KYUNGPOOK Math. J. 58(2018), 149-165
https://doi.org/10.5666/KMJ.2018.58.1.149
pISSN 1225-6951 eISSN 0454-8124

Simons′ Type Formula for Kaehlerian Slant Submanifolds in Complex Space Forms

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Abstract. A. Bejancu [2] was the first who instigated the new concept in differential geometry, i.e., CR-submanifolds. On the other hand, CR-submanifolds were generalized by B. Y. Chen [7] as slant submanifolds. Further, he gave the notion of a Kaehlerian slant submanifold as a proper slant submanifold.

This article has two objectives. For the first objective, we derive Simons′ type formula for a minimal Kaehlerian slant submanifold in a complex space form. Then, by applying this formula, we give a complete classification of a minimal Kaehlerian slant submanifold in a complex space form and also obtain its some immediate consequences. The second objective is to prove some results about semi-parallel submanifolds.

1. Introduction

In 1990, B. Y. Chen defined the slant submanifolds in complex manifolds as a natural generalization of holomorphic and totally real submanifolds [7]. Examples of slant submanifolds of $\mathcal{C}^2$ and $\mathcal{C}^4$ were given by B. Y. Chen and Tazawa [8, 9]. As far as contact geometry is concerned, this study was extended by A. Lotta [14] in almost contact geometry and further studied by Cabrerizo et al. in 2000. The theory of slant submanifolds became very rich area of research for geometers. B. Sahin, S. Maeda, J. L. Cabrerizo, A. Carriazo, L. M. Fernandez, M. Fernandez and many more geometers studied slant submanifolds in different ambient spaces [4, 5, 6, 10, 16, 19, 20, 21].

In 1901, the notion of totally geodesic submanifolds was introduced by J. Hadamard. He defined (totally) geodesic submanifolds of a Riemannian manifold as submanifolds such that each geodesic of them is a geodesic of the ambient space. This condition is equivalent

* Corresponding Author.
Received May 31, 2017; accepted February 8, 2018.
2010 Mathematics Subject Classification: 53C15, 53B25, 53C40.
Key words and phrases: slant submanifolds, semi-parallel submanifolds, Kaehlerian slant submanifolds, Kaehler manifolds, complex space forms.

to the vanishing on the second fundamental form on the submanifolds. Totally geodesic submanifolds are the simplest and the most fundamental submanifolds of Riemannian manifolds.

In 1981, A. Bejancu, M. Kon and K. Yano derived Simons′ type formula for CR-submanifolds of a complex space form [3]. Many authors derived Simons′ type formula for different submanifolds in ambient spaces [11, 12, 26, 27].

Our work is structured as follows : in *Section 2*, we recall some fundamental formulas for submanifolds of a Kaehler manifold, and in particular for slant submanifolds of a Kaehler manifold, and then we give the basic definitions. In *Section 3*, we first prove a preliminary lemma and quote some important lemmas for later use. Next, we construct Simons′ type formula for a minimal Kaehlerian slant submanifold of a complex space form. In *Section 4*, we classify a minimal Kaehlerian slant submanifold of a complex space form by using derived formula. Moreover, we give some immediate consequences of the main result of this article. *Section 5* deals with some results of semi-parallel submanifolds in a Kahler manifold with certian conditions. In *Section 6*, we construct some examples of slant, semi-slant, hemi-slant, bi-slant and CR submanifolds in almost complex manifolds.

2. Some Preliminaries on Kaehler Manifolds

In this section we give some notations which shall be used throughout this article. We recall some necessary facts and formulas from the theory of Kaehler manifolds and their submanifolds. Furthermore, we give some basic definitions.

Let $\mathcal{M}$ be an n-dimensional Riemannian manifold isometrically immersed in a Kaehler manifold $\overline{\mathcal{M}}$ of real dimension $2n$ with almost comple structure J. The Riemannian metric for $\mathcal{M}$ and $\overline{\mathcal{M}}$ is denoted by the same symbol g. Let $T\mathcal{M}$ and $T^{\perp}\mathcal{M}$ denote the Lie algebra of vector field and set of all normal vector fields on $\mathcal{M}$ respectively. The operator of covariant differentiation with respect to the Levi-Civita connection in $\mathcal{M}$ and $\overline{\mathcal{M}}$ is denoted by ∇ and $\overline{\nabla}$, respectively. The Gauss and Weingarten formulas are, respectively, given as

$$\overline{\nabla}_X Y = \nabla_X Y + \Omega(X,Y), \tag{2.1}$$

and

$$\overline{\nabla}_X V = -\Lambda_V(X) + \nabla^{\perp}_X V \tag{2.2}$$

for any vector fields $X, Y \in T\mathcal{M}$ and $V \in T^{\perp}\mathcal{M}$. Here Ω is the second fundamental form, Λ is the shape operator of $\mathcal{M}$ and $\nabla^{\perp}$ is the operator of covariant differentiation with respect to the linear connection induced in the normal bundle $T^{\perp}\mathcal{M}$ of $\mathcal{M}$. The second fundamental form and the shape operator are related by the following equation

$$g(\Omega(X,Y),V) = g(\Lambda_V(X),Y) \tag{2.3}$$

for any vector fields $X, Y \in T\mathcal{M}$ and $V \in T^{\perp}\mathcal{M}$.

Denote by $\overline{R}$ and R the curvature tensor of $\overline{\mathcal{M}}$ and $\mathcal{M}$, respectively. Then the equation of Gauss is given by

$$\begin{aligned} \overline{R}(X,Y,Z,W) &= R(X,Y,Z,W) - g(\Lambda_{\Omega(Y,Z)}X, W) \\ &\quad + g(\Lambda_{\Omega(X,Z)}Y, W) \end{aligned} \tag{2.4}$$

for any vector fields $X, Y, Z, W \in T\mathcal{M}$.

Let $x \in \mathcal{M}$ and $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_x\mathcal{M}$ and $\{e_{n+1}, \ldots, e_{2n}\}$ be an orthonormal basis of $T_x^{\perp}\mathcal{M}$. The mean curvature vector $\mathcal{H}$ of a submanifold $\mathcal{M}$ at x is given by the following relation

$$\mathcal{H}(x) = \frac{1}{n}\sum_{i=1}^{n}\Omega(e_i, e_i).$$

Also, we set

$$\hbar_{ij}^{r} = g(\Omega(e_i, e_j), e_r), \qquad i, j \in \{1, \ldots, n\}, \quad r \in \{n+1, \ldots, 2n\}$$

and

$$\|\Omega\|^2 = \sum_{i,j=1}^{n} g(\Omega(e_i, e_j), \Omega(e_i, e_j)).$$

Definition 2.1. A submanifold $\mathcal{M}$ of $\overline{\mathcal{M}}$ is said to be [6]

(i) *totally umbilical* if its second fundamental form satisfies the following relation

$$\Omega(X, Y) = g(X, Y)\mathcal{H}$$

for any vector fields $X, Y \in T\mathcal{M}$.

