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ON ASYMPTOTIC OF EXTREMES FROM GENERALIZED MAXWELL DISTRIBUTION

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Abstract. In this paper, with optimal normalized constants, the asymptotic expansions of the distribution and density of the normalized maxima from generalized Maxwell distribution are derived. For the distributional expansion, it shows that the convergence rate of the normalized maxima to the Gumbel extreme value distribution is proportional to $1/\log n$. For the density expansion, on the one hand, the main result is applied to establish the convergence rate of the density of extreme to its limit. On the other hand, the main result is applied to obtain the asymptotic expansion of the moment of maximum.

1. Introduction

Let $(X_n,\ n \geq 1)$ be a sequence of independent and identically distributed (iid) random variables with common cumulative distribution function (cdf) F_k which obeying the generalized Maxwell distribution (denoted by $F_k \sim GMD(k)$). Let $M_n = \max(X_k,\ 1 \leq k \leq n)$ denote the partial maximum of $(X_n,\ n \geq 1)$.

As the generalization of the classic Maxwell distribution, the generalized Maxwell distribution was introduced by Vodă[16]. The probability density function (pdf) of GMD(k) is given by

$$f_k(x) = \frac{k}{2^{1/(2k)}\sigma^{2+1/k}\Gamma(1+1/(2k))} x^{2k} \exp\left(-\frac{x^{2k}}{2\sigma^2}\right),\ x > 0,$$

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where k, σ is positive and $\Gamma(\cdot)$ represents the Gamma function. For $k = 1$, GMD(1) reduces to the classic Maxwell distribution.

Recently, several properties associated with GMD(k) have been investigated in the literature. Huang and Chen [4] established the Mills inequality, the Mills type ratio and distributional tail representation of GMD(k), and showed that F_k belongs to the domain of attraction Λ of the Gumbel extreme value distribution, i.e., there exist normalizing constants $a_n > 0$ and $b_n \in \mathbb{R}$, such that

$$\begin{aligned}\lim_{n\to\infty} P\left((M_n - b_n)/a_n \le x\right) &= \lim_{n\to\infty} F_k^n(a_n x + b_n)\\ &= \Lambda(x),\end{aligned}$$

where $\Lambda(x) = \exp\{-e^{-x}\}$. Liu and Liu [9] established the uniform convergence rate of normalized maxima for GMD(1), i.e., the classic Maxwell distribution (MD for short). Kumar and Chandra [6] developed Sequential probability ratio test for testing the hypothesis concerning the parameter of the GMD(k). Plucińska [14] considered the properties of Hermite polynomials from sample having GMD(k), which have applications in various statistical problems connected with expansions in series.

Besides, the generalized Maxwell distribution has recently been a popular model in Chemical Engineering Science, Engineering Technology and Physics and other fields, for example, as model for computing fluid flows using the lattice Boltzmann method and deriving a different class of multiple relaxation-time LB models (see [2]), the diffusion of mixtures of hydrocarbons in zeolites (see [5]), polymorphic friction simulation and compensation and quick simulation and control purposes, being both easy to implement and of high fidelity (see [1]), control purpose, based on a physically motivated friction model, i.e., a generic friction model which simulates the contact physics at asperity level (see [7]) and so forth.

The aim of this paper is to establish the asymptotic expansion for the distribution and density of normalized maxima of GMD(k) random variables. As byproduct, we derive the high-order expansion of the moment of extreme. The uniform convergence rates and asymptotic expansions of the distribution and density of normalized M_n, the maximum of independent and identically distributed random variables for some given cdf F, have been of considerable interest. Hall [3] derived optimal rates of uniform convergence for the cdf of M_n as F follows the standard normal cdf. Nair [10] obtained asymptotic expansions for the distribution and moments of M_n as F is the standard normal cdf. Omey [11] gave the rate of convergence for the density of normalized sample maxima to the appropriate limit density. Peng et al. [13] established optimal uniform convergence rates for the cdf of M_n as F obeys the general error distribution. For other related works, see [8] and [12].

In order to gain the asymptotic expansions of normalized maxima from GMD(k), we cite some results from [4]. They gave the Mills type ratio of

GMD(k) as follows: for $k > 0$,

$$\frac{1 - F_k(x)}{f_k(x)} \sim \frac{\sigma^2}{k} x^{1-2k} \quad \text{as } x \to \infty. \tag{1.1}$$

It also follows from [4] that

$$1 - F_k(x) = c(x) \exp\left(- \int_1^x \frac{g(t)}{f(t)} \, \mathrm{d}t \right) \tag{1.2}$$

for large x, where

$$c(x) \to \frac{\exp\left(-1/(2\sigma^2)\right)}{2^{1/(2k)} \sigma^{1/k} \Gamma(1 + 1/(2k))} \quad \text{as } x \to \infty,$$

$$\begin{aligned} f(x) &= k^{-1} \sigma^2 x^{1-2k}, \\ g(x) &= 1 - k^{-1} \sigma^2 x^{-2k}. \end{aligned} \tag{1.3}$$

Note that $f'(x) \to 0$ and $g(x) \to 1$ as $x \to \infty$ and we can choose the appropriate normalizing constants a_n and b_n in such way that b_n satisfies the equation

$$1 - F_k(b_n) = n^{-1} \tag{1.4}$$

with

$$a_n = f(b_n) = k^{-1} \sigma^2 b_n^{1-2k} \tag{1.5}$$

such that

$$\lim_{n \to \infty} F_k^n(a_n x + b_n) = \Lambda(x).$$

The remainder of this paper is organized as follows. Section 2 gives the main result on asymptotic expansions for the distribution, density and moment of partial maxima of the GMD(k). Some auxiliary lemmas needed to prove the main results and related proofs including the proofs of main results are given in Section 3.

2. Main results

Next, we derive an asymptotic expansion for the distribution of normalized maxima from the GMD(k). The distributional expansion could be used to show that the convergence rate of M_n to the Gumbel extreme value distribution is of the order of $O((\log n)^{-1})$.

Theorem 2.1. *Let $F_k(x)$ represent the cdf of* GMD(k). *For normalizing constants a_n and b_n given, respectively, by* (1.4) *and* (1.5), *we have*

$$b_n^{2k} \left[b_n^{2k} \left(F_k^n(a_n x + b_n) - \Lambda(x) \right) - l_k(x) \Lambda(x) \right] \to \left(w_k(x) + \frac{l_k^2(x)}{2} \right) \Lambda(x) \ \text{ as } n \to \infty,$$

where $l_k(x)$ and $w_k(x)$ are, respectively, given by

$$l_k(x) = k^{-1} \sigma^2 \left[(2k-1) x^2 - 2x \right] e^{-x}/2$$

and

$$w_k(x) = -k^{-2}\sigma^4\left[3(2k-1)^2x^4 - 4(2k+1)(2k-1)x^3 + 24x^2 - 48kx\right]e^{-x}/24.$$

Remark 2.1. By the definition of b_n, it is easy to check that $b_n^{-2k} = O(1/\log n)$. Hence, Theorem 2.1 shows that the convergence rate of $F_k^n(a_nx + b_n)$ tending to its extreme value limit is proportional to $1/\log n$. Further, the convergence rate of $b_n^{2k}(F_k^n(a_nx + b_n) - \Lambda(x))$ tending to its limit is also proportional to $1/\log n$.

