap w^1_{j_1} = \emptyset$. Since $v^1_{i_1} = v^1_i$ and $w^1_{j_1} = w^1_j$, $v^1_i \cap w^1_j = \emptyset$. So, $v^2_i \cap w^2_j = \emptyset$. If $w^2_j \cap v^2_1 = \emptyset$, then by (I_2), $w^2_j = w^2_{j_1}$. Since $v^2_i = v^2_{i_1}$, $w^2_{j_1} \cap v^2_{i_1} = \emptyset$. Hence, $j \notin \triangle v^2_{i_1}$. If $|w^2_j \cap v^2_1| = 1$, since $v^2_i \cap (w^2_j \cup w^2_{m^1}) = \emptyset$, by Remark 2.9, $|v^2_{i_1} \cap w^2_{j_1}| = k$, where $k = |v^2_i \cap (\cup_{r \in I(w_{m^1})} w^2_r)|$. If $k = 0$, then $j \notin \triangle v^2_{i_1}$. If $k > 0$, then $j \in \triangle v^2_{i_1} - \triangle v^1_{i_1}$. Since $|w^2_j \cap v^2_1| = 1$, by the minimality of m^1 and (2) in Proposition 2.3, $j > m^1 > r$, where $r \in I(w_{m^1})$. Since $k > 0$, there is $r \in I(w_{m^1})$ with $|v^2_i \cap w^2_r| = 1$. By the minimality of m^1, $w^2_r \cap v^2_1 = \emptyset$. By (I_2), $w^2_r = w^2_{r_1}$. Since $v^2_i = v^2_{i_1}$, $|v^2_{i_1} \cap w^2_{r_1}| = 1$. Since $j > r$, j is not the minimal label among all arcs of $\{w^2_{l_1} \mid 1 \le l \le n \text{ and } l \ne m^1\}$ on S^2 with $v^2_{i_1} \cap w^2_{j_1} \ne \emptyset$.

Case 2 in Lemma 2.11. $|v^1_i \cap w^1_{m^1}| = 1$ *for some* $2 \le i \le p$.

Since $v^1_i = v^2_i$ and $w^1_{m^1} = w^2_{m^1}$, $|v^2_i \cap w^2_{m^1}| = 1$. Since $|v^1_i \cap w^1_{m^1}| = 1$, $v^1_i \ne v^1_{i_1}$. By Remark 2.7, for each $1 \le l \le n$ and $l \ne m^1$, $|v^1_{i_1} \cap w^1_{l_1}| \le 1$. Hence, for each $j \in \triangle v^1_{i_1}$, $|w^1_{j_1} \cap v^1_{i_1}| = 1$. By Remark 2.7, $|w^1_j \cap (v^1_1 \cup v^1_i)| = 1$. So, $|w^2_j \cap (v^2_1 \cup v^2_i)| = 1$. If $w^2_j \cap v^2_1 = \emptyset$ and $|w^2_j \cap v^2_i| = 1$, then by (I_2), $w^2_j = w^2_{j_1}$. Since $v^2_i = v^2_{i_1}$, $|w^2_{j_1} \cap v^2_{i_1}| = 1$. Then, $j \in \triangle v^2_{i_1}$. So, $\triangle v^1_{i_1} \subseteq \triangle v^2_{i_1}$. If $|w^2_j \cap v^2_1| = 1$ and $w^2_j \cap v^2_i = \emptyset$, then $|v^2_i \cap (w^2_{m^1} \cup w^2_j)| = 1$. By Remark 2.9, $|w^2_{j_1} \cap v^2_{i_1}| = k + 1$, where $k = |v^2_i \cap (\cup_{r \in I(w_{m^1})} w^2_r)|$. Hence, $j \in \triangle v^2_{i_1}$. So, $\triangle v^1_{i_1} \subseteq \triangle v^2_{i_1}$.

For each $j \notin \triangle v^1_{i_1}$, $w^1_{j_1} \cap v^1_{i_1} = \emptyset$. By Remark 2.7, $|w^1_j \cap (v^1_1 \cup v^1_i) = 0$ or 2. So, $|w^2_j \cap (v^2_1 \cup v^2_i)| = 0$ or 2. If $w^2_j \cap (v^2_1 \cup v^2_i) = \emptyset$, then by (I_2), $w^2_j = w^2_{j_1}$. Since $v^2_i = v^2_{i_1}$, $w^2_{j_1} \cap v^2_{i_1} = \emptyset$. Hence, $j \notin \triangle v^2_{i_1}$. If $|w^2_j \cap (v^2_1 \cup v^2_i)| = 2$, then $|v^2_i \cap (w^2_{m^1} \cup w^2_j)| = 2$. By Remark 2.9, $|w^2_{j_1} \cap v^2_{i_1}| = k$, where $k = |v^2_i \cap (\cup_{r \in I(w_{m^1})}) w^2_r|$. If $k = 0$, then $j \notin \triangle v^2_{i_1}$. If $k > 0$, then $j \in \triangle v^2_{i_1} - \triangle v^1_{i_1}$. By the same argument as in Case 1 in Lemma 2.11, j is not minimal label among all arcs of $\{w^2_{l_1} \mid 1 \le l \le n \text{ and } l \ne m^1\}$ on S^2 with $v^2_{i_1} \cap w^2_{j_1} \ne \emptyset$. □