(ii) *totally geodesic* if

$$\Omega(X, Y) = 0$$

for any vector fields $X, Y \in T\mathcal{M}$.

(iii) *minimal* if $\mathcal{H} = 0$, i.e., trace $\Omega \equiv 0$.

For Kaehler manifolds, we have the following facts [28] :

$$J^2 = -I, \qquad g(JX, JY) = g(X, Y), \qquad \textit{and} \qquad \overline{\nabla}J = 0$$

for any vector fields $X, Y \in T\mathcal{M}$. Furthermore, the covariant derivative of the complex structure J is defined as

$$(\overline{\nabla}_X J)Y = \overline{\nabla}_X JY - J\overline{\nabla}_X Y$$

for any vector fields $X, Y \in T\mathcal{M}$.

Now for any vector field $X \in T\mathcal{M}$, we put

$$JX = \rho X + \digamma X, \tag{2.5}$$

where ρX and $\digamma X$ denote the tangential and normal components of JX, respectively. Then ρ is an endomorphism of $T\mathcal{M}$, and $\digamma$ is the normal bundle valued 1−form on $T\mathcal{M}$. In the same way, for any vector field $V \in T^{\perp}\mathcal{M}$, we put

$$JV = \iota V + \omega V, \tag{2.6}$$

where ιV and ωV denote tangential and normal components of JV, respectively. It is easy to see that ρ and ω are skew-symmetric and

$$g(\digamma X, V) = -g(X, \iota V) \tag{2.7}$$

for any vector fields $X \in T\mathcal{M}$ and $V \in T^{\perp}\mathcal{M}$. Moreover, the covariant derivative of tangential and normal components of (2.5) and (2.6) is given as [28]

$$(\overline{\nabla}_X \rho)Y = \nabla_X \rho Y - \rho \nabla_X Y,$$

$$(\overline{\nabla}_X F)Y = \nabla^{\perp}_X F Y - F \nabla_X Y,$$

$$(\overline{\nabla}_X \iota)V = \nabla_X \iota V - \iota \nabla^{\perp}_X V,$$

$$(\overline{\nabla}_X \omega)V = \nabla^{\perp}_X \omega V - \omega \nabla^{\perp}_X V$$

for any vector fields $X, Y \in T\mathcal{M}$ and $V \in T^{\perp}\mathcal{M}$.

If the ambient manifold $\overline{\mathcal{M}}$ is of constant holomorphic sectional curvature c, then $\overline{\mathcal{M}}$ is called a *complex space form* and is denoted by $\overline{\mathcal{M}}(c)$. Thus, the Riemannian curvature tensor $\overline{R}$ of $\overline{\mathcal{M}}(c)$ is given as [28]

$$\begin{aligned} \overline{R}(X,Y,Z,W) &= \frac{c}{4}\Big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ &\quad + g(JY,Z)g(JX,W) - g(JX,Z)g(JY,W) \\ &\quad + 2g(X,JY)g(JZ,W)\Big] \end{aligned} \tag{2.8}$$

for any vector fields X, Y, Z of $\overline{\mathcal{M}}(c)$. By using (2.4) we can write the Riemannian curvature tensor R of $\mathcal{M}$ as [28]

$$\begin{aligned} R(X,Y,Z,W) &= \frac{c}{4}\Big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ &\quad + g(\rho Y,Z)g(\rho X,W) - g(\rho X,Z)g(\rho Y,W) \\ &\quad + 2g(X,\rho Y)g(\rho Z,W)\Big] + g(\Lambda_{\Omega(Y,Z)}X,W) \\ &\quad - g(\Lambda_{\Omega(X,Z)}Y,W) \end{aligned} \tag{2.9}$$

for any vector fields $X, Y, Z \in T\mathcal{M}$.

There are certain important classes of submanifolds viz totally real submanifolds, holomorphic submanifolds and CR-submanifolds (generalization of totally real and holomorphic submanifolds). The CR-submanifolds were generalized by B. Y. Chen as slant submanifolds [7]. The definition of slant submanifold in an almost Hermitian manifold as follows :

Definition 2.2.([6]) Let $\mathcal{M}$ be a Reimannian submanifold of an almost Hermitian manifold $(\overline{\mathcal{M}}, J, g)$, then $\mathcal{M}$ is said to be a *slant submanifold* if for each $x \in \mathcal{M}$ and $X_x \in T\mathcal{M}$, the angle $\theta(X_x)$ between JX_x and the tangent space $T_x\mathcal{M}$ is constant.

Remark 2.1. If $\mathcal{M}$ is a slant submanifold of $(\overline{\mathcal{M}}, J, g)$ and $x \in \mathcal{M}$, then for a local orthonormal frame $\{e_1, \ldots, e_n\}$ of $\mathcal{M}$ we should have $|g(Je_i, e_j)| = \cos\theta =$ constant.

A slant submanifold is said to be *proper* if it is neither complex nor totally real. B. Y. Chen [6] gives the notion of a Kaehlerian slant submanifold as a proper slant submanifold such that the canonical endomorphism ρ (defined above) is parallel, that is, $\overline{\nabla}\rho = 0$. In fact, a Kaehlerian slant submanifold is a Kaehler manifold with respect to the induced metric and the almost complex structure $\widetilde{J} = (\sec\theta)\rho$, where θ is the slant angle.

For slant submanifolds, the following facts are known [6] :

$$
\left\{
\begin{array}{c}
\rho^2 X = -\cos^2\theta X, \\
\iota F X = -\sin^2\theta X, \\
\omega F X = -F\rho X, \\
g(\rho X, \rho Y) = \cos^2\theta g(X, Y), \\
g(F X, F Y) = \sin^2\theta g(X, Y)
\end{array}
\right.
\tag{2.10}
$$

for any vector fields $X, Y \in T\mathcal{M}$, where θ is the slant angle of $\mathcal{M}$ in $\overline{\mathcal{M}}$.

Definition 2.3. There are some other important classes of submanifolds which are determined by the behavior of tangent bundle of the submanifold under the action of an almost complex structure J of $\overline{\mathcal{M}}$:

(i) A submanifold $\mathcal{M}$ of $\overline{\mathcal{M}}$ is called a *CR-submanifold* [2] of $\overline{\mathcal{M}}$ if there exists a differentiable distribution D (holomorphic) on $\mathcal{M}$ whose orthogonal complementary distribution $D^\perp$ is totally real.

(ii) A submanifold $\mathcal{M}$ of $\overline{\mathcal{M}}$ is called *semi-slant submanifold* [18] of $\overline{\mathcal{M}}$ if there exists a pair of orthogonal distributions D and D_θ such that D is holomorphic and D_θ is proper slant.

(iii) A submanifold $\mathcal{M}$ of $\overline{\mathcal{M}}$ is called *hemi-slant submanifold* (or pseudo-slant) [22] of $\overline{\mathcal{M}}$ if there exists a pair of orthogonal distributions $D^\perp$ and D_θ such that $D^\perp$ is totally real and D_θ is proper slant.