Remark 2.2. As mentioned in the Introduction, we get classic Maxwell pdf when $k = 1$, and then Theorem 2.1 shows also that the asymptotic expansion of the distribution of normalized maximum from classic Maxwell distribution is

$$\bar{b}_n^2\left[\bar{b}_n^2\left(F_1^n(\bar{a}_nx+\bar{b}_n)-\Lambda(x)\right)-l_1(x)\Lambda(x)\right]\to\left(w_1(x)+\frac{l_1^2(x)}{2}\right)\Lambda(x) \text{ as } n\to\infty,$$

with normalizing constants $\bar{a}_n$ and $\bar{b}_n$ determined by

$$1 - F_1(\bar{b}_n) = n^{-1} \text{ and } \bar{a}_n = \sigma^2\bar{b}_n^{-1},$$

where $l_1(x)$ and $w_1(x)$ are, respectively, given by

$$l_1(x) = \frac{1}{2}\sigma^2(x^2 - 2x)e^{-x}$$

and

$$w_1(x) = -\frac{1}{8}\sigma^4\left(x^4 - 4x^3 + 8x^2 - 16x\right)e^{-x}.$$

In the sequel, provided

$$\beta_n(x) = \frac{dF_k^n(a_nx + b_n)}{\mathrm{d}x} = na_nF_k^{n-1}(a_nx + b_n)f_k(a_nx + b_n)$$

stand for the distribution density of $(M_n - b_n)/a_n$, and

$$\Theta_n(\beta_n, \Lambda'; x) = \beta_n(x) - \Lambda'(x).$$

By using [15, Proposition 2.5], we have $\Theta_n(\beta_n, \Lambda'; x) \to 0$ as $n \to \infty$.

In the following, we show the high-order expansion of density of maxima from the GMD(k) and its application to the asymptotic of the moments of maximum.

Theorem 2.2. *Let $F_k(x)$ denote the cdf of* GMD(k). *Then with the normalizing constants a_n and b_n defined by* (1.4) *and* (1.5), *we have*

$$b_n^{2k}\left(b_n^{2k}\left(F_k^n(a_nx + b_n) - \Lambda(x)\right)' - P(x)\Lambda^{'}(x)\right) \to Q(x)\Lambda^{'}(x) \text{ as } n\to\infty, \tag{2.1}$$

where $P(x)$ and $Q(x)$ are respectively determined by

$$P(x)=k^{-1}\sigma^2\left(\frac{1}{2}\left((2k-1)x^2 - 2x\right)e^{-x}+\left(-\frac{1}{2}(2k-1)x^2+(2k-3)x-1\right)\right) \tag{2.2}$$

and

$$
\begin{aligned}
Q(x) =& \frac{1}{8}k^{-2}\sigma^4\left((2k-1)x^2-2x\right)^2 e^{-2x}\\
&-\frac{1}{24}k^{-2}\sigma^4\left(9(2k-1)^2x^4-16(2k-1)(2k+1)x^3\right.\\
&\left.+6(12k+1)x^2-24(k+1)x\right)e^{-x}\\
&+k^{-2}\sigma^4\left(\frac{1}{8}(2k-1)^2x^4-\frac{1}{3}(2k-1)(4k-1)x^3\right.\\
&\left.+\frac{1}{2}(4k^2+1)x^2-2kx+2k\right).
\end{aligned}
\tag{2.3}
$$

Noting that $b_n^{-2k} \sim 1/(2\sigma^2 \log n)$, it follows Theorem 2.2 that we easily get the rate of convergence of the density of maxima to its limit below.

Corollary 2.1. *Let a_n and b_n be given by* (1.4) *and* (1.5) *and for $x > 0$. Then*

$$
(F_k^n(a_nx+b_n))' - \Lambda'(x) \sim \frac{e^{-x}\Lambda(x)P(x)}{2\sigma^2\log n}
$$

for large n.

Remark 2.3. When the parameter $k = 1$, i.e., the classic Maxwell case, we obtain the associated expansion of density for normalized maxima.

Remark 2.4. Since $b_n^{-2k} \sim 1/(2\sigma^2 \log n)$, by Theorem 2.2, we could obtain the convergence speed of $b_n^{2k}\left(F_k^n(a_nx+b_n)-\Lambda(x)\right)'$ converging to its appropriate limit is proportional to $1/\log n$.

In the end of the section, we utilize the asymptotic expansion of density to obtain the high-order expansion of the moment of normalized maxima.

In the sequel, for $r > 0$ let

$$
m_r(n) = E\left(\frac{M_n-b_n}{a_n}\right)^r = \int_{-\infty}^{+\infty} x^r\beta_n(x)\,\mathrm{d}x
$$

and

$$
m_r = EX^r = \int_{-\infty}^{+\infty} x^r\Lambda'(x)\,\mathrm{d}x
$$

respectively represent the rth moments of $(M_n - b_n)/a_n$ and $X \sim \Lambda(x) = \exp(-\exp(-x))$, and the norming constants a_n and b_n are defined by (1.4) and (1.5).

Theorem 2.3. *For $k > \frac{1}{2}$, we have*

$$
\begin{aligned}
&b_n^{2k}\left(b_n^{2k}\left(m_r(n)-m_r\right)+2^{-1}k^{-1}\sigma^2r\left((2k-1)m_{r+1}-2m_r\right)\right)\\
\to\ & -rk^{-2}\sigma^4\left\{\left(-\frac{1}{8}(2k-1)^2(r+3)+\frac{1}{3}(k-1)(2k-1)\right)m_{r+2}\right.
\end{aligned}
$$

$$+\frac{1}{2}\left((2k-1)(r+2)-1\right)m_{r+1}+\left(2k-\frac{1}{2}(r+1)\right)m_r\Big\}\ as\ n\to\infty,$$

where the normalizing constants a_n and b_n are defined by (1.4) *and* (1.5).

Remark 2.5. For the case of $k=1$, i.e., the classic Maxwell distribution case, the corresponding result is stated as follow:

$$\begin{aligned}&\bar{b}_n^2\left[\bar{b}_n^2(m_r(n)-m_r)+2^{-1}r\sigma^2(m_{r+1}-2m_r)\right]\\ &\to r\sigma^4\left[2^{-1}(r-3)m_r-2^{-1}(r+1)m_{r+1}+8^{-1}(r+3)m_{r+2}\right]\ \text{as}\ n\to\infty,\end{aligned}$$

where the normalizing constants $\bar{a}_n$ and $\bar{b}_n$ are determined by Remark 2.2.

3. Auxiliary results and related proofs

In order to obtain expansions for the distribution of the normalized extreme of GMD(k) random variables, we provide the following distributional tail decomposition of GMD(k).

Lemma 3.1. *Let $F_k(x)$ represent the cdf of* GMD(k). *For large x, we have*

$$\begin{aligned}1-F_k(x)&=f_k(x)\frac{\sigma^2}{k}x^{1-2k}\left(1+k^{-1}\sigma^2x^{-2k}+k^{-2}(1-2k)\sigma^4x^{-4k}+O(x^{-6k})\right)\\ &=\frac{\exp(-1/(2\sigma^2))}{2^{1/(2k)}\sigma^{1/k}\Gamma(1+1/(2k))}\big(1+k^{-1}\sigma^2x^{-2k}+k^{-2}(1-2k)\sigma^4x^{-4k}\\ &\quad+O(x^{-6k})\big)\exp\left(-\int_1^x\frac{g(t)}{f(t)}\,\mathrm{d}t\right)\end{aligned}\tag{3.1}$$

with $f(t)$ and $g(t)$ given by (1.3).