Remark 2.12. By the same proof as in Lemma 2.11, for $2 \le i \le p$, if $m^1 = \emptyset$ and the label v_1 on F_V is "$-$", then $\triangle v^1_{i_1} = \triangle v^2_{i_1}$; if $m^1 \ne \emptyset$, the label v_1 on F_V is "$-$" and the label w_{m^1} on F_V is "+", then $\triangle v^2_{i_1} \subseteq \triangle v^1_{i_1}$, and if $j \in \triangle v^1_{i_1} - \triangle v^2_{i_1}$, then j is not the minimal label among all arcs of $\{w^1_{l_1} \mid 1 \le l \le n \text{ and } l \ne m^1\}$ on S^1 with $v^1_{i_1} \cap w^1_{j_1} \ne \emptyset$.

Second, we consider the arc $v^1_{2_1}$ on S^1.

Lemma 2.13. *If m^2 is the minimal label among all arcs of $\{w^1_{j_1} \mid 1 \le j \le n$ and $j \ne m^1\}$ on S^1 with $|w^1_{m^2_1} \cap v^1_{2_1}| = 1$, then m^2 is the minimal label among all arcs of $\{w^2_{j_1} \mid 1 \le j \le n$ and $j \ne m^1\}$ on S^2 with $|w^2_{m^2_1} \cap v^2_{2_1}| = 1$.*

Proof. By Lemma 2.11, if $m^1 = \emptyset$, then $\triangle v^1_{2_1} = \triangle v^2_{2_1}$; if $m^1 \ne \emptyset$, then $\triangle v^1_{2_1} \subseteq \triangle v^2_{2_1}$, and if $j \in \triangle v^2_{2_1} - \triangle v^1_{2_1}$, then j is not the minimal label among all arcs of $\{w^2_{l_1} \mid 1 \le l \le n \text{ and } l \ne m^1\}$ on S^2 with $v^2_{2_1} \cap w^2_{j_1} \ne \emptyset$. So, if m^2 is minimal in $\triangle v^1_{2_1}$, then m^2 is minimal in $\triangle v^2_{2_1}$. By Remark 2.7, $|w^1_{m^2_1} \cap v^1_{2_1}| = 1$. By Remark 2.9, $|w^2_{m^2_1} \cap v^2_{2_1}| = 1$. □

By the same proof as above (see Remark 2.10), if $m^2 = \emptyset$ and the label v_2 on F_V is "+", then we label $v^1_{2_1}$ on S^1 with "×" and label $v^2_{2_1}$ on S^2 with "∘"; if $m^2 = \emptyset$ and the label v_2 on F_V is "−", then we label $v^1_{2_1}$ on S^1 with "∘" and label $v^2_{2_1}$ on S^2 with "×". For convenience, for each arc $v^k_{i_1}$ $(3 \le i \le p)$ and $w^k_{j_1}$ $(1 \le j \le n$ and $j \ne m^1)$ on S^k $(k = 1, 2)$, we denote them by $v^k_{i_2}$ and $w^k_{j_2}$. We may assume that $v^k_{i_2}$ and $v^k_{i_1}$ have the same label, and $w^k_{j_2}$ and $w^k_{j_1}$ have the same label. For each disk V_{i_1} $(3 \le i \le p$ or $i = x)$ and W_{j_1} $(1 \le j \le n$ and $j \ne m^1$, or $j = x)$, we denote them by V_{i_2} and W_{j_2}. If $v^k_{2_1}$ $(k = 1, 2)$ is retained, we also denote it by $v^k_{2_2}$. But in the future banding sum process, we do not consider $v^k_{2_2}$.