(iv) A submanifold $\mathcal{M}$ of $\overline{\mathcal{M}}$ is called *bi-slant submanifold* [25] of $\overline{\mathcal{M}}$ if there exists a pair of orthogonal distributions D_{θ_1} and D_{θ_2} such that D_{θ_1} is proper slant with slant angle θ_1 and D_{θ_2} is proper slant with slant angle θ_2.

3. Simons′ Type Formula

This section deals with the construction of Simons′ type formula while the main result of this article is printed out in the next section. We quote some basic lemmas without their proofs.

Lemma 3.1.([6], pg - 24) *Let $\mathcal{M}$ be a proper slant submanifold of a Kaehler manifold $\overline{\mathcal{M}}$. Then $\mathcal{M}$ is a Kaehlerian slant if and only if*

$$\Lambda_{FX}(Y) = \Lambda_{FY}(X) \tag{3.1}$$

for any vector fields $X, Y \in T\mathcal{M}$.

An immediate consequence of above quoted Lemma 3.1:

Lemma 3.2. *Let $\mathcal{M}$ be a Kaehlerian slant submanifold of a Kaehler manifold $\overline{\mathcal{M}}$. Then we have*

$$\Lambda_N\rho + \rho\Lambda_N = 0 \tag{3.2}$$

for any vector field $N \in T^\perp\mathcal{M}$.

Proof. From $\overline{\nabla}\rho = 0$ and together with $\overline{\nabla}J = 0$, we can conclude that $\overline{\nabla}F = 0$. Now by straightforward computation, we find that

$$\Omega(X, \rho Y) - \Lambda_{FY}X = J\Omega(X, Y)$$

for any vector field $X, Y \in T\mathcal{M}$. Similarly, we have

$$\Omega(Y, \rho X) - \Lambda_{FX} Y = J\Omega(Y, X)$$

for any vector fields $X, Y \in T\mathcal{M}$. Combining these relations and using the fact that $\Lambda_{FY} X = \Lambda_{FX} Y$, we get the following

$$\Omega(Y, \rho X) = \Omega(X, \rho Y)$$

for any vector fields $X, Y \in T\mathcal{M}$, from which it follows that $\Lambda_N \rho + \rho \Lambda_N = 0$. □

Lemma 3.3.([6], pg - 23) *Let $\mathcal{M}$ be a submanifold of a Kaehler manifold $\overline{\mathcal{M}}$. Then $\overline{\nabla} F = 0$ if and only if*

$$\Lambda_{\omega N}(X) = -\Lambda_N(\rho X) \tag{3.3}$$

*for any vector fields $N \in T^{\perp}\mathcal{M}$ and $X \in T\mathcal{M}$.*v

For the Laplacian of the second fundamental form Λ of an n-dimensional minimal submanifold $\mathcal{M}$ in an m-dimensional Riemannian manifold $\overline{\mathcal{M}}$, the following Simons$'$ type formula is well known ([23], pg - 81) :

$$\nabla^2 \Lambda = -\Lambda \circ \widetilde{\Lambda} - \underset{\sim}{\Lambda} \circ \Lambda + \overline{R}(\Lambda) + \overline{R'}, \tag{3.4}$$

where the operators $\widetilde{\Lambda}$ and $\underset{\sim}{\Lambda}$ are defined as follows

$$\widetilde{\Lambda} = {}^t\Lambda \circ \Lambda \quad and \quad \underset{\sim}{\Lambda} = \sum_{a=1}^{m-n} ad\Lambda_a ad\Lambda_a$$

for a normal frame $\{e_a\}, a = 1, \ldots, m-n$, and $\Lambda_a = \Lambda_{e_a}$. Here $\overline{R}(\Lambda)$ and $\overline{R'}$ are given by the followings

$$\begin{aligned} g(\overline{R}(\Lambda)^N(X), Y) \quad = \quad & \sum_{i=1}^{n} \big[2g(\overline{R}(e_i, Y)\Omega(X, e_i), N) + 2g(\overline{R}(e_i, X)\Omega(Y, e_i), N) \\ & -g(\Lambda_N(X), \overline{R}(e_i, Y)e_i) - g(\Lambda_N(Y), \overline{R}(e_i, X)e_i) \\ & +g(\overline{R}(e_i, \Omega(X, Y))e_i, N) - 2g(\Lambda_N(e_i), \overline{R}(e_i, X)Y)\big] \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} g(\overline{R'}^N(X), Y) \quad = \quad & \sum_{i=1}^{n} \big[g((\overline{\nabla}_X \overline{R})(e_i, Y)e_i, N) \\ & +g((\overline{\nabla}_{e_i} \overline{R})(e_i, X)Y, N)\big] \end{aligned} \tag{3.6}$$

for any vector fields $X, Y \in T\mathcal{M}$ and $N \in T^{\perp}\mathcal{M}$.

Using all Lemmas and equations (3.4), (3.5) and (3.6), we prove the following proposition to get the required Simons$'$ type formula for an $n-$dimensional minimal Kaehlerian slant submanifold of a complex space form $\overline{\mathcal{M}}(c)$ which plays an important role in working out our main result of this paper.

Proposition 3.1. *Let $\mathcal{M}$ be an n-dimensional minimal Kaehlerian slant submanifold of a complex space form $\overline{\mathcal{M}}(c)$. Then*

$$g(\nabla^2 \Lambda, \Lambda) = -g(\Lambda \circ \widetilde{\Lambda} + \underset{\sim}{\Lambda} \circ \Lambda, \Lambda) + \frac{c}{4}\big(n + 4 - 3\sin^2\theta\big)||\Lambda||^2. \tag{3.7}$$

Proof. Since $\overline{\mathcal{M}}(c)$ is locally symmetric, then (3.6) implies that $g(\overline{R'}^N(X),Y)=0$. Now, we compute $g(\overline{R}(\Lambda)^N(X),Y)$ by using (2.8) in (3.5), then a straight forward computation gives [3]

$$
\begin{aligned}
g(\overline{R}(\Lambda)^N(X),Y) \;=\; & \frac{c}{4}\Big[ng(\Lambda_N(X),Y)-2g(\Lambda_{\digamma Y}(X),\iota N)-2g(\Lambda_{\digamma X}(Y),\iota N)\\
&-4g(\Lambda_{\omega N}(X),\rho Y)-4g(\Lambda_{\omega N}(Y),\rho X)+3g(\rho X,\rho\Lambda_N(Y))\\
&+3g(\rho Y,\rho\Lambda_N(X))-6g(\Lambda_N(\rho X),\rho Y)\\
&-2\sum_{i=1}^{n}\big\{g(\Lambda_{\digamma e_i}(e_i),X)g(\digamma Y,N)+g(\Lambda_{\digamma e_i}(e_i),Y)g(\digamma X,N)\\
&+\frac{3}{2}g(\Lambda_{\digamma e_i}(X),Y)g(\digamma e_i,N)\big\}\Big]. \qquad (3.8)
\end{aligned}
$$

