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EXISTENCE OF EVEN NUMBER OF POSITIVE SOLUTIONS TO SYSTEM OF FRACTIONAL ORDER BOUNDARY VALUE PROBLEMS

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Abstract. We establish the existence and multiplicity of positive solutions to a coupled system of fractional order differential equations satisfying three-point boundary conditions by utilizing Avery–Henderson functional fixed point theorems and under suitable conditions.

1. Introduction

In the modeling of many phenomena in distinct areas of science and technology, the differential equations of an arbitrary real order will arise. Fractional calculus is the field of mathematical analysis which unifies the theories of integration and differentiation of any fractional order [5, 6, 12, 13, 14, 19]. Indeed we can find numerous applications in engineering and scientific disciplines like mathematical modeling of systems and processes in various fields such as physics, mechanics, control systems, flow in porous media, electromagnetics and viscoelasticity. We refer the reader to [10, 20] and references therein for some applications.

There has been a significant progress in the investigation on theory of fractional order differential equations constitutes with initial and boundary conditions in recent years. Due to its importance, researchers are concentrating on study the existence of solutions, positive solutions and multiple positive solutions for two-point, multi-point boundary value

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problems concerning the standard Riemann–Liouville fractional order derivative [3, 4, 7, 9, 11, 15, 16, 17].

Recently Prasad, Krushna and Wesen [18] established the existence of multiple positive solutions to the fractional order boundary value problem

$$D_{0^+}^{\alpha} y(t) + f\big(t, y(t)\big) = 0,\ t \in (0,1),$$
$$y^{(k)}(0) = 0,\ k = 0, 1, \cdots, n-2, \zeta D_{0^+}^{\beta} y(1) - \vartheta D_{0^+}^{\beta} y(\eta) = 0.$$

We wish to extend these results to the system of fractional order boundary value problems. Motivated by above mentioned papers, in this paper we consider the following three-point fractional order boundary value problems

$$D_{0^+}^{\gamma_1} w_1(t) + f_1\big(t, w_1(t), w_2(t)\big) = 0,\ t \in (0,1), \tag{1.1}$$

$$D_{0^+}^{\gamma_2} w_2(t) + f_2\big(t, w_1(t), w_2(t)\big) = 0,\ t \in (0,1), \tag{1.2}$$

$$\begin{cases} w_1^{(k)}(0) = 0,\ k = 0, 1, \cdots, n-2, \\ \zeta_1 D_{0^+}^{\beta_1} w_1(1) - \vartheta_1 D_{0^+}^{\beta_1} w_1(\eta) = 0, \end{cases} \tag{1.3}$$

$$\begin{cases} w_2^{(l)}(0) = 0,\ l = 0, 1, \cdots, n-2, \\ \zeta_2 D_{0^+}^{\beta_2} w_2(1) - \vartheta_2 D_{0^+}^{\beta_2} w_2\big(\eta\big) = 0, \end{cases} \tag{1.4}$$

where $\gamma_1, \gamma_2 \in (n-1, n]$ and $n \geq 3$, $\beta_i \in (1, \gamma_i), \zeta_i, \vartheta_i$ are positive constants, $f_i : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^+$ are continuous and $D_{0^+}^{\gamma_i}, D_{0^+}^{\beta_i}$, for $i = 1, 2$ are the standard Riemann–Liouville fractional order derivatives. By a positive solution of the fractional order boundary value problem (1.1)-(1.4), we mean $\Big(w_1(t), w_2(t)\Big) \in \Big(C^{\gamma_1}[0,1] \times C^{\gamma_2}[0,1]\Big)$ satisfying (1.1)-(1.4) with $w_i(t) \geq 0, i = 1, 2$ for all $t \in [0,1]$ and $(w_1, w_2) \neq (0,0)$.

The paper is organized as follows: In section 2, we construct the Green functions for the associated linear fractional order boundary value problems and estimate the bounds for these Green functions. In section 3, we establish the existence of at least two positive solutions for the system of fractional order boundary value problem (1.1)-(1.4) by using Avery–Henderson functional fixed point theorem. We also establish the existence of at least $2m$ positive solutions to the boundary value problem (1.1)-(1.4) for an arbitrary positive integer m. In the final section 4, as an application, we demonstrate our results with an example.