If $m^2 \ne \emptyset$, the label v_2 on F_V is "+", by Lemma 2.5, we may assume that the label w_{m^2} on F_V is "−", then V_{2_1} is a properly embedded disk in V^1 and $W_{m^2_1}$ is a properly embedded disk in W^2. For each disk V_{i_1} (the label v_i on F_V is "+"; $3 \le i \le p$ or $i = x$) in V^1, if $V_{i_1} \cap w^1_{m^2_1} = \emptyset$, then we do nothing and denote it by V_{i_2}; if $V_{i_1} \cap w^1_{m^2_1} \ne \emptyset$, then by the same argument as in (I_1), we band sum V_{i_1} and V_{2_1} along $w^1_{m^2_1}$, after banding sum, we denote it by V_{i_2}, such that $V_{i_2} \cap V_{2_1} = \emptyset$ and $\partial V_{i_2} \cap (v^1_{2_1} \cup w^1_{m^2_1}) = \emptyset$. For each disk W_{j_1} (the label w_j on F_V is "+"; $1 \le j \le n$ and $j \ne m^1, m^2$, or $j = x$) in W^1, we do nothing and denote it by W_{j_2}. For each disk W_{j_1} (the label w_j is "−"; $1 \le j \le n$ and $j \ne m^1, m^2$) in W^2, if $W_{j_1} \cap v^2_{2_1} = \emptyset$, then we do nothing and denote it by W_{j_2}; if $W_{j_1} \cap v^2_{2_1} \ne \emptyset$, then by the same argument as in (I_2), we band sum W_{j_1} and $W_{m^2_1}$ along $v^2_{2_1}$, after banding sum, we denote it by W_{j_2}, such that $W_{j_2} \cap W_{m^2_1} = \emptyset$ and $\partial W_{j_2} \cap (v^2_{2_1} \cup w^2_{m^2_1}) = \emptyset$. For each disk V_{i_1} (the label v_i is "−"; $3 \le i \le p$) in V^2, we do nothing and denote it by V_{i_2}.

Correspondingly, for each arc $v^k_{i_1}$ and $w^k_{j_1}$ $(3 \le i \le p; 1 \le j \le n$ and $j \ne m^1, m^2; k = 1, 2)$ on S^k before banding sum, we denote them by $v^k_{i_2}$ and $w^k_{j_2}$ after banding sum. Now we label $v^1_{2_1}$ on S^1 with "×", label $w^1_{m^2_1}$ on S^1 with "∘", label $v^2_{2_1}$ on S^2 with "∘" and label $w^2_{m^2_1}$ on S^2 with "×". Since both $w^1_{m^2_1}$ and $v^2_{2_1}$ are retained, for convenience, we denote them by $w^1_{m^2_2}$ and $v^2_{2_2}$. But in the future banding sum process, we do not need to consider them. So, we obtain four sets of pairwise disjoint properly embedded disks $\{V_{i_2} \mid$ the label v_i is "+" and $3 \le i \le p\} \cup \{V_{x_2}\}$ in V^1, $\{W_{j_2} \mid$ the label w_j is "+", $1 \le j \le n$ and $j \ne m^1, m^2\} \cup \{W_{x_2}\}$ in W^1, $\{V_{i_2} \mid$ the label v_i is "−" and $3 \le i \le p\}$ in V^2, and $\{W_{j_2} \mid$ the label w_j is "−", $1 \le j \le n$ and $j \ne m^1, m^2\}$ in W^2, satisfying the same properties as in Lemmas 2.6 and 2.8.

If $m^2 \ne \emptyset$, the label v_2 on F_V is "−", by Lemma 2.5, we may assume that the label w_{m^2} on F_V is "+", then V_{2_1} is a properly embedded disk in V^2 and

$W_{m_1^2}$ is a properly embedded disk in W^1. For each disk W_{j_1} (the label w_j is "$+$"; $1 \leq j \leq n$ and $j \neq m^1, m^2$, or $j = x$) in W^1, if $W_{j_1} \cap v_{2_1}^1 = \emptyset$, then we do nothing and denote it by W_{j_2}; if $W_{j_1} \cap v_{2_1}^1 \neq \emptyset$, then by the same argument as in (I_2), we band sum W_{j_1} and $W_{m_1^2}$ along $v_{2_1}^1$, after banding sum, we denote it by W_{j_2}, such that $W_{j_2} \cap W_{m_1^2} = \emptyset$ and $\partial W_{j_2} \cap (v_{2_1}^1 \cup w_{m_1^2}^1) = \emptyset$. For each disk V_{i_1} (the label v_i is "$+$"; $3 \leq i \leq p$ or $i = x$) in V^1, we do nothing and denote it by V_{i_2}. For each disk V_{i_1} (the label v_i is "$-$"; $3 \leq i \leq p$) in V^2, if $V_{i_1} \cap w_{m_1^2}^2 = \emptyset$, then we do nothing and denote it by V_{i_2}; if $V_{i_1} \cap w_{m_1^2}^2 \neq \emptyset$, then by the same argument as in (I_1), we band sum V_{i_1} and V_{2_1} along $w_{m_1^2}^2$, after banding sum, we denote it by V_{i_2}, such that $V_{i_2} \cap V_{2_1} = \emptyset$ and $\partial V_{i_2} \cap (v_{2_1}^2 \cup w_{m_1^2}^2) = \emptyset$. For each disk W_{j_1} (the label w_j is "$-$"; $1 \leq j \leq n$ and $j \neq m^1, m^2$) in W^2, we do nothing and denote it by W_{j_2}.