In the light of all above Lemmas and (2.10), the equation (3.8) becomes

$$
\begin{aligned}
g(\overline{R}(\Lambda)^N(X),Y) \;=\; & \frac{c}{4}\Big[\big(n+12\cos^2\theta\big)g(\Lambda_N(X),Y)-4g(\Lambda_{\digamma Y}(X),\iota N)\\
&+4g(\Lambda_N(\rho X),\rho Y)+4g(\Lambda_N(\rho Y),\rho X)\\
&-3\sin^2\theta g(\Lambda_N(X),Y)\Big].
\end{aligned}
$$

Further calculations reduce the above relation to

$$
\begin{aligned}
g(\overline{R}(\Lambda)^N(X),Y) \;=\; & \frac{c}{4}\big(n+4-7\sin^2\theta\big)g(\Lambda_N(X),Y)\\
&-cg(\Lambda_{\digamma Y}(X),\iota N). \qquad (3.9)
\end{aligned}
$$

With the help of (3.9) and (3.4) can be rewritten as

$$
g(\nabla^2\Lambda(X),Y)=-g(\Lambda\circ\widetilde{\Lambda}+\underset{\sim}{\Lambda}\circ\Lambda,\Lambda)+\frac{c}{4}\big(n+4-7\sin^2\theta\big)g(\Lambda_N X,Y)-cg(\Lambda_{\digamma Y}X,\iota N).
$$

Putting $X=e_i$ and $Y=\Lambda_N(e_i)$ and add for $i=1,\dots,n$, we get

$$
g(\nabla^2\Lambda,\Lambda)=-g(\Lambda\circ\widetilde{\Lambda}+\underset{\sim}{\Lambda}\circ\Lambda,\Lambda)+\frac{c}{4}\big(n+4-7\sin^2\theta\big)||\Lambda||^2-\sum_{i=1}^{n}cg(\Lambda_{\digamma e_i}\Lambda_N(e_i),\iota N).
$$

Again, by considering all above Lemmas and (2.10), we arrive at

$$
\begin{aligned}
g(\nabla^2\Lambda,\Lambda) \;=\; & -g(\Lambda\circ\widetilde{\Lambda}+\underset{\sim}{\Lambda}\circ\Lambda,\Lambda)+\frac{c}{4}\big(n+4-7\sin^2\theta\big)||\Lambda||^2\\
&+c\sin^2\theta||\Lambda||^2, \qquad (3.10)
\end{aligned}
$$

and hence the desired result (3.7) follows immediately from (3.10). This completes the proof of our Proposition. □

4. Kaehlerian Slant Submanifolds in Complex Space Forms

At this stage we are able to classify a Kaehlerian slant submanifold of a complex space form by using Simons′ type formula for the second fundamental form (3.7), which has been derived in the last section.

Theorem 4.1. *Let $\mathcal{M}$ be an n-dimensional compact and minimal Kaehlerian slant submanifold in a complex space form $\overline{\mathcal{M}}(c)$. Then any of the following can hold :*

(1) $\mathcal{M}$ *is totally geodesic,*

(2) $\mathcal{M}$ *is* $S^1 \times S^1$ *and* $n = 2$ *for* $\theta = \frac{\pi}{2}$ *and* $c = 4$,

(3) $||\Lambda||^2 > \frac{c}{4\left(2-\frac{1}{n}\right)}\left(n+4-3\sin^2\theta\right)$ *everywhere on* $\mathcal{M}$.

.

Proof. From Lemma 5.3.1 of J. Simons ([23], pg - 93), we get

$$g(\Lambda \circ \widetilde{\Lambda} + \underset{\sim}{\Lambda} \circ \Lambda, \Lambda) \leq \left(2 - \frac{1}{n}\right)||\Lambda||^4. \tag{4.1}$$

Since $\mathcal{M}$ is compact, we have

$$\int_{\mathcal{M}} g(\nabla^2\Lambda, \Lambda) = -\int_{\mathcal{M}} g(\nabla\Lambda, \nabla\Lambda). \tag{4.2}$$

Thus, from equations (4.1), (4.2) and Proposition , we see that

$$0 \leq \int_{\mathcal{M}} g(\nabla\Lambda, \nabla\Lambda) \leq \int_{\mathcal{M}} \left\{\left[\left(2 - \frac{1}{n}\right)||\Lambda||^2 - \frac{c}{4}\left(n+4-3\sin^2\theta\right)\right]||\Lambda||^2\right\}.$$

This is known as an *integral formula of Simons′ type.*

Suppose that

$$||\Lambda||^2 \leq \frac{c}{4\left(2-\frac{1}{n}\right)}\left(n+4-3\sin^2\theta\right)$$

everywhere on $\mathcal{M}$.

By combining last two inequalities, we find $\nabla\Lambda = 0$ and hence $\nabla||\Lambda||^2$ is constant, i.e., either $||\Lambda||^2 = 0$ or

$$||\Lambda||^2 = \frac{c}{4\left(2-\frac{1}{n}\right)}\left(n+4-3\sin^2\theta\right). \tag{4.3}$$

From $\Lambda = 0$, we conclude the statement (1), $\mathcal{M}$ is totally geodesic.

Now if we consider (4.3) and put $\theta = \frac{\pi}{2}, c = 4$, the equation reduces to

$$||\Lambda||^2 = \left(\frac{n+1}{2-\frac{1}{n}}\right)$$

which implies the statement (2), $\mathcal{M} = S^1 \times S^1$ and $n = 2$ [15].

Except for these possibilities,

$$||\Lambda||^2 > \frac{c}{4\left(2-\frac{1}{n}\right)}\left(n+4-3\sin^2\theta\right)$$

everywhere on $\mathcal{M}$, which is the statement (3).

This completes the proof of our Theorem. □

Moreover, we quote some results which are based on Theorem 4.1. As the ambient manifold is a complex space form $\overline{\mathcal{M}}(c)$, we can take a complex projective space CP^{2n} with constant holomorphic sectional curvature 4 of real dimension $2n$. Then we have the following result :

Corollary 4.1. *Let $\mathcal{M}$ be an n-dimensional complete, compact and minimal Kaehlerian slant submanifold of CP^{2n}. If*

$$||\Lambda||^2 < \left(\frac{n+4-3\sin^2\theta}{2-\frac{1}{n}}\right)$$

everywhere on $\mathcal{M}$. Then $\mathcal{M}$ is either a

(1) *complex projective space, or*

(2) *real projective space.*

The proof of above corollary directly follows from Theorem 4.1 (1) and Lemma 4 of [1].

Corollary 4.2. *Let $\mathcal{M}$ be an n-dimensional compact and minimal Kaehlerian slant submanifold of a complex space form $\overline{\mathcal{M}}(c)$. If*

$$||\Lambda||^2 > \frac{c}{4\left(2-\frac{1}{n}\right)}\left(n+4-3\sin^2\theta\right)$$

everywhere on $\mathcal{M}$. Then scalar curvature of $\mathcal{M}$ has the lower bound, i.e.,

$$\tau \leq \frac{c\,n\,(n+1)}{2(2-\frac{1}{n})}.$$

Here we give an example which shows that the statement (2) of Theorem 4.1 holds :

Example 4.1. Consider the flat clifford torus $\mathcal{M} = S^1 \times S^1$ in R^4 defined by

$$r(u,v) = \frac{1}{\sqrt{2}}\big(\cos u, \sin u, \cos v, \sin v\big).$$

It is minimal in $S^3 \subset R^4$ and hence austere. The cone cover $\mathcal{M}$

$$\mathcal{C}(\mathcal{M}) = \left\{\frac{1}{\sqrt{2}}(w\cos u, w\sin u, w\cos v, w\sin v) : u, v, w \in R\right\}$$

is easily proved austere in R^4.