Proof. By integration by parts, we have

$$\begin{aligned}1-F_k(x)&=f_k(x)\tfrac{\sigma^2}{k}x^{1-2k}\left(1+\tfrac{\sigma^2}{k}x^{-2k}+\tfrac{(1-2k)\sigma^4}{k^2}x^{-4k}+\tfrac{(1-2k)(1-4k)\sigma^6}{k^3}x^{-6k}\right)\\ &\quad+\frac{(1-2k)(1-4k)(1-6k)}{2^{1/(2k)}k^3\sigma^{1/k-6}\Gamma(1+1/(2k))}\int_x^\infty t^{-6k}\exp\left(-\frac{t^{2k}}{2\sigma^2}\right)\mathrm{d}t.\end{aligned}\tag{3.2}$$

It is easy to show by utilizing L'Hospital's rule that

$$\lim_{x\to\infty}\frac{\int_x^\infty t^{-6k}\exp(-\frac{t^{2k}}{2\sigma^2})\,\mathrm{d}t}{x^{1-6k}\exp(-\frac{x^{2k}}{2\sigma^2})}=0.\tag{3.3}$$

Thus, by (1.1), (1.2), (3.2) and (3.3), for large x, we can have

$$\begin{aligned}1-F_k(x)&=f_k(x)\frac{\sigma^2}{k}x^{1-2k}\left(1+\frac{\sigma^2}{k}x^{-2k}+\frac{(1-2k)\sigma^4}{k^2}x^{-4k}+O(x^{-6k})\right)\\ &=\frac{\exp(-\frac{1}{2\sigma^2})}{2^{\frac{1}{2k}}\sigma^{\frac{1}{k}}\Gamma(1+\frac{1}{2k})}\bigg(1+\frac{\sigma^2}{k}x^{-2k}+\frac{(1-2k)\sigma^4}{k^2}x^{-4k}\end{aligned}$$

$$+ O(x^{-6k})\Big) \exp\left(-\int_1^x \frac{g(t)}{f(t)}\,\mathrm{d}t\right).$$

The desired result follows. □

In order to prove Theorem 2.1, the following auxiliary result is needed.

Lemma 3.2. *Let $v_k(b_n; x) = n \log F_k(a_n x + b_n) + e^{-x}$. For normalizing constants a_n and b_n given, respectively, by* (1.4) *and* (1.5), *we have*

$$\lim_{n\to\infty} b_n^{2k}\left[b_n^{2k} v_k(b_n; x) - l_k(x)\right] = w_k(x), \tag{3.4}$$

where $l_k(x)$ and $w_k(x)$ are given by Theorem 2.1.

Proof. Obviously, $b_n \to \infty$ if and only if $n \to \infty$ since $1 - F_k(b_n) = n^{-1}$. The following facts can be gained by (1.1):

$$\lim_{n\to\infty} \frac{1 - F_k(a_n x + b_n)}{n^{-1}} = e^{-x} \tag{3.5}$$

and

$$\lim_{n\to\infty} \frac{1 - F_k(a_n x + b_n)}{b_n^{-2j}} = 0, \ j = 1, 2. \tag{3.6}$$

Set

$$B_k(n, x) = \frac{1 + \frac{\sigma^2}{k} b_n^{-2k} + \frac{(1-2k)\sigma^4}{k^2} b_n^{-4k} + O(b_n^{-6k})}{1 + \frac{\sigma^2}{k}(a_n x + b_n)^{-2k} + \frac{(1-2k)\sigma^4}{k^2}(a_n x + b_n)^{-4k} + O\left((a_n x + b_n)^{-6k}\right)}.$$

It is easy to check that $\lim_{n\to\infty} B_k(n, x) = 1$ and

$$B_k(n,x) - 1 = \left(\frac{2\sigma^4}{k} b_n^{-4k} x - \frac{\sigma^6}{k^2}((2k+1)x^2 - 4(1-2k)x) b_n^{-6k} x + O(b_n^{-6k})\right)(1 + o(1)).$$

Then

$$\lim_{n\to\infty} \frac{B_k(n, x) - 1}{b_n^{-2k}} = 0 \tag{3.7}$$

and

$$\lim_{n\to\infty} \frac{B_k(n, x) - 1}{b_n^{-4k}} = \frac{2\sigma^4}{k} x. \tag{3.8}$$

By (3.1), we have

$$\begin{aligned}
&\frac{1 - F_k(b_n)}{1 - F_k(a_n x + b_n)} e^{-x} \\
&= B_k(n, x) \exp\left(\int_0^x \left(\frac{k}{\sigma^2} a_n (a_n s + b_n)^{2k-1} - \frac{a_n}{a_n s + b_n} - 1\right) \mathrm{d}s\right) \\
&= B_k(n, x)\Bigg\{1 + \int_0^x \left(\frac{k}{\sigma^2} a_n (a_n s + b_n)^{2k-1} - \frac{a_n}{a_n s + b_n} - 1\right) \mathrm{d}s \\
&+ \frac{1}{2}\left(\int_0^x \left(\frac{k}{\sigma^2} a_n (a_n s + b_n)^{2k-1} - \frac{a_n}{a_n s + b_n} - 1\right) \mathrm{d}s\right)^2 (1 + o(1))\Bigg\}.
\end{aligned} \tag{3.9}$$

Combining with (3.5), (3.6), (3.7), (3.8) and (3.9), we have

$$
\begin{aligned}
(3.10)\quad & \lim_{n\to\infty} b_n^{2k} v_k(b_n;x) \\
&= \lim_{n\to\infty} \frac{\log F_k(a_n x+b_n) + (1-F_k(b_n))e^{-x}}{n^{-1}b_n^{-2k}} \\
&= \lim_{n\to\infty} \frac{-(1-F_k(a_n x+b_n)) - \frac{1}{2}(1-F_k(a_n x+b_n))^2(1+o(1))}{n^{-1}b_n^{-2k}} \\
&\quad + \lim_{n\to\infty} \frac{(1-F_k(b_n))e^{-x}}{n^{-1}b_n^{-2k}} \\
&= \lim_{n\to\infty} \frac{1-F_k(a_n x+b_n)}{n^{-1}} \frac{\frac{1-F_k(b_n)}{1-F_k(a_n x+b_n)}e^{-x}-1}{b_n^{-2k}} \\
&= e^{-x}\lim_{n\to\infty}\Bigg\{B_k(n,x)b_n^{2k}\bigg(\int_0^x\bigg(\frac{k}{\sigma^2}a_n(a_n s+b_n)^{2k-1} - \frac{a_n}{a_n s+b_n} - 1\bigg)\mathrm{d}s\bigg) \\
&\qquad\qquad (1+o(1)) + \frac{B_k(n,x)-1}{b_n^{-2k}}\Bigg\} \\
&= e^{-x}\lim_{n\to\infty}\int_0^x b_n^{2k}\bigg(\frac{k}{\sigma^2}a_n(a_n s+b_n)^{2k-1} - \frac{a_n}{a_n s+b_n} - 1\bigg)\,\mathrm{d}s \\
&= \frac{\sigma^2}{2k}\bigg((2k-1)x^2-2x\bigg)e^{-x} =: l_k(x),
\end{aligned}
$$

