JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **31**, No. 2, May 2018
http://dx.doi.org/10.14403/jcms.2018.31.1.231

ON HARDY AND PÓLYA-KNOPP'S INEQUALITIES

Ern Gun Kwon* and Min Ju Jo**

Abstract. Hardy's inequality is refined non-trivially as the form

$$\int_0^\infty \left\{\frac{1}{x}\int_0^x f(t)dt\right\}^p dx \leq Q_f \times \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx$$

for some $Q_f : 0 \leq Q_f \leq 1$. Pólya-Knopp's inequality is also refined by the similar form.

1. Introduction

The celebrated Hardy's inequality says

Theorem A ([2]). If $p > 1$, $f(x) \geq 0$, and $F(x) = \int_0^x f(t)dt$, then

$$\int_0^\infty \left(\frac{F}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p dx \tag{1.1}$$

unless $f \equiv 0$. The constant is the best possible.

There have been various proofs, generalizations and refinements of (1.1). This paper is on a refinement of (1.1). We know that we can not improve (1.1) by a smaller bounding constant independent of f. Our result says that there is a non-trivial constant Q_f, $0 \leq Q_f \leq 1$, depending on f such that (1.1) still remains true when Q_f is multiplied on the right hand side.

In this paper, all functions under consideration are real valued measurable and assumed to have, for notational convenience and without loss of generality, strictly positive finite L^p norms. Our results are described as follows.

Received January 08, 2018; Accepted March 13, 2018.
2010 Mathematics Subject Classification: Primary 26D15.
Key words and phrases: Minkowski's inequality, Hardy's inequality, Pólya-Knopp's inequality.
*This work was supported by a grant from 2017 Research Fund of Andong National University.
Correspondence should be addressed to Ern Gun Kwon, egkwon@anu.ac.kr.

THEOREM 1.1. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. If $f(x)$ is a positive function and $e(x)$ is a function satisfying $|e(x) - e(y)| \leq 1$ for all x, y, then*

$$\int_0^\infty \left\{ \frac{1}{x} \int_0^x f(t)dt \right\}^p dx \leq Q_f \times \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x)dx, \tag{1.2}$$

where $Q_f = Q(f, e, p)$ equals

$$\left[1 - \left\{ \frac{\int_0^\infty e(x) \left(\frac{1}{x} \int_0^x \widetilde{f}(t)dt \right)^p dx}{\int_0^\infty \left(\frac{1}{x} \int_0^x \widetilde{f}(t)dt \right)^p dx} - \frac{\int_0^\infty e(x) \widetilde{f}^p(x)dx}{\int_0^\infty \widetilde{f}^p(x)dx} \right\}^2 \right]^{\min\{\frac{1}{2}, \frac{p}{2q}\}}$$

and $\widetilde{f}$ is the decreasing rearrangement of f.

By a parallel pattern we have the following refinement, which reduces to Pólya-Knopp's inequality (see [2], [4]) when $Q_0 = 1$.

THEOREM 1.2. *If $f(x)$ is a positive function and $e(x)$ is a function satisfying $|e(x) - e(y)| \leq 1$ for all x, y, then*

$$\int_0^\infty \exp \left\{ \frac{1}{x} \int_0^x \ln f(t)dt \right\} dx \leq Q_0 \times e \int_0^\infty f(x)dx, \tag{1.3}$$

where $Q_0 = Q_0(f, e)$ equals

$$\left[1 - \left\{ \frac{\int_0^\infty e(x) \exp \left(\frac{1}{x} \int_0^x \ln \widetilde{f}(t)dt \right) dx}{\int_0^\infty \exp \left(\frac{1}{x} \int_0^x \ln \widetilde{f}(t)dt \right) dx} - \frac{\int_0^\infty e(x) \widetilde{f}(x)dx}{\int_0^\infty \widetilde{f}(x)dx} \right\}^2 \right]^{\frac{1}{2}}$$

and $\widetilde{f}$ is the decreasing rearrangement of f.