On the other hand, we have

$$\begin{aligned} e_1 &= (-\sin u, \cos u, 0, 0), \\ e_2 &= (0, 0, -\sin v, \cos v). \end{aligned}$$

If the almost complex structure J is defined as $J(a,b,c,d) = (-b,a,-d,c)$ on R^4, then $\mathcal{M}$ is a totally real surface in R^4 as $\theta = \frac{\pi}{2}$. Moreover, the length of second fundamental form of $2-$dimensional minimal submanifold $\mathcal{M}$ in R^4 is $||\Lambda||^2 = 2$ ([28], pg - 96). From this we conclude that the statement (2) of Theorem 4.1 holds.

5. Semi-parallel Submanifolds in Kaehler Manifolds

In this section, first we give the definition of semi-parallel submanifolds. Second, we see the role of semi-parallel condition in different submanifolds of a Kaehler manifold with certain conditions.

The covariant derivative of second fundamental form of $\mathcal{M}$ is given by [28]

$$(\widetilde{\nabla}\Omega)(Y,Z) = \nabla^{\perp}_{X}\Omega(Y,Z) - \Omega(\nabla_X Y, Z) - \Omega(Y, \nabla_X Z)$$

for any vector fields $X, Y, Z \in T\mathcal{M}$. Here $\widetilde{\nabla}$ is the *van der Waerden-Bortolotti connection* of $\mathcal{M}$.

We quote the condition of semi-parallel, that is, when a submanifold is called semi-parallel submanifold.

Definition 5.1.([29]) A submanifold $\mathcal{M}$ is called a *semi-parallel submanifold* for the connection $\widetilde{\nabla}$ if

$$\widetilde{R}(X,Y)\Omega = 0, \tag{5.1}$$

where

$$\begin{aligned}(\widetilde{R}(X,Y)\Omega)(Z,W) &= (\widetilde{\nabla}_X(\widetilde{\nabla}_Y\Omega))(Z,W) - (\widetilde{\nabla}_Y(\widetilde{\nabla}_X\Omega))(Z,W) \\ &\quad -(\widetilde{\nabla}^{\perp}_{[X,Y]}\Omega)(Z,W)\end{aligned} \tag{5.2}$$

for any vector fields $X, Y, Z \in T\mathcal{M}$.

Here we prepare some useful lemmas which shall be required in proving the interesting results of this section.

Lemma 5.1. *Let $\mathcal{M}$ be a submanifold of a Kaehler manifold $\overline{\mathcal{M}}$ such that $\overline{\nabla}F = 0$. Then*

$$R^{\perp}(X,Y)FZ = FR(X,Y)Z$$

for any vector fields $X, Y, Z \in T\mathcal{M}$. Here $R^{\perp}$ is the normal curvature of $\mathcal{M}$.

Proof. For any vector fields $X, Y, Z \in T\mathcal{M}$, we have

$$\begin{aligned}R^{\perp}(X,Y)FZ &= \nabla^{\perp}_X\nabla^{\perp}_Y FZ - \nabla^{\perp}_Y\nabla^{\perp}_X FZ - \nabla^{\perp}_{[X,Y]}FZ \\ &= \nabla^{\perp}_X(F\nabla_Y Z) - \nabla^{\perp}_Y(F\nabla_X Z) - F\nabla_{[X,Y]}Z \\ &= F\left[\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\right] \\ &= FR(X,Y)Z.\end{aligned}$$

This completes our assertion. □

Lemma 5.2. *Let $\mathcal{M}$ be a submanifold of a Kaehler manifold $\overline{\mathcal{M}}$. Then $\overline{\nabla}\omega = 0$ if and only if*

$$\Omega(X, \iota V) = -F\Lambda_V X$$

for any vector fields $V \in T^{\perp}\mathcal{M}, X \in T\mathcal{M}$.

Proof. By simple computation, we derive [28]

$$\overline{\nabla}\omega = -\Omega(X, \iota V) - F\Lambda_V X$$

for any vector fields $V \in T^{\perp}\mathcal{M}, X \in T\mathcal{M}$. From $\overline{\nabla}\omega = 0$, we get our assertion. Converse also holds. □

We now first prove the result in the case of Kaehlerian slant submanifolds by using semi-parallel condition:

Theorem 5.1. *Let $\mathcal{M}$ be an n-dimensional minimal semi-parallel Kaehlerian slant submanifold of a Kaehler manifold $\overline{\mathcal{M}}$. If $\mathcal{M}$ is of constant curvature k and $\overline{\nabla}\omega = 0$, then $\mathcal{M}$ is*

(1) *flat, i.e., $k = 0$, or*

(2) *totally geodesic.*

Proof. Since, $\overline{\nabla}\omega = 0$. Then, from Lemma 5.2, we use $\Omega(X, \iota V) = -F\Lambda_V X$ for any vector fields $X, Y \in T\mathcal{M}$ and $V \in T^{\perp}\mathcal{M}$. Let us put

$$\begin{aligned} V &= FY \\ \iota V &= \iota FY = -Y - \rho^2 Y = -\sin^2\theta\, Y, \end{aligned}$$

where we have used (2.10). Thus, we get

$$\Omega(X, Y) = \operatorname{cosec}^2\theta\, F\Lambda_{FY}X. \tag{5.3}$$

On the other hand, for any vector fields $X, Y, Z \in T\mathcal{M}$, we have

$$\begin{aligned} 0 = (\widetilde{R}(X,Y)\Omega)(Z,W) &= R^{\perp}(X,Y)\Omega(Z,W) \\ &\quad -\Omega(R(X,Y)Z,W) - \Omega(Z, R(X,Y)W). \end{aligned}$$

By using (5.3), we get

$$0 = \operatorname{cosec}^2\theta\big\{R^{\perp}(X,Y)F\Lambda_{FZ}W - F\Lambda_{FW}R(X,Y)Z - F\Lambda_{FZ}R(X,Y)W\big\}.$$

Taking into account Lemma and since $\mathcal{M}$ is of constant curvature k, i.e.,

$$R(X,Y)Z = k\big\{g(Y,Z)X - g(X,Z)Y\big\}$$

for any vector fields $X, Y, Z \in T\mathcal{M}$, we obtain

$$\begin{aligned} 0 &= k\operatorname{cosec}^2\theta\, F\big\{g(Y,\Lambda_{FZ}W)X - g(X,\Lambda_{FZ}W)Y - g(Y,Z)\Lambda_{FW}X \\ &\quad +g(X,Z)\Lambda_{FW}Y - g(Y,W)\Lambda_{FZ}X + g(X,W)\Lambda_{FZ}Y\big\}. \end{aligned}$$