where the last step follows by the dominated convergence theorem and

$$
\lim_{n\to\infty} b_n^{2k}\bigg(\frac{k}{\sigma^2}a_n(a_n s+b_n)^{2k-1}-1\bigg) = \frac{2k-1}{k}\sigma^2 s
$$

and

$$
\lim_{n\to\infty}\frac{a_n b_n^{2k}}{a_n s+b_n} = \frac{\sigma^2}{k}.
$$

By arguments similar to (3.10), we have

$$
\begin{aligned}
& \lim_{n\to\infty} b_n^{2k}\bigg[b_n^{2k}v_k(b_n;x) - l_k(x)\bigg] \\
&= \lim_{n\to\infty}\frac{\log F_k(a_n x+b_n) + n^{-1}e^{-x} - n^{-1}b_n^{-2k}l_k(x)}{n^{-1}b_n^{-4k}} \\
&= \lim_{n\to\infty}\frac{\log F_k(a_n x+b_n) + (1-F_k(b_n))e^{-x}(1-l_k(x)e^x b_n^{-2k})}{n^{-1}b_n^{-4k}} \\
&= \lim_{n\to\infty}\frac{-(1-F_k(a_n x+b_n)) + (1-F_k(b_n))e^{-x}(1-l_k(x)e^x b_n^{-2k})}{n^{-1}b_n^{-4k}} \\
&= \lim_{n\to\infty}\frac{1-F_k(a_n x+b_n)}{n^{-1}}\,\frac{\frac{1-F_k(b_n)}{1-F_k(a_n x+b_n)}e^{-x}(1-l_k(x)e^x b_n^{-2k})-1}{b_n^{-4k}}
\end{aligned}
$$

$$= e^{-x} \lim_{n\to\infty} \Bigg\{ B_k(n,x) b_n^{4k} \Bigg[\int_0^x \left(\frac{k}{\sigma^2} a_n (a_n s + b_n)^{2k-1} - \frac{a_n}{a_n s + b_n} - 1 \right) \mathrm{d}s - l_k(x) e^x b_n^{-2k} \Bigg]$$
$$+ \frac{1}{2} B_k(n,x) b_n^{4k} \Bigg[\int_0^x \left(\frac{k}{\sigma^2} a_n (a_n s + b_n)^{2k-1} - \frac{a_n}{a_n s + b_n} - 1 \right) \mathrm{d}s \Bigg]^2$$
$$- B_k(n,x) l_k(x) e^x b_n^{2k} \int_0^x \left(\frac{k}{\sigma^2} a_n (a_n s + b_n)^{2k-1} - \frac{a_n}{a_n s + b_n} - 1 \right) \mathrm{d}s + \frac{B_k(n,x) - 1}{b_n^{-4k}} \Bigg\}$$
$$= -\frac{\sigma^4}{24k^2} \Big[3(2k-1)^2 x^4 - 4(2k+1)(2k-1)x^3 + 24x^2 - 48kx \Big] e^{-x}$$
$$= w_k(x).$$

This arrives to the conclusion of Lemma 3.2. □

Lemma 3.3. *Let $F_k(x)$ stand for the cdf of* GMD(k). *Then with normalizing constants a_n and b_n determined by* (1.4) *and* (1.5), *for large n we have*

$$F_k^{n-1}(a_n x + b_n) = C_n(x) \Lambda(x) \tag{3.11}$$

here

$$C_n(x) = 1 + b_n^{-2k} l_k(x) + b_n^{-4k} \left(w_k(x) + \frac{1}{2} l_k^2(x) \right) (1 + o(1)),$$

$l_k(x)$ and $w_k(x)$ are given by Theorem 2.1.

Proof. By using Theorem 2.1, we have

$$F_k^n(a_n x + b_n) = \Lambda(x) + b_n^{-2k} l_k(x) \Lambda(x) + b_n^{-4k} \left(w_k(x) + \frac{1}{2} l_k^2(x) \right) \Lambda(x)(1 + o(1))$$
$$= \left[1 + b_n^{-2k} l_k(x) + b_n^{-4k} \left(w_k(x) + \frac{1}{2} l_k^2(x) \right) (1 + o(1)) \right] \Lambda(x). \tag{3.12}$$

Observing that

$$F_k^n(a_n x + b_n) \to \exp(-e^{-x}) \quad \text{as } n \to \infty,$$

then we have

$$n(1 - F_k(a_n x + b_n)) \to e^{-x},$$

hence,

$$1 - F_k(a_n x + b_n) = O(n^{-1}),$$

which implies

$$\frac{1}{F_k(a_n x + b_n)} = \frac{1}{1 - (1 - F_k(a_n x + b_n))} = 1 + O(n^{-1}). \tag{3.13}$$

Combining (3.12) and (3.13) together, we derive the desired result. □

Lemma 3.4. *Let $f_k(x)$ denote the pdf of* GMD(k). *Then*

$$f_k(x) = (1 - F_k(x)) D_n(x) \tag{3.14}$$

for large x, and with normalizing constants a_n and b_n determined by (1.4) *and* (1.5), *we have*

$$\frac{a_n f_k(a_n x+b_n)}{1-F_k(a_n x+b_n)} = 1 + A_1(x)b_n^{-2k} + A_2(x)b_n^{-4k} + O(b_n^{-6k}) \tag{3.15}$$

for large n, where

$$\begin{aligned}
&D_n(x) = k\sigma^{-2}x^{2k-1}\left(1-k^{-1}\sigma^2x^{-2k}+2k^{-1}\sigma^4x^{-4k}+O(x^{-6k})\right),\\
&A_1(x) = k^{-1}\sigma^2((2k-1)x-1) \text{ and}\\
&A_2(x) = k^{-2}\sigma^4((2k-1)(k-1)x^2+x+2k).
\end{aligned}$$

Proof. By Lemma 3.1, we have

$$\begin{aligned}
f_k(x) &= (1-F_k(x))k\sigma^{-2}x^{2k-1}\left(1+k^{-1}\sigma^2x^{-2k}+k^{-2}(1-2k)\sigma^4x^{-4k}+O(x^{-6k})\right)^{-1}\\
&= (1-F_k(x))k\sigma^{-2}x^{2k-1}\left(1-k^{-1}\sigma^2x^{-2k}+2k^{-1}\sigma^4x^{-4k}+O(x^{-6k})\right)\\
&=: (1-F_k(x))D_n(x)
\end{aligned}$$

for large x. Therefore, by (1.5), for large n we have

$$\begin{aligned}
&\frac{a_n f_k(a_n x+b_n)}{1-F_k(a_n x+b_n)}\\
&= a_nD_n(a_nx+b_n)\\
&= a_nk\sigma^{-2}(a_nx+b_n)^{2k-1}\left(1-k^{-1}\sigma^2(a_nx+b_n)^{-2k}+2k^{-1}\sigma^4(a_nx+b_n)^{-4k}\right.\\
&\qquad\left.+O((a_nx+b_n)^{-6k})\right)\\
&= (1+k^{-1}\sigma^2b_n^{-2k}x)^{2k-1} - k^{-1}\sigma^2b_n^{-2k}(1+k^{-1}\sigma^2b_n^{-2k}x)^{-1}\\
&\qquad + 2k^{-1}\sigma^4b_n^{-4k}(1+k^{-1}\sigma^2b_n^{-2k}x)^{-2k-1} + O(b_n^{-6k})\\
&= 1 + k^{-1}\sigma^2((2k-1)x-1)b_n^{-2k} + k^{-2}\sigma^4((2k-1)(k-1)x^2+x+2k)b_n^{-4k}\\
&\qquad + O(b_n^{-6k})\\
&=: 1 + A_1(x)b_n^{-2k} + A_2(x)b_n^{-4k} + O(b_n^{-6k}).
\end{aligned}$$