Proofs will be given in Section 3 after preliminary lemmas in Section 2. We refer to [1, 2] and [5] for basic inequalities.

2. Lemmas

2.1. Hölder's and Minkowski's inequalities

The following lemmas refine Hölder's inequality and Minkowski's inequality.

LEMMA 2.1. [3] *Let E be a measurable set, let $f(x)$ and $g(x)$ be nonnegative measurable functions with $\int_E f^p(x)dx < \infty$, $\int_E g^q(x)dx <$*

∞, and let $e(x)$ be a measurable function with $1 - e(x) + e(y) \geq 0$. If $p \geq q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_E f(x)g(x)dx \leq \left(\int_E g^q(x)dx\right)^{\frac{1}{q}-\frac{1}{p}} \times \left[\left(\int_E f^p(x)dx \int_E g^q(x)dx\right)^2 - H^2\right]^{\frac{1}{2p}},$$

where

$$H = \int_E f^p(x)e(x)dx \int_E g^q(x)dx - \int_E f^p(x)dx \int_E g^q(x)e(x)dx.$$

LEMMA 2.2. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. If $f_j(x)$, $j = 1, 2$, are positive functions and $e_j(x)$, $j = 1, 2$, are functions satisfying $|e_j(x) - e_j(y)| \leq 1$ for all x, y, then*

$$\left\{\int_0^\infty (f_1 + f_2)^p(x)dx\right\}^{\frac{1}{p}} \leq Q_1 \left\{\int_0^\infty f_1^p(x)dx\right\}^{\frac{1}{p}} + Q_2 \left\{\int_0^\infty f_2^p(x)dx\right\}^{\frac{1}{p}},$$

where $Q_j = Q(f_j, e_j, p)$ equals

$$\left[1 - \left\{\frac{\int_0^\infty e_j(x)(f_1 + f_2)^p(x)dx}{\int_0^\infty (f_1 + f_2)^p(x)dx} - \frac{\int_0^\infty e_j(x) f_j^p(x)dx}{\int_0^\infty f_j^p(x)dx}\right\}^2\right]^{\min\{\frac{1}{2p}, \frac{1}{2q}\}}.$$

See [6], [7], [8], and references therein for further refinements of Hölder's inequality of similar types.

2.2. Proof of Lemma 2.2

Lemma 2.2 is a direct consequence of Lemma 2.1: By Lemma 2.1

$$\int_0^\infty (f_1 + f_2)^{p-1}(x)\ f_j(x)\ dx \leq Q_j \left\{\int_0^\infty (f_1 + f_2)^p(x)dx\right\}^{\frac{1}{q}} \left(\int_0^\infty f_j^p(x)dx\right)^{\frac{1}{p}},$$

so that

$$\int_0^\infty (f_1 + f_2)^p(x)dx = \sum_{j=1}^2 \int_0^\infty (f_1 + f_2)^{p-1}(x)\ f_j(x)dx$$
$$\leq \left\{\int_0^\infty (f_1 + f_2)^p\,(x)dx\right\}^{\frac{1}{q}} \sum_{j=1}^2 Q_j \left\{\int_0^\infty f_j^p(x)dx\right\}^{\frac{1}{p}},$$

whence the result follows.

3. Proofs of Main Results

3.1. Proof of Theorem 1.1

Let $\widetilde{f}$ be the decreasing rearrangement of f and let $F(x) = \int_0^x \widetilde{f}(t)dt$. Then since $F'(x) = \widetilde{f}(x)$ is a decreasing function, $F''(x) = (\widetilde{f})'(x) < 0$. So F is a concave function.