Correspondingly, for each arc $v_{i_1}^k$ and $w_{j_1}^k$ ($3 \leq i \leq p; 1 \leq j \leq n$ and $j \neq m^1, m^2; k = 1, 2$) on S^k before banding sum, we denote them by $v_{i_2}^k$ and $w_{j_2}^k$ after banding sum. Now we label $v_{2_1}^1$ on S^1 with "$\circ$", label $w_{m_1^2}^1$ on S^1 with "$\times$", label $v_{2_1}^2$ on S^2 with "$\times$" and label $w_{m_1^2}^2$ on S^2 with "$\circ$". Since both $v_{2_1}^1$ and $w_{m_1^2}^2$ are retained, we denote them by $v_{2_2}^1$ and $w_{m_2^2}^2$. But in the future banding sum process, we do not need to consider them. So, we obtain four sets of pairwise disjoint properly embedded disks $\{V_{i_2} \mid \text{the label } v_i \text{ is "}+\text{" and } 3 \leq i \leq p\} \cup \{V_{x_2}\}$ in V^1, $\{W_{j_2} \mid \text{the label } w_j \text{ is "}+\text{"}, 1 \leq j \leq n \text{ and } j \neq m^1, m^2\} \cup \{W_{x_2}\}$ in W^1, $\{V_{i_2} \mid \text{the label } v_i \text{ is "}-\text{" and } 3 \leq i \leq p\}$ in V^2, and $\{W_{j_2} \mid \text{the label } w_j \text{ is "}-\text{"}, 1 \leq j \leq n \text{ and } j \neq m^1, m^2\}$ in W^2, satisfying the same properties as in Lemmas 2.6 and 2.8.

We continue this procedure as above, there are p steps. For each step l ($1 \leq l \leq p$), by the same argument as above, before banding sum, there are four sets of pairwise disjoint arcs $\{v_{i_{l-1}}^k \mid l \leq i \leq p\} \cup \{v_{i_{l-1}}^k \mid 1 \leq i \leq l-1 \text{ and } v_{i_{l-1}}^k \text{ is labelled with "}\circ\text{"}\}$ and $\{w_{j_{l-1}}^k \mid 1 \leq j \leq n \text{ and } j \neq m^1, \dots, m^{l-1}\} \cup \{w_{j_{l-1}}^k \mid j = m^1, \dots, m^{l-1} \text{ and } w_{j_{l-1}}^k \text{ is labelled with "}\circ\text{"}\}$ on S^k ($k = 1, 2$). Let $\triangle v_{i_{l-1}}^k = \{j \mid v_{i_{l-1}}^k \cap w_{j_{l-1}}^k \neq \emptyset\}$ ($l \leq i \leq p; 1 \leq j \leq n$ and $j \neq m^1, \dots, m^{l-1}; k = 1, 2$). Then, we have:

Lemma 2.14. *For $l \leq i \leq p$, if $j \in \triangle v_{i_{l-1}}^1 - \triangle v_{i_{l-1}}^2$, then j is not the minimal label among all arcs of $\{w_{h_{l-1}}^1 \mid 1 \leq h \leq n \text{ and } h \neq m^1, \dots, m^{l-1}\}$ on S^1 with $v_{i_{l-1}}^1 \cap w_{j_{l-1}}^1 \neq \emptyset$; if $j \in \triangle v_{i_{l-1}}^2 - \triangle v_{i_{l-1}}^1$, then j is not the minimal label among all arcs of $\{w_{h_{l-1}}^2 \mid 1 \leq h \leq n \text{ and } h \neq m^1, \dots, m^{l-1}\}$ on S^2 with $v_{i_{l-1}}^2 \cap w_{j_{l-1}}^2 \neq \emptyset$.*

Proof. Note that $\triangle v_{i_{l-1}}^1 \nsubseteq \triangle v_{i_{l-1}}^2$ and $\triangle v_{i_{l-1}}^2 \nsubseteq \triangle v_{i_{l-1}}^1$ for $l \geq 3$. Recall the step 2 in Lemma 13, we do not need to consider $\triangle v_{i_1}^1 \subseteq \triangle v_{i_1}^2$ and $\triangle v_{i_1}^2 \subseteq \triangle v_{i_1}^1$, if $j \in \triangle v_{i_1}^2 - \triangle v_{i_1}^1$, i.e., $j \in \triangle v_{i_1}^2$ and $j \notin \triangle v_{i_1}^1$, then $w_j^2 \cap v_1^2 \neq \emptyset$ and for some i, $v_i^2 \cap w_{m^1}^2 = \emptyset$ and $v_i^2 \cap (\cup_{r \in I(w_{m^1})} w_r^2) \neq \emptyset$, see Figure 5. We may assume that $v_i^2 \cap w_r^2 \neq \emptyset$ for some $r \in I(w_{m^1})$. So, $j > m^1 > r$. After banding sum, $w_{j_1}^2 \cap v_{i_1}^2 \neq \emptyset$ on S^2, see Figure 5, and $w_{j_1}^1 \cap v_{i_1}^1 = \emptyset$ on S^1, see Figure 6.