The wanted result is deduced. ☐

Lemma 3.5. *Let $C_n(x)$ and $D_n(x)$ respectively be defined by* (3.11) *and* (3.14). *Then, for $k>\frac{1}{2}$, $-c\log b_n < x < db_n^{\frac{2}{3}k}$ and large n, we have*

$$\begin{aligned}
&|a_nC_n(x)D_n(a_nx+b_n)| < 2,\\
&\left|b_n^{2k}(a_nC_n(x)D_n(a_nx+b_n)-1)\right|\\
&\le 1 + k^{-1}\sigma^2\left((2k-1)|x| + \frac{1}{2}((2k-1)x^2+2|x|)e^{-x}\right), \quad \text{and}\\
&\left|b_n^{2k}\left(b_n^{2k}(a_nC_n(x)D_n(a_nx+b_n)-1)-l_{k1}(x)\right)\right|\\
&\le 1 + k^{-2}\sigma^4\left((2k-1)|k-1|x^2+|x|+2k+\frac{1}{2}\left(\frac{1}{4}(2k-1)^2x^4\right.\right.
\end{aligned}$$

$$+\frac{4}{3}(2k-1)|k-1||x|^3+|6k-5|x^2+2|1-2k||x|\Big)e^{-x}$$
$$+\frac{1}{8}((2k-1)x^2+2|x|)^2e^{-2x}\Big),$$

with $0<c,d<1$, *here the normalizing constants* a_n *and* b_n *are given by* (1.4) *and* (1.5), *and*

$$l_{k1}=k^{-1}\sigma^2\left(\frac{1}{2}\left((2k-1)x^2-2x\right)e^{-x}+(2k-1)x-1\right).$$

Proof. These results are directly from Lemmas 3.3 and 3.4. The details are omitted. □

In order to derive later lemmas, we need the following result.

It follows from [4] that we have the Mills type inequality of the GMD(k) as follows:

$$\frac{\sigma^2}{k}x^{1-2k}<\frac{1-F_k(x)}{f_k(x)}<\frac{\sigma^2}{k}x^{1-2k}\left(1+\left(\frac{\sigma^2}{k}x^{2k}-1\right)^{-1}\right) \tag{3.16}$$

for all $x>0$ and $k>\frac{1}{2}$, where σ is positive.

Lemma 3.6. *For any constants* $0<c,d<1$ *and arbitrary nonnegative integers* i,j, *we have*

$$\lim_{n\to\infty}b_n^i\int_{db_n^{\frac{2}{3}k}}^{\infty}|x|^j\exp(-i_0x)\Lambda(x)\mathrm{d}x=0,\ i_0=1,2,3,\dots, \tag{3.17}$$

$$\lim_{n\to\infty}b_n^i\int_{db_n^{\frac{2}{3}k}}^{\infty}|x|^j\beta_n(x)\mathrm{d}x=0, \tag{3.18}$$

$$\lim_{n\to\infty}b_n^i\int_{-\infty}^{-c\log b_n}|x|^j\exp(-i_0x)\Lambda(x)\mathrm{d}x=0,\ i_0=1,2,3,\dots, \tag{3.19}$$

and

$$\lim_{n\to\infty}b_n^i\int_{-\infty}^{-c\log b_n}|x|^j\beta_n(x)\mathrm{d}x=0, \tag{3.20}$$

with the normalizing constant b_n *determined by* (1.4).

Proof. Firstly, we consider the Eq. (3.17). Noting that the inequalities $1-x<e^{-x}<1$ for $x>0$, we have

$$\begin{aligned}&b_n^i\int_{db_n^{\frac{2}{3}k}}^{\infty}|x|^j\exp(-i_0x)\Lambda(x)\mathrm{d}x\\&<b_n^i\int_{db_n^{\frac{2}{3}k}}^{\infty}|x|^j\exp(-i_0x)\mathrm{d}x\\&<b_n^i\exp\left(-\frac{1}{2}i_0db_n^{\frac{2}{3}k}\right)\int_{db_n^{\frac{2}{3}k}}^{\infty}|x|^j\exp\left(-\frac{1}{2}i_0x\right)\mathrm{d}x\end{aligned}$$

$$\to 0 \text{ as } n \to \infty.$$

Eq. (3.17) is complete.

Secondly, we consider Eq. (3.18). By (3.16) and Lemma 3.5, we have

$$\begin{aligned}
& b_n^i \int_{db_n^{\frac{2}{3}k}}^{\infty} |x|^j \beta_n(x)\mathrm{d}x \\
&= b_n^i \int_{db_n^{\frac{2}{3}k}}^{\infty} |x|^j a_n(1-F_k(b_n))^{-1} F_k^{n-1}(a_n x + b_n) f_k(a_n x + b_n)\mathrm{d}x \\
&= b_n^i \int_{db_n^{\frac{2}{3}k}}^{\infty} |x|^j a_n C_n(x) D_n(a_n x + b_n) \frac{(1-F_k(a_n x + b_n))}{1-F_k(b_n)} \Lambda(x)\mathrm{d}x \\
&< 2b_n^i \int_{db_n^{\frac{2}{3}k}}^{\infty} |x|^j \frac{(1-F_k(a_n x + b_n))}{1-F_k(b_n)} \Lambda(x)\mathrm{d}x \\
&< 4b_n^i \int_{db_n^{\frac{2}{3}k}}^{\infty} |x|^j \exp(-x)\Lambda(x)\mathrm{d}x \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Eq. (3.18) is finished.

Next, we consider Eq. (3.19).

$$\begin{aligned}
& b_n^i \int_{-\infty}^{-c\log b_n} |x|^j \exp(-i_0 x)\Lambda(x)\mathrm{d}x \\
&= b_n^i \int_{c\log b_n}^{\infty} t^j \exp(i_0 t)\exp(-\exp(t))\mathrm{d}t \\
&< b_n^i \exp\left(-\frac{1}{2} b_n^c\right) \int_1^{\infty} t^j \exp(i_0 t) \exp\left(-\frac{1}{2}\exp(t)\right)\mathrm{d}t \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

by methods similar to Eq. (3.18). Eq. (3.19) is proved.