2. Preliminaries

In this section we give some definitions, the Green functions for the associated linear fractional order boundary value problems and the bounds for these Green functions, which are useful in the proof of our main results.

Definition 2.1. [14] The Riemann–Liouville fractional integral of order $p > 0$ of a function $f : [0, +\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{p} f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau) d\tau,$$

provided the right-hand side is defined.

Definition 2.2. [14] The Riemann–Liouville fractional derivative of order $p > 0$ of a function $f : [0, +\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{p} f(t) = \frac{1}{\Gamma(k-p)} \frac{d^k}{dt^k} \int_0^t (t-\tau)^{k-p-1} f(\tau) d\tau, \ (k-1 \le p < k),$$

provided the right-hand side is defined.

Lemma 2.3. *Let* $\Delta_1 = \Gamma(\gamma_1)\mathcal{N}_1 \neq 0$. *If* $h_1(t) \in C[0,1]$, *then the fractional order differential equations*

$$D_{0^+}^{\gamma_1} w_1(t) + h_1(t) = 0, \ t \in (0,1), \tag{2.1}$$

satisfying the boundary conditions (1.3) *has a unique solution*

$$w_1(t) = \int_0^1 G_1(t,s) h_1(s) ds,$$

where $G_1(t,s)$ *is the Green's function for the problem* (2.1), (1.3) *and is given by*

(2.2)

$$G_1(t,s) = \begin{cases} \underset{t\in[0,\eta]}{G_1(t,s)} = \begin{cases} G_{11}(t,s), & 0 \le t \le s \le \eta < 1, \\ G_{12}(t,s), & 0 \le s \le \min\{t,\eta\} < 1, \end{cases} \\ \underset{t\in[\eta,1]}{G_1(t,s)} = \begin{cases} G_{13}(t,s), & 0 \le \max\{t,\eta\} \le s \le 1, \\ G_{14}(t,s), & 0 < \eta \le s \le t \le 1, \end{cases} \end{cases}$$

$$
\begin{aligned}
G_{11}(t,s) =& \frac{1}{\Delta_1}\Big[\vartheta_1 t^{\gamma_1-1}(1-s)^{\gamma_1-\beta_1-1} - \zeta_1 t^{\gamma_1-1}(\eta-s)^{\gamma_1-\beta_1-1}\Big],\\
G_{12}(t,s) =& \frac{1}{\Delta_1}\Big[\vartheta_1 t^{\gamma_1-1}(1-s)^{\gamma_1-\beta_1-1} - \mathcal{N}_1(t-s)^{\gamma_1-1}\\
&- \zeta_1 t^{\gamma_1-1}(\eta-s)^{\gamma_1-\beta_1-1}\Big],\\
G_{13}(t,s) =& \frac{1}{\Delta_1}\Big[\vartheta_1 t^{\gamma_1-1}(1-s)^{\gamma_1-\beta_1-1}\Big],\\
G_{14}(t,s) =& \frac{1}{\Delta_1}\Big[\vartheta_1 t^{\gamma_1-1}(1-s)^{\gamma_1-\beta_1-1} - \mathcal{N}_1(t-s)^{\gamma_1-1}\Big], \text{ and}\\
\mathcal{N}_1 =& \vartheta_1 - \zeta_1\eta^{\gamma_1-\beta_1-1}.
\end{aligned}
$$

For details refer to [18].

Lemma 2.4. *Let* $\mathcal{N}_1 > 0$. *Then the Green's function* $G_1(t,s)$ *given in* (2.2) *satisfies*

(a) $G_1(t,s) \geq 0,\ \forall\ t,s \in [0,1]$,
(b) $G_1(t,s) \leq G_1(1,s),\ \forall\ t,s \in [0,1]$,*4*
(c) $G_1(t,s) \geq \tau^{\gamma_1-1}G_1(1,s),\ \forall\ t \in [\tau,1], s \in [0,1]$.

For details refer to [18].