Putting $X = W = e_i$ into last relation and add for $i = 1, 2, \ldots, n$. Further, we use Lemma 3.1, we find that

$$k\ (n+1)\operatorname{cosec}^2\theta\big\{F\Lambda_{FZ}Y\big\} = 0,$$

which gives $k = 0$ or $\operatorname{cosec}^2\theta\, F\Lambda_{FZ}Y = 0$.
Hence, our assertions follow because of (5.3). □

Using a similar method, we can prove the same result for totally real submanifolds in Kaehler manifolds :

Theorem 5.2.([13]) *Let $\mathcal{M}$ be an n-dimensional minimal semi-parallel totally real submanifold of a Kaehler manifold $\overline{\mathcal{M}}$. If $\mathcal{M}$ is of constant curvature k and $\overline{\nabla}\omega = 0$, then $\mathcal{M}$ is*

(1) *flat, i.e.,* $k = 0$*, or*

(2) *totally geodesic.*

Now considering the particular case of totally real submanifolds, that is, a totally real submanifold of maximum dimension ($n = m$) is called a *Lagrangian submanifold*, where $n = \dim(\mathcal{M})$ and $m = \dim(\overline{\mathcal{M}})$:

Theorem 5.3. *Let* $\mathcal{M}$ *be an n-dimensional minimal semi-parallel Lagrangian submanifold of a Kaehler manifold* $\overline{\mathcal{M}}$*. If* $\mathcal{M}$ *is of constant curvature* $k(\neq 0)$ *and* $\overline{\nabla}\omega = 0$*, then* $\mathcal{M}$ *is*

(1) *flat, i.e.,* $k = 0$*, or*

(2) *totally geodesic.*

Next we prove the result in the case of bi-slant submanifolds by using semi-parallel condition. Moreover, we see the same result in the case of hemi-slant, semi-slant and CR submanifolds also. For two distributions D_1 and D_2 on $\mathcal{M}$, we say that $\mathcal{M}$ is $(D_1, D_2)-$*mixed totally geodesic* if for all $X \in D_1$ and $Y \in D_2$, we have $\Omega(X, Y) = 0$.

Theorem 5.4. *Let* $\mathcal{M}$ *be an n-dimensional minimal semi-parallel bi-slant submanifold of a Kaehler manifold* $\overline{\mathcal{M}}$*. If* $\mathcal{M}$ *is of constant curvature* $k, \overline{\nabla}\omega = 0$ *and* $\overline{\nabla}F = 0$*, then* $\mathcal{M}$ *is*

(1) flat, i.e., $k = 0$*, or*

(2) $(D_{\theta_1}, D_{\theta_2})-$*mixed totally geodesic.*

Moreover, D_{θ_2} *is invariant.*

Proof. We put $\dim(\mathcal{M}) = n = 2n_1 + 2n_2 = \dim(D_{\theta_1}) + \dim(D_{\theta_2})$. From Lemma , we find that

$$\Omega(X, Y) = \text{cosec}^2\theta_1 \; F\Lambda_{FY}X \tag{5.4}$$

for any $X \in T\mathcal{M}$ and $Y \in D_{\theta_1}$.

In the same way, we can say that

$$\Omega(X, Y) = \text{cosec}^2\theta_2 \; F\Lambda_{FY}X \tag{5.5}$$

for any $X \in T\mathcal{M}$ and $Y \in D_{\theta_2}$.

On the other hand, for any $X, W \in D_{\theta_1}$ and $Y, Z \in D_{\theta_2}$, we have

$$\begin{aligned} 0 = (\widetilde{R}(X, Y)\Omega)(Z, W) \quad &= \quad R^{\perp}(X, Y)\Omega(Z, W) \\ &\quad -\Omega(R(X, Y)Z, W) - \Omega(Z, R(X, Y)W). \end{aligned}$$

By using (5.4) and (5.5), we get

$$\begin{aligned} 0 \quad = \quad & \text{cosec}^2\theta_2 R^{\perp}(X, Y)F\Lambda_{FZ}W - \text{cosec}^2\theta_1 F\Lambda_{FW}R(X, Y)Z \\ & -\text{cosec}^2\theta_2 F\Lambda_{FZ}R(X, Y)W. \end{aligned}$$

Taking into account Lemma and since $\mathcal{M}$ is of constant curvature k, i.e.,

$$R(X, Y)Z = k\{g(Y, Z)X - g(X, Z)Y\}$$

for any vector fields $X, Y, Z \in T\mathcal{M}$, we derive

$$\begin{aligned} 0 \quad = \quad & k \; \text{cosec}^2\theta_2 \; F\{g(Y, \Lambda_{FZ}W)X - g(X, \Lambda_{FZ}W)Y - g(Y, W)\Lambda_{FZ}X \\ & +g(X, W)\Lambda_{FZ}Y\} - k \; \text{cosec}^2\theta_1 \; F\{g(Y, Z)\Lambda_{FW}X - g(X, Z)\Lambda_{FW}Y\}. \end{aligned}$$

Putting $X = W = e_i$ and add for $i = 1, 2, \dots, 2n_1$, we arrive at

$$0 = \sum_{i=1}^{2n_1} \Big\{ k \operatorname{cosec}^2\theta_2 \, F\{g(Y, \Lambda_{FZ} e_i)e_i - g(e_i, \Lambda_{FZ} e_i)Y + 2n_1 \Lambda_{FZ} Y\} - k \operatorname{cosec}^2\theta_1 \, F\{g(Y, Z)\Lambda_{Fe_i} e_i\} \Big\}.$$

Taking inner product with FU for $U \in D_{\theta_1}$ on both sides of above equation, we obtain

$$k \,\{g(U, \Lambda_{FZ} Y) + 2n_1 g(\Lambda_{FZ} Y, U) - g(Y, Z) \sum_{i=1}^{2n_1} g(\Lambda_{Fe_i} e_i, U)\} = 0.$$

Let us use Y is orthogonal to Z in the last equation, which gives

$$k \,(1 + 2n_1) g(\Lambda_{FZ} Y, U) = 0.$$

for any $Z, Y \in D_{\theta_2}$ and $U \in D_{\theta_1}$. From this, we conclude that either $\mathcal{M}$ is $(D_{\theta_1}, D_{\theta_2})$-mixed totally geodesic or D_{θ_2} is invariant. Moreover, $\mathcal{M}$ is flat. Hence, our assertions follow. □

The proofs of following theorems directly follow from the above Theorem 5.4:

Theorem 5.5. *Let $\mathcal{M}$ be an n-dimensional minimal semi-parallel hemi-slant submanifold of a Kaehler manifold $\overline{\mathcal{M}}$. If $\mathcal{M}$ is of constant curvature $k, \overline{\nabla}\omega = 0$ and $\overline{\nabla}F = 0$, then $\mathcal{M}$ is*

(1) *flat, i.e., $k = 0$, or*
(2) *$(D^{\perp}, D_\theta)-$mixed totally geodesic.*

Moreover, D_θ is invariant.