Last, we consider Eq. (3.20). Observing that the fact $1-\alpha x < (1-x)^\alpha < 1-\alpha x + \frac{\alpha(\alpha-1)}{2}x^2$ as $0 < x < \frac{1}{2}$, $\alpha > 2$, then for $-\infty < x < -c\log b_n$, by (3.16) we have

$$\begin{aligned}
n a_n f_k(a_n x + b_n) &= \frac{a_n f_k(a_n x + b_n)}{1-F_k(b_n)} \\
&< \exp\left(-\frac{b_n^{2k}}{2\sigma^2}\left((1+k^{-1}\sigma^2 b_n^{-2k} x)^{2k} - 1\right)\right) \\
&< e^{-x}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1-F_k(a_n x + b_n)}{1-F_k(b_n)} > {} & (1+k^{-1}\sigma^2 b_n^{-2k} x)^{1-2k} \left(1 + \left(\frac{\sigma^2}{k} b_n^{2k} - 1\right)^{-1}\right) \\
& \exp\left(-\frac{b_n^{2k}}{2\sigma^2}\left((1+k^{-1}\sigma^2 b_n^{-2k} x)^{2k} - 1\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&> (1+k^{-1}\sigma^2 b_n^{-2k}x)^{1-2k}\left(1+\left(\frac{\sigma^2}{k}b_n^{2k}-1\right)^{-1}\right)\\
&\quad \exp\left(-x-\frac{1}{2}(2-k^{-1})\sigma^2 b_n^{-2k}x^2\right)\\
&> \frac{1}{2}e^{-x}
\end{aligned}
$$

for large n. Thus,

$$
\begin{aligned}
& b_n^i\int_{-\infty}^{-c\log b_n}|x|^j\beta_n(x)\mathrm{d}x\\
&= b_n^i\int_{-\infty}^{-c\log b_n}|x|^j a_n(1-F_k(b_n))^{-1}F_k^{n-1}(a_nx+b_n)f_k(a_nx+b_n)\mathrm{d}x\\
&< b_n^i\int_{-\infty}^{-c\log b_n}|x|^j a_n(1-F_k(b_n))^{-1}f_k(a_nx+b_n)\\
&\qquad \exp\left[-(n-1)(1-F_k(a_nx+b_n))\right]\mathrm{d}x\\
&< b_n^i\int_{-\infty}^{-c\log b_n}|x|^j\exp(-x)\exp\left(-\frac{1}{2}\exp(-x)\right)\mathrm{d}x\\
&\to 0 \text{ as } n\to\infty.
\end{aligned}
$$

Eq. (3.20) is derived.

Combining those results above, we complete the proof. □

Lemma 3.7. *Set*

$$H_k(b_n;x)=\frac{1-F_k(a_nx+b_n)}{1-F_k(b_n)}e^x-1.$$

For large n *and* $-c\log b_n<x<db_n^{\frac{2}{3}k}$*, some integrable functions independent of* n *dominate* $x^r b_n^{2k}\Theta_n(\beta_n,\Lambda';x)$ *and* $x^r b_n^{2k}\big(b_n^{2k}\Theta_n(\beta_n,\Lambda';x)-P(x)\Lambda'(x)\big)$ *with* $r>0$ *and* $0<c,d<1$*, here* a_n *and* b_n *are defined by* (1.4) *and* (1.5)*, and* $P(x)$ *is determined by* (2.2)*.*

Proof. Rescript

$$
\begin{aligned}
b_n^{2k}\Theta_n(\beta_n,\Lambda';x) &= b_n^{2k}(a_nC_n(x)D_n(a_nx+b_n)-1)\Lambda'(x)\\
&\quad + b_n^{2k}a_nC_n(x)D_n(a_nx+b_n)H_k(b_n;x)\Lambda'(x).
\end{aligned}
$$

Easily check that $\int_{-\infty}^{\infty}y^i\exp(-sy)\exp(-\exp(-y))dy=(-1)^i\Gamma^{(i)}(s)<\infty$ for $s\in\mathbb{R}^+$ and nonnegative integers i. Lemma 3.5 proves that

$$b_n^{2k}(a_nC_n(x)D_n(a_nx+b_n)-1)\Lambda'(x)$$

is bounded by some integrable function independent of n. Next we show that $b_n^{2k}H_k(b_n;x)$ is bounded by $p(x)$, here $p(x)$ is a polynomial on x.

Rescript

$$b_n^{2k}H_k(b_n;x) = b_n^{2k}(E_k(n,x)-1) - b_n^{2k}E_k(n,x)\int_0^x \delta_n(s)\mathrm{d}s(1+o(1)) \tag{3.21}$$
$$=: \mathcal{A}_n(x) - \mathcal{B}_n(x),$$

here

$$\delta_n(x) = \frac{k}{\sigma^2}a_n(a_nx+b_n)^{2k-1} - \frac{a_n}{a_nx+b_n} - 1.$$

For $-c\log b_n < x < db_n^{\frac{2}{3}k}$, by Lemma 3.2 and (1.5) we have

$$|\mathcal{A}_n(x)| < 1 \tag{3.22}$$

and

$$|\mathcal{B}_n(x)| < 1+\frac{1}{2}(2-k^{-1})\sigma^2x^2+\frac{2}{3}(2-k^{-1})\left|1-k^{-1}\right|\sigma^4|x|^3+\frac{1}{k\sigma^2-c}|x| \tag{3.23}$$

for large n.

By (3.22) and (3.23), $b_n^{2k}H_k(b_n;x)$ is bounded by one integrable function independent of n.

Rescript

$$\begin{aligned}
&b_n^{2k}\left(b_n^{2k}\Theta_n(\beta_n,\Lambda';x) - P(x)\Lambda'(x)\right)\\
&= b_n^{2k}\left(b_n^{2k}\left(a_n(1-F_k(b_n))^{-1}F_k^{n-1}(a_nx+b_n)f_k(a_nx+b_n) - \Lambda'(x)\right)\right.\\
&\qquad \left.-P(x)\Lambda'(x)\right)\\
&= b_n^{2k}\left(b_n^{2k}\left(\left(\frac{(1-F_k(a_nx+b_n))}{1-F_k(b_n)}e^x - 1 + 1\right)a_nC_n(x)D_n(a_nx+b_n) - 1\right)\right.\\
&\qquad \left.-(l_{k1}(x)+l_{k2}(x))\right)\Lambda'(x)\\
&= b_n^{2k}\Bigg(b_n^{2k}a_nC_n(x)D_n(a_nx+b_n)\left(H_k(b_n;x) - b_n^{-2k}l_{k2}(x)\right)\\
&\qquad + b_n^{2k}\left(a_nC_n(x)D_n(a_nx+b_n) - 1 - b_n^{-2k}l_{k1}(x)\right)\\
&\qquad + \left(a_nC_n(x)D_n(a_nx+b_n)-1\right)l_{k2}(x)\Bigg)\Lambda'(x),
\end{aligned}$$

here

$$l_{k2}(x) = -\frac{1}{2}k^{-1}\sigma^2((2k-1)x^2+2x).$$

It follows from Lemma 3.5 that only need to calculate the bound of

$$b_n^{2k}\Bigg(b_n^{2k}a_nC_n(x)D_n(a_nx+b_n)\left(H_k(b_n;x) - b_n^{-2k}l_{k2}(x)\right)\Bigg).$$

Rescript

$$\begin{aligned}
&b_n^{2k}\Bigg(b_n^{2k}a_nC_n(x)D_n(a_nx+b_n)\left(H_k(b_n;x) - b_n^{-2k}l_{k2}(x)\right)\Bigg)\\
&= b_n^{4k}(E_k(n,x)-1) - b_n^{2k}\left(E_k(n,x)b_n^{2k}\int_0^x \delta(s)\mathrm{d}s + b_n^{-2k}l_{k2}(x)\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}E_k(n,x)b_n^{4k}\left(\int_0^x \delta(s)\mathrm{d}s\right)^2(1+o(1))\\
&=: \mathcal{I}_n(x)-\mathcal{J}_n(x)+\mathcal{K}_n(x).
\end{aligned}
$$

As to the case of $0<x<db_n^{\frac{2}{3}k}$, we have

$$
|\mathcal{I}_n(x)|<4k^{-2}\sigma^4(k+4(2k-1)\sigma^2)x \tag{3.24}
$$

for large n, by Lemma 3.2 and $1-\alpha x<(1+x)^{-\alpha}<1$ for $\alpha>0$ and $x>0$.