Lemma 2.5. *Let* $\Delta_2 = \Gamma(\gamma_2)\mathcal{N}_2 \neq 0$. *If* $h_2(t) \in C[0,1]$, *then the fractional order differential equations*

$$D_{0^+}^{\gamma_2} w_2(t) + h_2(t) = 0,\ t \in (0,1), \tag{2.3}$$

satisfying the boundary conditions (1.4), *has a unique solution*

$$w_2(t) = \int_0^1 G_2(t,s)h_2(s)ds,$$

where $G_2(t,s)$ *is the Green's function for the problem* (2.3), (1.4) *and is given by*

(2.4)

$$
G_2(t,s) = \begin{cases} \underset{t\in[0,\eta]}{G_2(t,s)} = \begin{cases} G_{21}(t,s), & 0 \leq t \leq s \leq \eta < 1,\\ G_{22}(t,s), & 0 \leq s \leq \min\{t,\eta\} < 1,\end{cases}\\ \underset{t\in[\eta,1]}{G_2(t,s)} = \begin{cases} G_{23}(t,s), & 0 \leq \max\{t,\eta\} \leq s \leq 1,\\ G_{24}(t,s), & 0 < \eta \leq s \leq t \leq 1,\end{cases}\end{cases}
$$

$$
\begin{aligned}
G_{21}(t,s) =& \frac{1}{\Delta_2}\Big[\vartheta_2 t^{\gamma_2-1}(1-s)^{\gamma_2-\beta_2-1} - \zeta_2 t^{\gamma_2-1}(\eta-s)^{\gamma_2-\beta_2-1}\Big],\\
G_{22}(t,s) =& \frac{1}{\Delta_2}\Big[\vartheta_2 t^{\gamma_2-1}(1-s)^{\gamma_2-\beta_2-1} - \mathcal{N}_2(t-s)^{\gamma_2-1}\\
&\qquad - \zeta_2 t^{\gamma_2-1}(\eta-s)^{\gamma_2-\beta_2-1}\Big],\\
G_{23}(t,s) =& \frac{1}{\Delta_2}\Big[\vartheta_2 t^{\gamma_2-1}(1-s)^{\gamma_2-\beta_2-1}\Big],\\
G_{24}(t,s) =& \frac{1}{\Delta_2}\Big[\vartheta_2 t^{\gamma_2-1}(1-s)^{\gamma_2-\beta_2-1} - \mathcal{N}_2(t-s)^{\gamma_2-1}\Big], \text{ and}\\
\mathcal{N}_2 =& \vartheta_2 - \zeta_2\eta^{\gamma_2-\beta_2-1}.
\end{aligned}
$$

For details refer to [18].

Lemma 2.6. *Let $\mathcal{N}_2 > 0$. Then the Green's function $G_2(t,s)$ given in (2.4) satisfies*

(a) $G_2(t,s) \geq 0,\ \forall\, t,s \in [0,1]$,
(b) $G_2(t,s) \leq G_2(1,s),\ \forall\, t,s \in [0,1]$,
(c) $G_2(t,s) \geq \tau^{\gamma_2-1}G_2(1,s),\ \forall\, t \in [\tau,1], s \in [0,1]$.

For details refer to [18].

Let ψ be a nonnegative continuous functional on a cone P of the real Banach space B. Then for a positive real number c', the sets are defined as

$$P(\psi,c') = \big\{y \in P : \psi(y) < c'\big\} \text{ and } P_a = \big\{y \in P : \|y\| < a\big\}.$$

In obtaining multiple positive solutions of the fractional order boundary value problem (1.1)-(1.4), the following Avery–Henderson functional fixed point theorem is fundamental.

Theorem 2.7. [1] *Let P be a cone in the real Banach space B. Suppose α and γ are increasing, nonnegative continuous functionals on P and θ is nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some positive numbers c' and k, $\gamma(y) \leq \theta(y) \leq \alpha(y)$ and $\|y\| \leq k\gamma(y)$, for all $y \in \overline{P(\gamma,c')}$. Suppose that there exist positive numbers a' and b' with $a' < b' < c'$ such that $\theta(\lambda y) \leq \lambda\theta(y)$, for all $0 \leq \lambda \leq 1$ and $y \in \partial P(\theta,b')$. Further, let $T : \overline{P(\gamma,c')} \to P$ be a completely continuous operator such that*

(B1) *$\gamma(Ty) > c'$, for all $y \in \partial P(\gamma,c')$,*
(B2) *$\theta(Ty) < b'$, for all $y \in \partial P(\theta,b')$,*
(B3) *$P(\alpha,a') \neq \emptyset$ and $\alpha(Ty) > a'$ for all $y \in \partial P(\alpha,a')$.*

Then, T has at least two fixed points $y_1, y_2 \in \overline{P(\gamma, c')}$ such that $a' < \alpha(y_1)$ with $\theta(y_1) < b'$ and $b' < \theta(y_2)$ with $\gamma(y_2) < c'$.