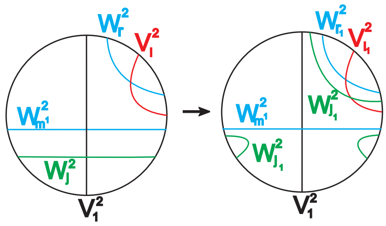

FIGURE 5. $w_{j_1}^2 \cap v_{i_1}^2 \neq \emptyset$ on S^2

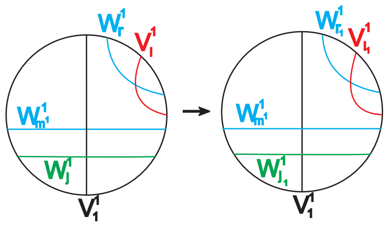

FIGURE 6. $w_{j_1}^1 \cap v_{i_1}^1 = \emptyset$ on S^1

Since $v_i^2 \cap w_r^2 \neq \emptyset$ for some $r \in I(w_{m^1})$ and $r < j$, then after banding sum, j is not the minimal label among all arcs of $\{w_{h_1}^2 \,|\, 1 \leq h \leq n \text{ and } h \neq m^1\}$ on S^2 with $v_{i_1}^2 \cap w_{j_1}^2 \neq \emptyset$ and $j \notin \triangle v_{i_1}^1$. If $j \in (\triangle v_{i_1}^1 \cap \triangle v_{i_1}^2)$ and $|w_{j_1}^2 \cap v_{i_1}^2| > 1$, then by the same argument, j is not the minimal label among all arcs of $\{w_{h_1}^2 \,|\, 1 \leq h \leq n$ and $h \neq m^1\}$ on S^2 with $v_{i_1}^2 \cap w_{j_1}^2 \neq \emptyset$, see Figure 7, also, j is not the minimal label among all arcs of $\{w_{h_1}^1 \,|\, 1 \leq h \leq n \text{ and } h \neq m^1\}$ on S^1 with $v_{i_1}^1 \cap w_{j_1}^1 \neq \emptyset$, see Figure 8. If $j \in (\triangle v_{i_1}^1 \cap \triangle v_{i_1}^2)$ and $|w_{j_1}^2 \cap v_{i_1}^2| \leq 1$, then by Remarks 9 and 11, $|w_{j_1}^1 \cap v_{i_1}^1| = |w_{j_1}^2 \cap v_{i_1}^2| = 1$. So, $m^2 \in (\triangle v_{i_1}^1 \cap \triangle v_{i_1}^2)$ and $|w_{m_1^2}^1 \cap v_{i_1}^1| = |w_{m_1^2}^2 \cap v_{i_1}^2| = 1$. If $j \in \triangle v_{i_1}^1 - \triangle v_{i_1}^2$, then by the same arguments, j is not the minimal label among all arcs of $\{w_{h_1}^1 \,|\, 1 \leq h \leq n$ and $h \neq m^1\}$ on S^1 with $v_{i_1}^1 \cap w_{j_1}^1 \neq \emptyset$ and $j \notin \triangle v_{i_1}^2$, and $m^2 \in (\triangle v_{i_1}^1 \cap \triangle v_{i_1}^2)$ and $|w_{m_1^2}^1 \cap v_{i_1}^1| = |w_{m_1^2}^2 \cap v_{i_1}^2| = 1$.