As to the case of $-c\log b_n<x<0$, we have

$$
|\mathcal{I}_n(x)|<4k^{-2}\sigma^4(k+4(2k-1)\sigma^2)|x| \tag{3.25}
$$

for large n, by Lemma 3.2 and $1+\alpha x<(1+x)^{\alpha}<1$ for $\alpha>1$ and $-1<x<0$.

Similarly, as for the bound of $\mathcal{J}_n(x)$ and $\mathcal{K}_n(x)$, we have

$$
|\mathcal{J}_n(x)|<\frac{2}{3}(2-k^{-1})\left|1-k^{-1}\right|\sigma^4|x|^3+\frac{1}{2}k^{-1}\sigma^2\left|k\sigma^{-2}-c\right|^{-1}x^2 \tag{3.26}
$$

and

$$
\begin{aligned}
|\mathcal{K}_n(x)|<1+&\frac{1}{2}\bigg(\frac{1}{2}\left(2-k^{-1}\right)\sigma^2x^2+\frac{2}{3}\left(2-k^{-1}\right)\left|1-k^{-1}\right|\sigma^4|x|^3\\
&+\left|k\sigma^{-2}-c\right|^{-1}|x|\bigg)^2
\end{aligned} \tag{3.27}
$$

for large n, as $-c\log b_n<x<db_n^{\frac{2}{3}k}$.

Combining (3.24)-(3.27) together, the desired result is gained. The proof is finished. □

Proof of Theorem 2.1. By (3.10), we have $v_k(b_n;x)\to 0$ and

$$
\left|\sum_{i=3}^{\infty}\frac{v_k^{i-3}(b_n;x)}{i!}\right|<\exp\left(|v_k(b_n;x)|\right)\to 1 \text{ as } n\to\infty.
$$

By applying Lemma 3.2, we have

$$
\begin{aligned}
&b_n^{2k}\left[b_n^{2k}\left(F_k^n(a_nx+b_n)-\Lambda(x)\right)-l_k(x)\Lambda(x)\right]\\
&=b_n^{2k}\left[b_n^{2k}\left(\exp\left(v_k(b_n;x)\right)-1\right)-l_k(x)\right]\Lambda(x)\\
&=\left[b_n^{2k}\left(b_n^{2k}v_k(b_n;x)-l_k(x)\right)+b_n^{4k}v_k^2(b_n;x)\left(\frac{1}{2}+v_k(b_n;x)\sum_{i=3}^{\infty}\frac{v_k^{i-3}(b_n;x)}{i!}\right)\right]\Lambda(x)\\
&\to\left(w_k(x)+\frac{l_k^2(x)}{2}\right)\Lambda(x) \text{ as } n\to\infty.
\end{aligned}
$$

We obtain the desired result. □

Proof of Theorem 2.2. Let $E_k(n,x) = 1/B_k(n,x)$, by (3.7) and (3.8), we have

$$\lim_{n\to\infty} b_n^{2k}(E_k(n,x) - 1) = 0 \tag{3.28}$$

and

$$\lim_{n\to\infty} b_n^{4k}(E_k(n,x) - 1) = -2k^{-1}\sigma^4 x. \tag{3.29}$$

It follows from (3.9) that

$$\begin{aligned} &\frac{1 - F_k(a_n x + b_n)}{1 - F_k(b_n)} e^x \\ &= E_k(n,x)\left\{1 - \int_0^x \delta_n(s)\,\mathrm{d}s + \frac{1}{2}\left(\int_0^x \delta_n(s)\,\mathrm{d}s\right)^2 (1 + o(1))\right\}, \end{aligned} \tag{3.30}$$

where

$$\delta_n(x) = \frac{k}{\sigma^2} a_n (a_n x + b_n)^{2k-1} - \frac{a_n}{a_n x + b_n} - 1.$$

By (3.11) and (3.14), we have

$$\begin{aligned} &a_n C_n(x) D_n(a_n x + b_n) \\ =& \left(1 + b_n^{-2k} l_k(x) + b_n^{-4k}\left(w_k(x) + \frac{1}{2} l_k^2(x)\right)(1 + o(1))\right) \\ &\times \left(1 + A_1(x) b_n^{-2k} + A_2(x) b_n^{-4k} + O(b_n^{-6k})\right) \\ =& 1 + (l_k(x) + A_1(x)) b_n^{-2k} + \left(w_k(x) + \frac{1}{2} l_k^2(x) + l_k(x) A_1(x) + A_2(x)\right) b_n^{-4k} \\ &+ O(b_n^{-6k}). \end{aligned} \tag{3.31}$$

By Lemmas 3.3 and 3.4 and combining (3.28)-(3.31) together, we have

$$\begin{aligned} &\Theta_n(\beta_n, \Lambda'; x) \\ &= \beta_n(x) - \Lambda'(x) \\ &= (1 - F_k(b_n))^{-1} a_n F_k^{n-1}(a_n x + b_n) f_k(a_n x + b_n) - \Lambda'(x) \\ &= (1 - F_k(b_n))^{-1} a_n C_n(x) \Lambda(x) (1 - F_k(a_n x + b_n)) D_n(a_n x + b_n) - \Lambda'(x) \\ &= \Bigg(E_k(n,x)(l_k(x) + A_1(x)) b_n^{-2k} \\ &\qquad + E_k(n,x)\left(w_k(x) + \frac{1}{2} l_k^2(x) + l_k(x) A_1(x) + A_2(x)\right) b_n^{-4k} \\ &\qquad - E_k(n,x)\left[1 + (l_k(x) + A_1(x)) b_n^{-2k} \right. \\ &\qquad\quad \left. + \left(w_k(x) + \frac{1}{2} l_k^2(x) + l_k(x) A_1(x) + A_2(x)\right) b_n^{-4k}\right] \\ &\qquad\quad \times \int_0^x \delta_n(s)\,\mathrm{d}s + \frac{1}{2} E_k(n,x)\Big[1 + (l_k(x) + A_1(x)) b_n^{-2k} \end{aligned}$$

$$+\left(w_k(x)+\frac{1}{2}l_k^2(x)+l_k(x)A_1(x)+A_2(x)\right)b_n^{-4k}+O(b_n^{-6k})\Bigg]$$

$$\left(\int_0^x \delta_n(s)\,\mathrm{d}s\right)^2(1+o(1))+E_k(n,x)-1\Bigg)\Lambda'(x). \tag{3.32}$$