3. Main Results

In this section we establish the existence of at least two positive solutions to a coupled system of fractional order boundary value problem (1.1)-(1.4) by using Avery–Henderson functional fixed point theorem. We also establish the existence of at least $2m$ positive solutions to the problem (1.1)-(1.4) for an arbitrary positive integer m.

Consider the Banach space $\mathcal{B} = \mathcal{E} \times \mathcal{E}$, where $\mathcal{E} = \Big\{w_1 : w_1 \in C[0,1]\Big\}$ endowed with the norm $\|(w_1, w_2)\| = \|w_1\|_0 + \|w_2\|_0$, for $(w_1, w_2) \in \mathcal{B}$ and the norm is defined as $\|w_1\|_0 = \max_{t\in[0,1]} |w_1(t)|$.

Define a cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \Big\{(w_1, w_2) \in \mathcal{B} : w_1(t) \geq 0,\ w_2(t) \geq 0,\ t \in [0,1] \text{ and } \min_{t\in I} \sum_{i=1}^{2} w_i(t) \geq \eta \|(w_1, w_2)\|\Big\},$$

where $I = [\tau, 1]$ and

$$\eta = \min\Big\{\tau^{\gamma_1 - 1}, \tau^{\gamma_2 - 1}\Big\}. \tag{3.1}$$

Let

$$\mathcal{S}_1 = \min\left\{\int_0^1 G_1(1,s)ds, \int_0^1 G_2(1,s)ds\right\}, \text{ and}$$

$$\mathcal{S}_2 = \max\left\{\int_{s\in I} \eta G_1(1,s)ds, \int_{s\in I} \eta G_2(1,s)ds\right\}.$$

Let us define the nonnegative, increasing, continuous functionals γ, θ and α on the cone $\mathcal{P}$ by

$$\gamma(w_1, w_2) = \min_{t\in I} \Big\{|w_1(t)| + |w_2(t)|\Big\},$$

$$\theta(w_1, w_2) = \max_{t\in I} \Big\{|w_1(t)| + |w_2(t)|\Big\}, \text{ and}$$

$$\alpha(w_1, w_2) = \max_{t\in[0,1]} \Big\{|w_1(t)| + |w_2(t)|\Big\}.$$

For any $(w_1, w_2) \in \mathcal{P}$, we have

$$\gamma(w_1, w_2) \leq \theta(w_1, w_2) \leq \alpha(w_1, w_2) \tag{3.2}$$

and

$$\begin{cases} \|(w_1, w_2)\| \leq \dfrac{1}{\eta} \min\limits_{t \in I} \Big\{|w_1(t)| + |w_2(t)|\Big\} = \dfrac{1}{\eta}\gamma(w_1, w_2) \\ \qquad\qquad\qquad\qquad\qquad \leq \dfrac{1}{\eta}\theta(w_1, w_2) \leq \dfrac{1}{\eta}\alpha(w_1, w_2). \end{cases} \tag{3.3}$$

3.1. Existence of at least two positive solutions

THEOREM 3.1. *Suppose there exist $0 < \Upsilon' < \Phi' < \Psi'$ such that f_i, for $i = 1, 2$ satisfies the following conditions:*

(A_1) $f_i\Big(t, w_1, w_2\Big) > \frac{\Psi'}{2\mathcal{S}_2}$, *$t \in I$ and $w_1, w_2 \in \left[\Psi', \frac{\Psi'}{\eta}\right]$,*

(A_2) $f_i\Big(t, w_1, w_2\Big) < \frac{\Phi'}{2\mathcal{S}_1}$, *$t \in [0, 1]$ and $w_1, w_2 \in \left[0, \frac{\Phi'}{\eta}\right]$,*

(A_3) $f_i\Big(t, w_1, w_2\Big) > \frac{\Upsilon'}{2\mathcal{S}_2}$, *$t \in I$ and $w_1, w_2 \in \left[\Upsilon', \frac{\Upsilon'}{\eta}\right]$.*