Fix $k > 1$ and $0 < A < \infty$ for a moment. Then by the concavity of F

$$F'(x) \geq \frac{F(kx) - F(x)}{kx - x},$$

that is

$$F(kx) \leq F(x) + (k-1)xF'(x),$$

whence

$$\begin{aligned}\left\{\frac{1}{k}\int_0^A x^{-p}F^p(x)dx\right\}^{\frac{1}{p}} &= \left\{\frac{1}{k}\int_0^{\frac{A}{k}} k^{-p}y^{-p}F^p(ky)kdy\right\}^{\frac{1}{p}} \\ &\leq \frac{1}{k}\left\{\int_0^A x^{-p}F^p(kx)dx\right\}^{\frac{1}{p}} \\ &\leq \frac{1}{k}\left\{\int_0^A x^{-p}\left\{F(x) + (k-1)xF'(x)\right\}^p dx\right\}^{\frac{1}{p}}.\end{aligned}$$

Applying Lemma 2.2, we have

$$\begin{aligned}\left\{\frac{1}{k}\int_0^A x^{-p}F^p(x)dx\right\}^{\frac{1}{p}} \leq &\frac{1}{k}\left\{\int_0^A x^{-p}F^p(x)dx\right\}^{\frac{1}{p}} Q_1 \\ &+ \frac{1}{k}\left\{\int_0^A x^{-p}\left\{(k-1)xF'(x)\right\}^p dx\right\}^{\frac{1}{p}} Q_2,\end{aligned}$$

where Q_1 and Q_2 equals respectively

$$\left[1 - \left\{\frac{\int_0^A e_1(x)\left(G_1 + G_2\right)^p(x)\, dx}{\int_0^A \left(G_1 + G_2\right)^p(x)\, dx} - \frac{\int_0^A e_1(x)G_1^p(x)\, dx}{\int_0^A G_1^p(x)\, dx}\right\}^2\right]^\delta$$

and

$$\left[1-\left\{\frac{\int_0^A e_2(x)\,(G_1+G_2)^p\,(x)\;dx}{\int_0^A (G_1+G_2)^p\,(x)\;dx}-\frac{\int_0^A e_2(x)G_2^p(x)\;dx}{\int_0^A G_2^p(x)\;dx}\right\}^2\right]^\delta$$

$$=\left[1-\left\{\frac{\int_0^A e_2(x)\,(G_1+G_2)^p\,(x)\;dx}{\int_0^A (G_1+G_2)^p\,(x)\;dx}-\frac{\int_0^A e_2(x)\,(F'(x))^p\;\;dx}{\int_0^A (F'(x))^p\;\;dx}\right\}^2\right]^\delta$$

with

$$G_1(x)=\frac{F(x)}{x},\;\;G_2(x)=(k-1)F'(x),\;\;\delta=\min\left\{\frac{1}{2p},\frac{1}{2q}\right\}.$$

Equivalently, we have

$$\left\{\int_0^A x^{-p}F^p(x)dx\right\}^{\frac{1}{p}}\le\frac{(k-1)Q_2}{k\left(k^{-\frac{1}{p}}-k^{-1}Q_1\right)}\left\{\int_0^A (F'(x))^p dx\right\}^{\frac{1}{p}}.$$

Now, let $k\to 1$, then

$$\frac{k-1}{k\left(k^{-\frac{1}{p}}-k^{-1}Q_1\right)}\le\frac{k-1}{k\left(k^{-\frac{1}{p}}-k^{-1}\right)}\to\frac{p}{p-1}$$

and

$$Q_2\to\left[1-\left\{\frac{\int_0^A e_2(x)x^{-p}F^p(x)dx}{\int_0^A x^{-p}F^p(x)dx}-\frac{\int_0^A e_2(x)(F'(x))^p dx}{\int_0^A (F'(x))^p dx}\right\}^2\right]^\delta,$$

whence

$$\left\{\int_0^A x^{-p}F^p(x)dx\right\}^{\frac{1}{p}}\le\frac{p}{p-1}\left\{\int_0^A (F'(x))^p dx\right\}^{\frac{1}{p}}\times$$
$$\left[1-\left\{\frac{\int_0^A e_2(x)x^{-p}F^p(x)dx}{\int_0^A x^{-p}F^p(x)dx}-\frac{\int_0^A e_2(x)(F'(x))^p dx}{\int_0^A (F'(x))^p dx}\right\}^2\right]^\delta.$$