Theorem 5.6. *Let $\mathcal{M}$ be an n-dimensional minimal semi-parallel semi-slant submanifold of a Kaehler manifold $\overline{\mathcal{M}}$. If $\mathcal{M}$ is of constant curvature $k, \overline{\nabla}\omega = 0$ and $\overline{\nabla}F = 0$, then $\mathcal{M}$ is*

(1) *flat, i.e., $k = 0$, or*
(2) *$(D, D_\theta)-$mixed totally geodesic.*

Theorem 5.7. *Let $\mathcal{M}$ be an n-dimensional minimal semi-parallel CR submanifold of a Kaehler manifold $\overline{\mathcal{M}}$. If $\mathcal{M}$ is of constant curvature $k, \overline{\nabla}\omega = 0$ and $\overline{\nabla}F = 0$, then $\mathcal{M}$ is*

(1) *flat, i.e., $k = 0$, or*
(2) *$(D, D^{\perp})-$mixed totally geodesic.*

6. Some examples

In this section we construct some examples of slant submanifolds in an almost complex manifold inspirited by B. Y. Chen [6]. Let (R^{2m}, J) be the Euclidean $2m-$space and endowed with the Euclidean metric g, where J is an almost complex structure on R^{2m}. The Euclidean metric g is defined by

$$g\big((x_1, \dots, x_m, y_1, \dots, y_m), (z_1, \dots, z_m, w_1, \dots, w_m)\big) = x_1 z_1 + \dots + x_m z_m + y_1 w_1 + \dots + y_m w_m.$$

Example 6.1. Consider a 2−dimensional submanifold M in (R^4, J) given by

$$r(u, s) = (-s\sin u, \sin s, s\cos u, \cos s).$$

Then at any point $p \in M$, we have

$$dr_p = \begin{bmatrix} -s\cos u & -\sin u \\ 0 & \cos s \\ -s\sin u & \cos u \\ 0 & -\sin s \end{bmatrix}$$

Let $\{e_1, e_2\}$ be a local orthonormal frame on M.

$$\begin{aligned} e_1 &= (-\cos u, 0, -\sin u, 0), \\ e_2 &= \frac{(-\sin u, \cos s, \cos u, -\sin s)}{\sqrt{2}}. \end{aligned}$$

Since $J(a, b, c, d) = (-c, -d, a, b)$ satisfies $J^2 = -I$, then we have

$$\begin{aligned} Je_1 &= (\sin u, 0, -\cos u, 0), \\ Je_2 &= \frac{(-\cos u, \sin s, -\sin u, \cos s)}{\sqrt{2}}. \end{aligned}$$

Here we see that

$$\left|g(Je_i, e_j)\right| = \frac{1}{\sqrt{2}}, \quad i, j = 1, 2.$$

Hence, M is a proper slant surface in R^4 with the slant angle $\theta = \frac{\pi}{4}$.

On the other hand, given metric g satisfies $g(Je_1, Je_2) = g(e_1, e_2)$, then R^4 is an almost Hermitian manifold.

In the same manner, we can quote more examples of slant submanifolds.

Example 6.2. Consider a 2−dimensional submanifold M in (R^4, J) given by

$$r(u, v) = (u\sin\alpha, v\cos\beta, u\cos\alpha, v\sin\beta),$$

where α and β are constant. Then M is a slant surface with the slant angle $\theta = \cos^{-1}\left(\left|\sin(\alpha+\beta)\right|\right)$, where $J(a, b, c, d) = (-b, a, -d, c)$.

Example 6.3. Consider a 2-dimensional submanifold M in (R^4, J) given by

$$r(u, v) = (u + v, u + v, u, v).$$

Then M is a proper slant surface with the slant angle $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$, where $J(a, b, c, d) = (-b, a, -d, c)$.

If the almost complex structure J is defined as $J(a, b, c, d) = (-c, -d, a, b)$ on R^4, then M is a totally real surface in R^4 as $\theta = \frac{\pi}{2}$.

Example 6.4. Consider a 2−dimensional submanifold M in (R^4, J) given by

$$r(u, v) = (u, u, v\cos t, v\sin t),$$

where t is any constant. Then M is a slant surface with the slant angle $\theta = \cos^{-1}\left(\left|\frac{\cos t+\sin t}{\sqrt{\sqrt{2}}}\right|\right)$, where $J(a,b,c,d) = (c,d,-a,-b)$.

Example 6.5. Consider a 2−dimensional submanifold M in (R^8, J) given by

$$r(u,v,w,z) = (u, v, k\sin w, k\sin z, kw, kz, k\cos w, k\cos z).$$

Then M is a Kaehlerian slant submanifold with the slant angle $\theta = \cos^{-1}(k)$ for any positive number k. It is the well-known example of Kaehlerian slant submanifolds given by B. Y. Chen.

Example 6.6. If M is a slant submanifold in an almost Hermitian manifold $\overline{M}$, then $M \times R$ is a slant submanifold in the almost contact metric manifold $M \times R$ with the usual product structure [7].

Remark 6.1. It is known that in complex geometry, proper slant submanifolds are always even dimensional, while in contact geometry, proper slant submanifolds are always odd dimensional.

Now, let us quote some examples of CR, semi-slant, hemi-slant and bi-slant submanifolds of almost complex manifold.

Example 6.7. Let $f : M \to R^8$ defined by

$$f(t,s,u,v) = (t\cos u, s\cos u, t\cos v, s\cos v, t\sin u, s\sin u, t\sin v, s\sin v).$$

Then we have

$$\begin{aligned} e_1 &= (\cos u, 0, \cos v, 0, \sin u, 0, \sin v, 0),\\ e_2 &= (0, \cos u, 0, \cos v, 0, \sin u, 0, \sin v),\\ e_3 &= (-t\sin u, -s\sin u, 0, 0, t\cos u, s\cos u, 0, 0),\\ e_4 &= (0, 0, -t\sin v, s\sin v, 0, 0, t\cos v, s\cos v). \end{aligned}$$

Thus, M is a CR-submanifold of R^8 such that $D^\perp = span\{e_3, e_4\}$ and $D = span\{e_1, e_2\}$, where $J(a,b,c,d,e,f,g,h) = (b,-a,d,-c,f,-e,h,-g)$.

Example 6.8. Let $f : M \to R^{10}$ defined by

$$f(u,v,s,t) = (t\cos u, s\cos u, t\cos v, s\cos v, t\sin u, s\sin u, t\sin v, s\sin v, u, v).$$

Then we have

$$\begin{aligned} e_1 &= (\cos u, 0, \cos v, 0, \sin u, 0, \sin v, 0, 0, 0),\\ e_2 &= (0, \cos u, 0, \cos v, 0, \sin u, 0, \sin v, 0, 0),\\ e_3 &= (-t\sin u, -s\sin u, 0, 0, t\cos u, s\cos u, 0, 0, 1, 0),\\ e_4 &= (0, 0, -t\sin v, s\sin v, 0, 0, t\cos v, s\cos v, 0, 1). \end{aligned}$$

Thus, M is a semi-slant submanifold of R^{10} such that $D_\theta = span\{e_3, e_4\}$ with slant angle $\cos^{-1}\left(\frac{1}{s^2+t^2+1}\right)$ and $D = span\{e_1, e_2\}$, where $J(a,b,c,d,e,f,g,h,i,j) = (b,-a,d,-c,f,-e,h,-g,j,-i)$.