Then the fractional order boundary value problem (1.1)-(1.4) *has at least two positive solutions (x_1, y_1) and (x_2, y_2) such that*

$$\Upsilon' < \max_{t \in [0,1]} \Big\{|x_1(t)| + |y_1(t)|\Big\} \textit{ with } \max_{t \in I} \Big\{|x_1(t)| + |y_1(t)|\Big\} < \Phi',$$
$$\Phi' < \max_{t \in I} \Big\{|x_2(t)| + |y_2(t)|\Big\} \textit{ with } \min_{t \in I} \Big\{|x_2(t)| + |y_2(t)|\Big\} < \Psi'.$$

Proof. Let $T_1, T_2 : \mathcal{P} \to \mathcal{E}$ and $T : \mathcal{P} \to \mathcal{B}$ be the operators defined by

$$\begin{cases} T_1\big(w_1, w_2\big)(t) = \displaystyle\int_0^1 G_1(t, s) f_1\Big(s, w_1(s), w_2(s)\Big) ds, \\ T_2\big(w_1, w_2\big)(t) = \displaystyle\int_0^1 G_2(t, s) f_2\Big(s, w_1(s), w_2(s)\Big) ds, \end{cases}$$

and

$$T\big(w_1, w_2\big)(t) = \Big(T_1(w_1, w_2)(t), T_2(w_1, w_2)(t)\Big), \text{ for } (w_1, w_2) \in \mathcal{B}. \tag{3.4}$$

It is obvious that a fixed point of the operator T is the solution of the fractional order boundary value problem (1.1)-(1.4). We seek two fixed points of the operator T. Primarily we show that the operator T

is a self map. Let $(w_1, w_2) \in \mathcal{P}$. Clearly $T_i\big(w_1, w_2\big)(t) \geq 0$ for $i = 1, 2$ and $t \in [0, 1]$. Also for $(w_1, w_2) \in \mathcal{P}$,

$$\|T_1\big(w_1, w_2\big)\|_0 \leq \int_0^1 G_1(1, s) f_1\Big(s, w_1(s), w_2(s)\Big) ds,$$

and

$$\begin{aligned}
\min_{t \in I} T_1\big(w_1, w_2\big)(t) &= \min_{t \in I} \int_0^1 G_1(t, s) f_1\Big(s, w_1(s), w_2(s)\Big) ds \\
&\geq \eta \int_0^1 G_1(1, s) f_1\Big(s, w_1(s), w_2(s)\Big) ds \\
&\geq \eta \|T_1\big(w_1, w_2\big)\|_0.
\end{aligned}$$

Similarly $\min_{t \in I} T_2\big(w_1, w_2\big)(t) \geq \eta \|T_2\big(w_1, w_2\big)\|_0$. Therefore,

$$\begin{aligned}
\min_{t \in I} \Big\{ T_1\big(w_1, w_2\big)(t) + T_2\big(w_1, w_2\big)(t) \Big\} & \\
&\geq \eta \|T_1\big(w_1, w_2\big)\|_0 + \eta \|T_2\big(w_1, w_2\big)\|_0 \\
&= \eta \Big(\|T_1\big(w_1, w_2\big)\|_0 + \|T_2\big(w_1, w_2\big)\|_0 \Big) \\
&= \eta \Big\| \Big(T_1\big(w_1, w_2\big), T_2\big(w_1, w_2\big) \Big) \Big\| \\
&= \eta \|T\big(w_1, w_2\big)\|.
\end{aligned}$$

Hence $T\big(w_1, w_2\big) \in \mathcal{P}$ and so $T : \mathcal{P} \to \mathcal{P}$. Further the operator T is completely continuous by an application of Arzela–Ascoli theorem. From (3.2) and (3.3), for each $(w_1, w_2) \in \mathcal{P}$, $\gamma\big(w_1, w_2\big) \leq \theta\big(w_1, w_2\big) \leq \alpha\big(w_1, w_2\big)$ and $\|\big(w_1, w_2\big)\| \leq \dfrac{1}{\eta} \gamma\big(w_1, w_2\big)$. Also for any $\lambda \in [0, 1]$ and $\big(w_1, w_2\big) \in \mathcal{P}$,

$$\begin{cases}
\theta\Big(\lambda(w_1, w_2)\Big) = \max_{t \in I} \Big\{ \lambda\Big(|w_1(t)| + |w_2(t)| \Big) \Big\} \\
\qquad\qquad\quad = \lambda \max_{t \in I} \Big\{ |w_1(t)| + |w_2(t)| \Big\} = \lambda \theta\big(w_1, w_2\big).
\end{cases}$$

It is clear that $\theta\big(0, 0\big) = 0$. We now show that the remaining conditions of Theorem 2.7 are satisfied.