By the same arguments, for $l \leq i \leq p$, if $j \in \triangle v_{i_{l-1}}^1 - \triangle v_{i_{l-1}}^2$, then j is not the minimal label among all arcs of $\{w_{h_{l-1}}^1 \,|\, 1 \leq h \leq n \text{ and } h \neq m^1, \ldots, m^{l-1}\}$ on S^1 with $v_{i_{l-1}}^1 \cap w_{j_{l-1}}^1 \neq \emptyset$ and $j \notin \triangle v_{i_{l-1}}^2$; if $j \in (\triangle v_{i_{l-1}}^1 \cap \triangle v_{i_{l-1}}^2)$ and $|w_{j_{l-1}}^1 \cap v_{i_{l-1}}^1| > 1$, then by the same argument, j is not the minimal label among all arcs of $\{w_{h_{l-1}}^1 \,|\, 1 \leq h \leq n \text{ and } h \neq m^1, \ldots, m^{l-1}\}$ on S^1 with $v_{i_{l-1}}^1 \cap w_{j_{l-1}}^1 \neq \emptyset$, also, j is not the minimal label among all arcs of $\{w_{h_{l-1}}^2 \,|\, 1 \leq h \leq n$ and

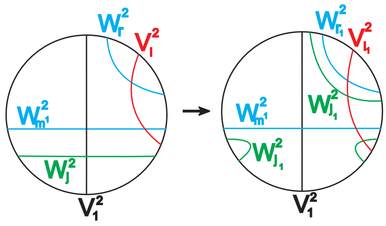

FIGURE 7. $|w^2_{j_1} \cap v^2_{i_1}| > 1$ on S^2

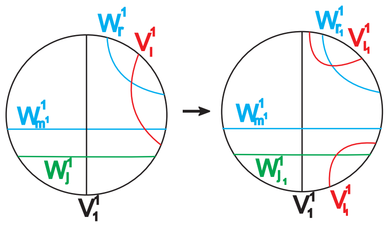

FIGURE 8. $w^1_{j_1}$ and $v^1_{i_1}$ on S^1

$h \neq m^1, \ldots, m^{l-1}\}$ on S^2 with $v^2_{i_{l-1}} \cap w^2_{j_{l-1}} \neq \emptyset$; if $j \in (\triangle v^1_{i_{l-1}} \cap \triangle v^2_{i_{l-1}})$ and $|w^2_{j_{l-1}} \cap v^2_{i_{l-1}}| > 1$, then by the same argument, j is not the minimal label among all arcs of $\{w^2_{h_{l-1}} \,|\, 1 \leq h \leq n$ and $h \neq m^1, \ldots, m^{l-1}\}$ on S^2 with $v^2_{i_{l-1}} \cap w^2_{j_{l-1}} \neq \emptyset$, also, j is not the minimal label among all arcs of $\{w^1_{h_{l-1}} \,|\, 1 \leq h \leq n$ and $h \neq m^1, \ldots, m^{l-1}\}$ on S^1 with $v^1_{i_{l-1}} \cap w^1_{j_{l-1}} \neq \emptyset$; if $j \in (\triangle v^1_{i_{l-1}} \cap \triangle v^2_{i_{l-1}})$, $|w^1_{j_{l-1}} \cap v^1_{i_{l-1}}| \leq 1$ and $|w^2_{j_{l-1}} \cap v^2_{i_{l-1}}| \leq 1$, then $|w^1_{j_{l-1}} \cap v^1_{i_{l-1}}| = |w^2_{j_{l-1}} \cap v^2_{i_{l-1}}| = 1$. So, $m^l \in (\triangle v^1_{i_{l-1}} \cap \triangle v^2_{i_{l-1}})$ and $|w^1_{m^l_{l-1}} \cap v^1_{i_{l-1}}| = |w^2_{m^l_{l-1}} \cap v^2_{i_{l-1}}| = 1$. If $j \in \triangle v^2_{i_{l-1}} - \triangle v^1_{i_{l-1}}$, then by the same arguments, j is not the minimal label among all arcs of $\{w^1_{h_{l-1}} \,|\, 1 \leq h \leq n$ and $h \neq m^1, \ldots, m^{l-1}\}$ on S^2 with $v^1_{i_{l-1}} \cap w^1_{j_{l-1}} \neq \emptyset$ and $j \notin \triangle v^1_{i_{l-1}}$, and $m^l \in (\triangle v^1_{i_{l-1}} \cap \triangle v^2_{i_{l-1}})$ and $|w^1_{m^l_{l-1}} \cap v^1_{i_{l-1}}| = |w^2_{m^l_{l-1}} \cap v^2_{i_{l-1}}| = 1$. □

For step l, we consider the arc $v^1_{l_{l-1}}$ on S^1. By the proof of Lemma 2.14, we have:

Lemma 2.15. *If m^l is the minimal label among all arcs of $\{w^1_{j_{l-1}} \,|\, 1 \leq j \leq n$ and $j \neq m^1, \ldots, m^{l-1}\}$ on S^1 with $|w^1_{m^l_{l-1}} \cap v^1_{l_{l-1}}| = 1$, then m^l is the minimal label among all arcs of $\{w^2_{j_{l-1}} \,|\, 1 \leq j \leq n$ and $j \neq m^1, \ldots, m^{l-1}\}$ on S^2 with $|w^2_{m^l_{l-1}} \cap v^2_{l_{l-1}}| = 1$.*