Thus, by applying (3.32) we have

(3.33)

$$\begin{aligned}
&\lim_{n\to\infty} b_n^{2k}\Theta_n(\beta_n,\Lambda';x)\\
&=\lim_{n\to\infty}\left(l_k(x)+A_1(x)-\int_0^x b_n^{2k}\delta_n(s)\,\mathrm{d}s\right)\Lambda'(x)\\
&=k^{-1}\sigma^2\left(\frac{1}{2}\left((2k-1)x^2-2x\right)e^{-x}+\left(-\frac{1}{2}(2k-1)x^2+(2k-3)x-1\right)\right)\Lambda'(x)\\
&=: P(x)\Lambda'(x).
\end{aligned}$$

Combining (3.32) and (3.33) together, we have

$$\begin{aligned}
&\lim_{n\to\infty} b_n^{2k}\left(b_n^{2k}\Theta_n(\beta_n,\Lambda';x)-P(x)\Lambda'(x)\right)\\
=\ &\lim_{n\to\infty} b_n^{2k}\Bigg((E_k(n,x)-1)(l_k(x)+A_1(x))\\
&+E_k(n,x)\left(w_k(x)+\frac{1}{2}l_k^2(x)+l_k(x)A_1(x)+A_2(x)\right)b_n^{-2k}\\
&-b_n^{2k}\int_0^x\left(\delta_n(s)-\left(\frac{2k-1}{k}\sigma^2 s-\frac{\sigma^2}{k}\right)b_n^{-2k}\right)\mathrm{d}s\\
&-E_k(n,x)\left[l_k(x)+A_1(x)+\left(w_k(x)+\frac{1}{2}l_k^2(x)+l_k(x)A_1(x)+A_2(x)\right)b_n^{-2k}\right]\\
&\int_0^x\delta_n(s)\,\mathrm{d}s+\frac{1}{2}E_k(n,x)b_n^{2k}\Bigg[1+(l_k(x)+A_1(x))b_n^{-2k}\\
&+\left(w_k(x)+\frac{1}{2}l_k^2(x)+l_k(x)A_1(x)+A_2(x)\right)b_n^{-4k}+O(b_n^{-6k})\Bigg]\\
&\left(\int_0^x\delta_n(s)\,\mathrm{d}s\right)^2(1+o(1))+b_n^{2k}(E_k(n,x)-1)\Bigg)\Lambda'(x)\\
=\ &\lim_{n\to\infty}\Bigg(b_n^{4k}(E_k(n,x)-1)+w_k(x)+\frac{1}{2}l_k^2(x)+l_k(x)A_1(x)+A_2(x)\\
&-b_n^{4k}\int_0^x\left(\delta_n(s)-\left(\frac{2k-1}{k}\sigma^2 s-\frac{\sigma^2}{k}\right)b_n^{-2k}\right)\mathrm{d}s\\
&-(l_k(x)+A_1(x))b_n^{2k}\int_0^x\delta_n(s)\,\mathrm{d}s+\frac{1}{2}b_n^{4k}\left(\int_0^x\delta_n(s)\,\mathrm{d}s\right)^2(1+o(1))\Bigg)\Lambda'(x)\\
=\ &\frac{1}{8}k^{-2}\sigma^4\left((2k-1)x^2-2x\right)^2e^{-2x}-\frac{1}{24}k^{-2}\sigma^4\left(9(2k-1)^2x^4\right.
\end{aligned}$$

$$
\begin{aligned}
&-16(2k-1)(2k+1)x^3 + 6(12k+1)x^2 - 24(k+1)x\big) e^{-x} \\
&+ k^{-2}\sigma^4 \left(\frac{1}{8}(2k-1)^2x^4 - \frac{1}{3}(2k-1)(4k-1)x^3\right. \\
&\left.+\frac{1}{2}(4k^2+1)x^2 - 2kx + 2k\right) \Lambda'(x) \\
=:&\ Q(x)\Lambda'(x).
\end{aligned}
$$

The conclusion follows. □

Proof of Theorem 2.3. By Lemmas 3.5-3.7 and the dominated convergence theorem, we have

$$
\begin{aligned}
& b_n^{2k}(m_r(n) - m_r) \\
= & \int_{-\infty}^{\infty} x^r b_n^{2k} \Theta_n(\beta_n, \Lambda'; x) \mathrm{d}x \\
= & \int_{-\infty}^{-c\log b_n} x^r b_n^{2k} \Theta_n(\beta_n, \Lambda'; x) \mathrm{d}x + b_n^{2k} \int_{-c\log b_n}^{db_n^{\frac{2}{3}k}} x^r b_n^{2k} \Theta_n(\beta_n, \Lambda'; x) \mathrm{d}x \\
& + \int_{db_n^{\frac{2}{3}k}}^{\infty} x^r b_n^{2k} \Theta_n(\beta_n, \Lambda'; x) \mathrm{d}x \\
\to & \int_{-\infty}^{\infty} x^r P(x) \Lambda'(x) \mathrm{d}x \\
= & -2^{-1}k^{-1}\sigma^2 r\left((2k-1)m_{r+1} - 2m_r\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& b_n^{2k}\left(b_n^{2k}(m_r(n) - m_r) + 2^{-1}k^{-1}\sigma^2 r\left((2k-1)m_{r+1} - 2m_r\right)\right) \\
= & \int_{-\infty}^{\infty} x^r b_n^{2k}\left(b_n^{2k}\Theta_n(\beta_n, \Lambda'; x) - P(x)\Lambda'(x)\right) \mathrm{d}x \\
= & \int_{db_n^{\frac{2}{3}k}}^{+\infty} x^r b_n^{2k}\left(b_n^{2k}\Theta_n(\beta_n, \Lambda'; x) - P(x)\Lambda'(x)\right) \mathrm{d}x \\
& + \int_{-c\log b_n}^{db_n^{\frac{2}{3}k}} x^r b_n^{2k}\left(b_n^{2k}\Theta_n(\beta_n, \Lambda'; x) - P(x)\Lambda'(x)\right) \mathrm{d}x \\
& + \int_{-\infty}^{-c\log b_n} x^r b_n^{2k}\left(b_n^{2k}\Theta_n(\beta_n, \Lambda'; x) - P(x)\Lambda'(x)\right) \mathrm{d}x \\
\to & \int_{-\infty}^{\infty} x^r Q(x)\Lambda'(x) \mathrm{d}x \\
= & -rk^{-2}\sigma^4\left\{\left(-\frac{1}{8}(2k-1)^2(r+3) + \frac{1}{3}(k-1)(2k-1)\right) m_{r+2}\right. \\
& \left.+\frac{1}{2}\left((2k-1)(r+2)-1\right) m_{r+1} + \left(2k - \frac{1}{2}(r+1)\right) m_r\right\}
\end{aligned}
$$

as $n \to \infty$ with

$$\int_{-\infty}^{+\infty} x^{r+1} e^{-2x} \Lambda(x)\,\mathrm{d}x = -(r+1)m_r + m_{r+1}$$

and

$$\int_{-\infty}^{+\infty} x^{r+1} e^{-3x} \Lambda(x)\,\mathrm{d}x = r(r+1)m_{r-1} - 3(r+1)m_r + 2m_{r+1}.$$

The proof is complete. □

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