Primarily we shall verify that condition $(B1)$ of Theorem 2.7 is satisfied. Since $(w_1, w_2) \in \partial\mathcal{P}\big(\gamma, \Psi'\big)$, from (3.3) we have that

$$\Psi' = \min_{t \in I} \Big\{ |w_1(t)| + |w_2(t)| \Big\} \leq \|(w_1, w_2)\| \leq \frac{\Psi'}{\eta}.$$

Then

$$\begin{aligned}
\gamma\Big(T\big(w_1,w_2\big)(t)\Big) &= \min_{t\in I}\sum_{i=1}^{2}\bigg[\int_0^1 G_i(t,s)f_i\Big(s,w_1(s),w_2(s)\Big)ds\bigg]\\
&\geq \sum_{i=1}^{2}\bigg[\int_{s\in I}\eta G_i(1,s)f_i\Big(s,w_1(s),w_2(s)\Big)ds\bigg]\\
&> \sum_{i=1}^{2}\bigg[\int_{s\in I}\eta G_i(1,s)\Big(\frac{\Psi'}{2\mathcal{S}_2}\Big)ds\bigg]\\
&= \sum_{i=1}^{2}\bigg[\frac{\Psi'}{2\mathcal{S}_2}\int_{s\in I}\eta G_i(1,s)ds\bigg]\\
&= \frac{\Psi'}{2}+\frac{\Psi'}{2}=\Psi',
\end{aligned}$$

using the condition (A_1).

Now we shall show that condition $(B2)$ of Theorem 2.7 is satisfied. Since $(w_1,w_2)\in\partial\mathcal{P}\big(\theta,\Phi'\big)$, from (3.3) we have that $0\leq\big(w_1(t),w_2(t)\big)\leq\|\big(w_1,w_2\big)\|\leq\dfrac{\Phi'}{\eta}$, for $t\in[0,1]$. Thus

$$\begin{aligned}
\theta\Big(T\big(w_1,w_2\big)(t)\Big) &= \max_{t\in I}\sum_{i=1}^{2}\bigg[\int_0^1 G_i(t,s)f_i\Big(s,w_1(s),w_2(s)\Big)ds\bigg]\\
&\leq \sum_{i=1}^{2}\bigg[\int_0^1 G_i(t,s)f_i\Big(s,w_1(s),w_2(s)\Big)ds\bigg]\\
&< \sum_{i=1}^{2}\bigg[\int_0^1 G_i(1,s)\Big(\frac{\Phi'}{2\mathcal{S}_1}\Big)ds\bigg]\\
&= \sum_{i=1}^{2}\bigg[\frac{\Phi'}{2\mathcal{S}_1}\int_0^1 G_i(1,s)ds\bigg]\\
&= \frac{\Phi'}{2}+\frac{\Phi'}{2}=\Phi',
\end{aligned}$$

by the condition (A_2).

Finally utilizing the condition (A_3), we shall prove that condition $(B3)$ of Theorem 2.7 is satisfied. Since $(0,0)\in\mathcal{P}$ and $\Upsilon'>0$, we get $\mathcal{P}\big(\alpha,\Upsilon'\big)\neq\phi$. Since $(w_1,w_2)\in\partial\mathcal{P}\big(\alpha,\Upsilon'\big)$,

$$\Upsilon' = \max_{t\in[0,1]} \Big\{|w_1(t)| + |w_2(t)|\Big\} \le \|(w_1, w_2)\| \le \frac{\Upsilon'}{\eta}.$$