If $m^l = \emptyset$ and the label v_l on F_V is "$+$", then we label $v^1_{l_{l-1}}$ on S^1 with "$\times$" and label $v^2_{l_{l-1}}$ on S^2 with "$\circ$"; if $m^l = \emptyset$ and the label v_l on F_V is "$-$", then we label $v^1_{l_{l-1}}$ on S^1 with "$\circ$" and label $v^2_{l_{l-1}}$ on S^2 with "$\times$". For convenience, for each arc $v^k_{i_{l-1}}$ ($l \le i \le p$) and $w^k_{j_{l-1}}$ ($1 \le j \le n$ and $j \ne m^1, \dots, m^{l-1}$) on S^k ($k = 1, 2$), we denote them by $v^k_{i_l}$ and $w^k_{j_l}$. We may assume that $v^k_{i_l}$ and $v^k_{i_{l-1}}$ have the same label, and $w^k_{j_l}$ and $w^k_{j_{l-1}}$ have the same label. For each disk $V_{i_{l-1}}$ ($l \le i \le p$ or $i = x$) and $W_{j_{l-1}}$ ($1 \le j \le n$ and $j \ne m^1, \dots, m^{l-1}$, or $j = x$), we denote them by V_{i_l} and W_{j_l}. If $v^k_{l_{l-1}}$ ($k = 1, 2$) is retained, we also denote it by $v^k_{l_l}$. But in the future banding sum process, we do not consider $v^k_{l_l}$.

If $m^l \ne \emptyset$, the label v_l on F_V is "$+$", then by Lemma 2.5, we may assume that the label w_{m^l} on F_V is "$-$". By Lemma 2.15 and the same argument as above (see (I_1)), after banding sum, we label $v^1_{l_{l-1}}$ on S^1 with "$\times$", label $w^1_{m^l_{l-1}}$ on S^1 with "$\circ$", label $v^2_{l_{l-1}}$ on S^2 with "$\circ$", and label $w^2_{m^l_{l-1}}$ on S^2 with "$\times$"; if $m^1 \ne \emptyset$, the label v_l on F_V is "$-$", then by Lemma 2.5, we may assume that the label w_{m^l} on F_V is "$+$". By Lemma 2.15 and the same argument as above (see (I_2)), after banding sum, we label $v^1_{l_{l-1}}$ on S^1 with "$\circ$", label $w^1_{m^l_{l-1}}$ on S^1 with "$\times$", label $v^2_{l_{l-1}}$ on S^2 with "$\times$", and label $w^2_{m^l_{l-1}}$ on S^2 with "$\circ$". If the arc $v^k_{l_{l-1}}$ (resp. $w^k_{m^l_{l-1}}$) on S^k ($k = 1, 2$) is labelled with "$\circ$", then we denote it by $v^k_{l_l}$ (resp. $w^k_{m^l_l}$), but in the future banding sum process, we do not consider it. By the same argument as in Lemmas 2.6 and 2.8, after banding sum, we have:

Lemma 2.16. *There are four sets of pairwise disjoint arcs* $\{v^k_{i_l} \mid l+1 \le i \le p\} \cup \{v^k_{i_l} \mid 1 \le i \le l$ *and* $v^k_{i_l}$ *is labelled with* "$\circ$"$\}$ *and* $\{w^k_{j_l} \mid 1 \le j \le n$ *and* $j \ne m^1, \dots, m^l\} \cup \{w^k_{j_l} \mid j = m^1, \dots, m^l$ *and* $w^k_{j_l}$ *is labelled with* "$\circ$"$\}$ *on* S^k ($k = 1, 2$)*, and four sets of pairwise disjoint disks* $\{V_{i_l} \mid$ *the label* v_i *is* "$+$" *and* $l+1 \le i \le p\} \cup \{V_{x_l}\}$ *in* V^1, $\{W_{j_l} \mid$ *the label* w_j *is* "$+$", $1 \le j \le n$ *and* $j \ne m^1, \dots, m^l\} \cup \{W_{x_l}\}$ *in* W^1, $\{V_{i_l} \mid$ *the label* v_i *is* "$-$" *and* $l+1 \le i \le p\}$ *in* V^2, *and* $\{W_{j_l} \mid$ *the label* w_j *is* "$-$", $1 \le j \le n$ *and* $j \ne m^1, \dots, m^l\}$ *in* W^2, *satisfying the following conditions*:

(1) *If* V_{i_l} *lies in* V^1 *and* W_{j_l} *lies in* W^1, *then* $V_{i_l} \cap F^1_V = (v^1_{i_l} \cap F^1_V) \cup_{r \in I(v_i)} (v^1_{r_l} \cap F^1_V)$, $W_{j_l} \cap F^1_V = (w^1_{j_l} \cap F^1_V) \cup_{r \in I(w_j)} (w^1_{r_l} \cap F^1_V)$, $V_{x_l} \cap F^1_V = \cup_{r \in I(v)} (v^1_{r_l} \cap F^1_V)$, $W_{x_l} \cap F^1_V = \cup_{r \in I(w)} (w^1_{r_l} \cap F^1_V)$, $V_{i_l} \cap W_{j_l} = V_{i_l} \cap W_{j_l} \cap F^1_V$, $V_{i_l} \cap W_{x_l} = V_{i_l} \cap W_{x_l} \cap F^1_V$, $V_{x_l} \cap W_{j_l} = V_{x_l} \cap W_{j_l} \cap F^1_V$, $V_{x_l} \cap W_{x_l} = \{x\} \cup (V_{x_l} \cap W_{x_l} \cap F^1_V)$;

(2) *If* V_{i_l} *lies in* V^2 *and* W_{j_l} *lies in* W^2, *then* $V_{i_l} \cap F^2_V = (v^2_{i_l} \cap F^2_V) \cup_{r \in I(v_i)} (v^2_{r_l} \cap F^2_V)$, $W_{j_l} \cap F^2_V = (w^2_{j_l} \cap F^2_V) \cup_{r \in I(w_j)} (w^2_{r_l} \cap F^2_V)$, $V_{i_l} \cap W_{j_l} = V_{i_l} \cap W_{j_l} \cap F^2_V$.

For step p, as in Lemma 2.16, after banding sum, we obtain three sets of pairwise disjoint arcs $\{v^k_{i_p} \mid 1 \le i \le p$ and $v^k_{i_p}$ is labelled with "$\circ$"$\}$ and $\{w^k_{j_p} \mid 1 \le j \le n$ and $j \ne m^1, \dots, m^p\} \cup \{w^k_{j_p} \mid j = m^1, \dots, m^p$ and $w^k_{j_p}$ is labelled with "$\circ$"} on S^k $(k = 1, 2)$, and three sets of pairwise disjoint disks $\{V_{x_p}\}$ in V^1, $\{W_{j_p} \mid$ the label w_j is "+", $1 \le j \le n$ and $j \ne m^1, \dots, m^p\} \cup \{W_{x_p}\}$ in W^1, and $\{W_{j_p} \mid$ the label w_j is "$-$", $1 \le j \le n$ and $j \ne m^1, \dots, m^p\}$ in W^2, satisfying the following condition:

$(*)$ If W_{j_p} lies in W^1, then $W_{j_p} \cap F_V^1 = (w_{j_p}^1 \cap F_V^1) \cup_{r \in I(w_j)} (w_{r_p}^1 \cap F_V^1)$, $V_{x_p} \cap F_V^1 = \cup_{r \in I(v)} (v_{r_p}^1 \cap F_V^1)$, $W_{x_p} \cap F_V^1 = \cup_{r \in I(w)} (w_{r_p}^1 \cap F_V^1)$, $V_{x_p} \cap W_{j_p} = V_{x_p} \cap W_{j_p} \cap F_V^1$, $V_{x_p} \cap W_{x_p} = \{x\} \cup (V_{x_p} \cap W_{x_p} \cap F_V^1)$.

For each arc $w_{j_p}^k$ $(1 \le j \le n$ and $j \ne m^1, \dots, m^p)$ on S^k $(k = 1, 2)$, if the label w_j on F_V is "$+$", then we label $w_{j_p}^1$ on S^1 with "$\times$", and label $w_{j_p}^2$ on S^2 with "$\circ$"; if the label w_j on F_V is "$-$", then we label $w_{j_p}^1$ on S^1 with "$\circ$", and label $w_{j_p}^2$ on S^2 with "$\times$". For each $r \in I(v)$, by (5) in Proposition 2.3, the label $v_{r_p}^1$ on S^1 is "$-$". Then, $v_{r_p}^1$ is labelled with "$\circ$". Hence, $v_{r_p}^1$ is retained. So, V_{x_p} is a properly embedded disk in V^1. For each $r \in I(w)$, by (6) in Proposition 2.3, the label $w_{r_p}^1$ on S^1 is "$-$". Then, $w_{r_p}^1$ is labeled with "$\circ$". Hence, $w_{r_p}^1$ is retained. So, W_{x_p} is a properly embedded disk in W^1. Since both $v_{r_p}^1$ and $w_{r_p}^1$ are retained, $v_{r_p}^1 \cap w_{r_p}^1 = \emptyset$. By $(*)$, $V_{x_p} \cap W_{x_p} = x$. So, $M^1 = V^1 \cup_{S^1} W^1$ is stabilized. □

By Proposition 2.4, Theorem 1.2 holds. □

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