Example 6.9. Let $f : M \to R^6$ defined by

$$f(u, v, t) = (u \cos v, u \sin v, u, 0, t, t).$$

Then we have

$$\begin{aligned} e_1 &= (\cos v, \sin v, 1, 0, 0, 0), \\ e_2 &= (-u \sin v, u \cos v, 0, 0, 0, 0), \\ e_3 &= (0, 0, 0, 0, 1, 1). \end{aligned}$$

Thus, M is a proper hemi-slant submanifold of R^6 such that $D_\theta = span\{e_1, e_2\}$ with slant angle $\frac{\pi}{4}$ and $D^\perp = span\{e_3\}$, where $J(a, b, c, d, e, f) = (b, -a, c, -d, f, -e)$.

Example 6.10. Let $f : M \to R^8$ defined by

$$f(u, v, t, s) = (u \cos \alpha, v, u \sin \alpha, 0, s, t \cos \beta, t \sin \beta),$$

where α and β are constant. Then we have

$$\begin{aligned} e_1 &= (\cos \alpha, 0, \sin \alpha, 0, 0, 0, 0, 0), \\ e_2 &= (0, 1, 0, 0, 0, 0, 0, 0), \\ e_3 &= (0, 0, 0, 0, 1, 0, 0, 0), \\ e_4 &= (0, 0, 0, 0, 0, \cos \beta, \sin \beta). \end{aligned}$$

Thus, M is a bi-slant submanifold of R^8 such that $D_{\theta_1} = span\{e_1, e_2\}$ with slant angle α and $D_{\theta_2} = span\{e_3, e_4\}$ with slant angle β, where $J(a, b, c, d, e, f, g, h, i, j) = (b, -a, d, -c, f, -e, h, -g, j, -i)$.

References

[1] K. Abe, *Applications of Riccati type differential equation to Riemannian manifolds with totally geodesic distributions*, Tohoku Math. J., **25**(1973), 425–444.

[2] A. Bejancu, *Geometry of CR-submanifolds*, D. Reidel Publ. Comp., Dordrecht, Holland, 1986.

[3] A. Bejancu, M. Kon and K. Yano, *CR-submanifolds of a complex space form*, J. Differential Geom., **16**(1981), 137–145.

[4] J. L. Cabrerizo, A. Carriazo and L. M. Fernandez, *Slant submanifolds in Sasakian Manifolds*, Glasg. Math. J., **42**(2000), 125–138.

[5] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez, *Structure on Slant submanifold of a contact manifold*, Indian J. Pure Appl. Math., **31** (2000), 857–864.

[6] B. Y. Chen, *Geometry of slant submanifolds*, Katholieke Universiteit, Leuven, 1990.

[7] B. Y. Chen, *Slant immersions*, Bull. Austral. Math. Soc., **41**(1990), 135–147.

[8] B. Y. Chen and Y. Tazawa, *Slant surfaces of codimension two*, Ann. Fac. Sci. Toulouse Math. (5), **11**(1990), 29–43.

[9] B. Y. Chen and Y. Tazawa, *Slant submanifolds in complex Eulidean spaces*, Tokyo J. Math., **14**(1991), 101–120.

[10] R. S. Gupta, S. M. Khursheed Haider and A. Sharfuddin, *Slant submanifolds of a trans-Sasakian manifold*, Bull. Math. Soc. Sci. Math. Roumanie (N. S.), **47**(2004), 45–57.

[11] T. Ikawa, *On submanifolds in a space with Sasakian* 3*−structure*, Colloq. Math., **47**(1982), 47–57.

[12] T. Ikawa, *On some* C*-totally real submanifolds in a space with Sasakian* 3*-structure*, Colloq. Math., **49**(1984), 27–32.

[13] O. Kassabov, *On totally real submanifolds*, Bull. Soc. Math. Belg., Ser. B, **38(2)**(1986), 136–143.

[14] A. Lotta, *Slant submanifolds in contact geometry*, Bul. Math. Soc. Sc. Math. Roumanie, **39(87)**(1996), 183–198.

[15] G. D. Ludden, M. Okumura and K. Yano, *A totally real surface in* CP^2 *that is not totally geodesic*, Proc. Amer. Math. Soc., **53**(1975), 186–190.

[16] S. Maeda, Y. Ohnita and S. Udagawa, *On slant immersions into Kähler manifolds*, Kodai Math. J., **16**(1993), 205–219.

[17] A. Mihai and I. Mihai, *CR-submanifolds in complex and Sasakian space forms*, In: Dragomir S., Shahid M., Al-Solamy F. (eds) Geometry of Cauchy-Riemann Submanifolds. Springer, Singapore, (2016), 217–266.

[18] N. Papaghiuc, *Semi-slant submanifolds of a Kaehlerian manifold*, An. Stiint. Al. I. Cuza Univ. Iasi, **40**(1994), 55–61.

[19] B. Sahin, *Slant submersions from almost Hermitian manifolds*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), **54**(2011), 93–105.

[20] B. Sahin, *Slant submanifolds of an almost product Riemannian Manifold*, J. Korean Math. Soc., **43**(2006), 717–732.

[21] B. Sahin, *Slant submanifolds of Quaternion Kaehler manifolds*, Commun. Korean Math. Soc., **22**(2007), 123–135.

[22] B. Sahin, *Warped product submanifolds of Kaehler manifolds with a slant factor*, Ann. Polon. Math., **95**(2009), 207–226.

[23] J. Simons, *Minimal varieteis in Riemannian manifolds*, Ann. Math., **88**(1968), 62–105.

[24] H. M. Tastan, B. Sahin and S. Yanan, *Hemi-Slant submersions*, Mediterr. J. Math., **13(4)**(2016), 2171–2184.

[25] S. Uddin, B. Y. Chen and F. R. Al-Solamy, *Warped product bi-slant immersions in Kaehler manifolds*, Mediterr. J. Math., **14**(2017), 11 pp.

[26] S. Yamaguchi, M. Kon and T. Ikawa, *C-totally real submanifolds*, J. Differential Geometry, **11**1976, 59–64.

[27] S. Yamaguchi and T. Ikawa, *On compact minimal* C*-totally real submanifolds*, Tensor (N.S.), **29**(1975), 30–34.

[28] K. Yano and M. Kon, *Structures on manifolds*, Worlds Scientific, Singapore, 1984.

[29] A. Yildiz and C. Murathan, *Ricci generalized pseudo-parallel Kaehlerian submanifolds in complex space forms*, Bull. Malay. Math. Sci. Soc., **31**(2008), 153–163.