Therefore,

$$\begin{aligned}
\alpha\Big(T\big(w_1, w_2\big)(t)\Big) &= \max_{t\in[0,1]} \sum_{i=1}^{2} \left[\int_0^1 G_i(t,s) f_i\Big(s, w_1(s), w_2(s)\Big) ds\right] \\
&\ge \sum_{i=1}^{2} \left[\int_0^1 \eta G_i(t,s) f_i\Big(s, w_1(s), w_2(s)\Big) ds\right] \\
&\ge \sum_{i=1}^{2} \left[\int_{s\in I} \eta G_i(1,s) f_i\Big(s, w_1(s), w_2(s)\Big) ds\right] \\
&> \sum_{i=1}^{2} \left[\int_{s\in I} \eta G_i(1,s) \Big(\frac{\Upsilon'}{2\mathcal{S}_2}\Big) ds\right] \\
&= \sum_{i=1}^{2} \left[\frac{\Upsilon'}{2\mathcal{S}_2} \int_{s\in I} \eta G_i(1,s) ds\right] \\
&= \frac{\Upsilon'}{2} + \frac{\Upsilon'}{2} = \Upsilon'.
\end{aligned}$$

Thus all the conditions of Theorem 2.7 hold. Therefore the fractional order boundary value problem (1.1)-(1.4) has at least two positive solutions (x_1, y_1), $(x_2, y_2) \in \overline{\mathcal{P}(\gamma, c')}$. This completes the proof. □

3.2. Existence of $2m$ positive solutions

THEOREM 3.2. *Let m be an arbitrary positive integer. Suppose that there exist numbers $\Upsilon_r (r = 1, 2, \cdots, m+1)$ and $\Phi_s (s = 1, 2, \cdots, m)$ with $0 < \Phi_1 < \Phi_1 < \Upsilon_2 < \Phi_2 < \cdots < \Upsilon_m < \Phi_m < \Upsilon_{m+1}$ such that f_i, for $i = 1, 2$ satisfies the following conditions:*

$$(A_4) \begin{cases} f_i\Big(t, w_1, w_2\Big) > \dfrac{\Upsilon_r}{2\mathcal{S}_2}, \\ \qquad t \in I \text{ and } w_1, w_2 \in \left[\Upsilon_r, \dfrac{\Upsilon_r}{\eta}\right],\ r = 1, 2, \cdots, m+1, \end{cases}$$

$$(A_5) \begin{cases} f_i\Big(t, w_1, w_2\Big) < \dfrac{\Phi_s}{2\mathcal{S}_1}, \\ \qquad t \in [0,1] \text{ and } w_1, w_2 \in \left[0, \dfrac{\Phi_s}{\eta}\right],\ s = 1, 2, \cdots, m. \end{cases}$$

Then the fractional order boundary value problem (1.1)-(1.4) *has at least* $2m$ *positive solutions in* $\overline{\mathcal{P}}_{\Upsilon_{m+1}}$.

Proof. We use mathematical induction on m. We know from the conditions (A_4) and (A_5) that $T : \overline{\mathcal{P}}_{\Upsilon_2} \to \mathcal{P}_{\Upsilon_2}$, then it follows from Avery–Henderson functional fixed point theorem that the fractional order boundary value problem (1.1)-(1.4) has at least two positive solutions in $\overline{\mathcal{P}}_{\Upsilon_2}$ for $m = 1$.

Next we assume that this conclusion holds for $m = l$. In order to prove that this conclusion holds for $m = l + 1$. Suppose that there exist numbers $\Upsilon_r(r = 1, 2, \cdots, l + 2)$ and $\Phi_s(s = 1, 2, \cdots, l + 1)$ with $0 < \Upsilon_1 < \Phi_1 < \Upsilon_2 < \Phi_2 < \cdots < a_{l+1} < \Phi_{l+1} < \Upsilon_{l+2}$ such that f_i, for $i = 1, 2$ satisfies the following conditions:

$$\tag{3.5} \begin{cases} f_i\Big(t, w_1, w_2\Big) > \dfrac{\Upsilon_r}{2\mathcal{S}_2},\ t \in I \text{ and } w_1, w_2 \in \left[\Upsilon a_r, \dfrac{\Upsilon_r}{\eta}\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad r = 1, 2, \cdots, l + 2, \end{cases}$$

$$\tag{3.6} \begin{cases} f_i\Big(t, w_1, w_2\Big) < \dfrac{\Phi_s}{2\mathcal{S}_1},\ t \in [0, 1] \text{ and } w_1, w_2 \in \left[0, \dfrac{\Phi_s}{\eta}\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad s = 1, 2, \cdots, l + 1. \end{cases}$$