By letting $A\to\infty$, we thus have

$$\left\{\int_0^\infty\left(\frac{F(x)}{x}\right)^p dx\right\}^{\frac{1}{p}}\le\frac{p}{p-1}\left\{\int_0^\infty (F'(x))^p dx\right\}^{\frac{1}{p}}\times$$
$$\left[1-\left\{\frac{\int_0^\infty e_2(x)\left(\frac{F(x)}{x}\right)^p dx}{\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx}-\frac{\int_0^\infty e_2(x)(F'(x))^p dx}{\int_0^\infty (F'(x))^p dx}\right\}^2\right]^\delta.$$

Taking p-power and recovering $F(x) = \int_0^x \widetilde{f}(t)dt$, we finally obtain

$$\begin{aligned}\int_0^\infty \left\{\frac{1}{x}\int_0^x f(t)dt\right\}^p dx &\le \int_0^\infty \left\{\frac{1}{x}\int_0^x \widetilde{f}(t)dt\right\}^p dx\\ &\le \left(\frac{p}{p-1}\right)^p Q_f \int_0^\infty \widetilde{f}^p(x)dx\\ &= \left(\frac{p}{p-1}\right)^p Q_f \int_0^\infty f^p(x)dx,\end{aligned}$$

where

$$Q_f = \left[1 - \left\{\frac{\int_0^\infty e_2(x)\left(\frac{1}{x}\int_0^x \widetilde{f}(t)dt\right)^p dx}{\int_0^\infty \left\{\frac{1}{x}\int_0^x \widetilde{f}(t)dt\right\}^p dx} - \frac{\int_0^\infty e_2(x)\widetilde{f}^p(x)dx}{\int_0^\infty \widetilde{f}^p(x)dx}\right\}^2\right]^{\delta p}.$$

The proof of Theorem 1.1 is complete.

3.2. Proof of Theorem 1.2

Replace f by $f^{\frac{1}{p}}$ in (1.2) and let $p \to \infty$, then we obtain (1.3) by monotone convergence theorem.

3.3. Remarks

By somewhat different way $\widetilde{f}$ in Theorem 1.1 and Theorem 1.2 can be replaced by f also. Discrete version of (1.2) and (1.3) are also possible. Refinements of type (1.2) and (1.3) will be considered more expansively in a forthcoming paper of the present authors.

References

[1] E. F. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag, Berlin, 1983.

[2] G. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, 1952.

[3] K. Hu, *On an inequality and its applications*, Sci. Sinica, **24** (1981), no. 8, 1047-1055.

[4] K. Knopp, *Über reihen mit positiven gliedern*, J. London Math. Soc. **3** (1928), 205-211.

[5] E. G. Kwon, *On Carlesons inequality*, Jour. Inequal. Appl. **2018** (2018), 2018:91, DOI10.1186/s13660-018-1682-2.

[6] E. G. Kwon and E. K. Bae, *On a continuous form of Hölder inequality*, Jour. Math. Anal. Appl. **343** (2008), 585-592.

[7] E. G. Kwon and J. E. Bae, *On a generalized Hölder inequality*, Jour. Inequal. Appl. **2015** (2015), 2015:88 DOI10.1186/s13660-015-0612-9.

[8] E. G. Kwon and J. E. Bae, *On a refined Hölder's inequality*, Jour. Math. Inequal. **10** (2016), 261-268.

*
Department of Mathematics Education
Andong National University
Andong 36729, Republic of Korea
E-mail: egkwon@anu.ac.kr

**
Department of Mathematics
Graduate School, Andong National University
Andong 36729, Republic of Korea
E-mail: fatcat@hanmail.net