By assumption, the fractional order boundary value problem (1.1)-(1.4) has at least $2l$ positive solutions (x_i^*, y_i^*) $(i = 1, 2, \cdots, 2l)$ in $\overline{\mathcal{P}}_{\Upsilon_{l+1}}$. At the same time, it follows from Theorem 3.1, (3.5) and (3.6) that the fractional order boundary value problem (1.1)-(1.4) has at least two positive solutions (x_1, y_1), (x_2, y_2) in $\overline{\mathcal{P}}_{\Upsilon_{l+2}}$ such that $\Upsilon_{l+1} < \alpha\big(x_1, y_1\big)$ with $\theta\big(x_1, y_1\big) < \Phi_{l+1}$ and $\Phi_{l+1} < \theta\big(x_2, y_2\big)$ with $\gamma\big(x_2, y_2\big) < \Upsilon_{l+2}$. Obviously (x_1, y_1) and (x_2, y_2) are different from $(x_i^*, y_i^*)(i = 1, 2, \cdots, 2l)$ in $\overline{P}_{\Upsilon_{l+1}}$. Therefore the fractional order boundary value problem (1.1)-(1.4) has at least $2l + 2$ positive solutions in $\overline{P}_{\Upsilon_{l+2}}$, which shows that this conclusion also holds for $m = l + 1$. This completes the proof. □

4. Example

In this section we give an application of the result established in earlier section.

Consider the system of fractional order boundary value problem,

$$\tag{4.1} D_{0^+}^{2.8} w_1(t) + f_1\big(t, w_1, w_2\big) = 0,\ t \in (0, 1),$$

$$D_{0^+}^{2.9} w_2(t) + f_2\big(t, w_1, w_2\big) = 0,\ t \in (0,1), \tag{4.2}$$

$$w_1(0) = 0,\ w_1'(0) = 0,\ 9D_{0^+}^{1.7} w_1(1) - \frac{7}{2} D_{0^+}^{1.7} w_1\Big(\frac{1}{2}\Big) = 0, \tag{4.3}$$

$$w_2(0) = 0,\ w_2'(0) = 0,\ 9D_{0^+}^{1.7} w_2(1) - \frac{7}{2} D_{0^+}^{1.7} w_2\Big(\frac{1}{2}\Big) = 0, \tag{4.4}$$

where

$$f_1\big(t, w_1, w_2\big) = \frac{786(w_1 + w_2)^2}{73(w_1 + w_2)^2 + 49932},$$
$$f_2\big(t, w_1, w_2\big) = \frac{794(w_1 + w_2)^2}{75(w_1 + w_2)^2 + 52125}.$$

By direct calculations, one can determine $\eta = 0.0825$, $\mathcal{S}_1 = 0.4502$ and $\mathcal{S}_2 = 0.02628$. Choosing $\Upsilon' = 0.002, \Phi' = 0.5$ and $\Psi' = 4$, then $0 < \Upsilon' < \Phi' < \Psi'$ and f_i, for $i = 1, 2$ satisfies

- $f_i\big(t, w_1, w_2\big) > 76.1035 = \dfrac{\Psi'}{2\mathcal{S}_2},\ t \in \Big[\dfrac{1}{4}, \dfrac{3}{4}\Big]$ and $w_1, w_2 \in [4, 48.4848]$,
- $f_i\big(t, w_1, w_2\big) < 0.5553 = \dfrac{\Phi'}{2\mathcal{S}_1},\ t \in [0,1]$ and $w_1, w_2 \in [0, 6.0606]$,
- $f_i\big(t, w_1, w_2\big) > 0.0381 = \dfrac{\Upsilon'}{2\mathcal{S}_2},\ t \in \Big[\dfrac{1}{4}, \dfrac{3}{4}\Big]$ and $w_1, w_2 \in [0.002, 0.0012]$.

Then all the conditions of Theorem 3.1 hold. Thus with Theorem 3.1, the fractional order boundary value problem (4.1)-(4.4) has at least two positive solutions.

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