KYUNGPOOK Math. J. 58(2018), 137-148
https://doi.org/10.5666/KMJ.2018.58.1.137
pISSN 1225-6951 eISSN 0454-8124

On a Classification of Almost Kenmotsu Manifolds with Generalized $(k,\mu)'$-nullity Distribution

Gopal Ghosh, Pradip Majhi and Uday Chand De*
Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kol- 700019, West Bengal, India.
e-mail : ghoshgopal.pmath@gmail.com, mpradipmajhi@gmail.com *and* uc_de@yahoo.com

Abstract. In the present paper we prove that in an almost Kenmotsu manifold with generalized $(k,\mu)'$-nullity distribution the three conditions: (i) the Ricci tensor of M^{2n+1} is of Codazzi type, (ii) the manifold M^{2n+1} satisfies div $C = 0$, (iii) the manifold M^{2n+1} is locally isometric to $H^{n+1}(-4) \times \mathbb{R}^n$, are equivalent. Also we prove that if the manifold satisfies the cyclic parallel Ricci tensor, then the manifold is locally isometric to $H^{n+1}(-4) \times \mathbb{R}^n$.

1. Introduction

Geometry of Kenmotsu manifolds was originated by Kenmotsu [13] and became an interesting area of research in differential geometry. As a generalization of Kenmotsu manifolds, the notion of almost Kenmotsu manifolds was first introduced by Janssens and Vanhecke [12]. In recent years, some results regarding such manifolds we refer the reader to [5, 6, 7, 8, 9, 10, 13, 15, 16, 17, 18, 19, 21, 22]. Almost Kenmotsu manifolds satisfying the (k,μ) and $(k,\mu)'$-nullity conditions were introduced by Dileo and Pastore [10], where k and μ both are constants. In 2011, Pastore and Saltarelli in [14] extend the above nullity conditions to the corresponding generalized nullity conditions for which both k and μ are smooth functions. Recently some results on generalized (k,μ) and $(k,\mu)'$-almost Kenmotsu manifolds satisfying some conditions are obtained by Wang et al. [20, 21].

Gray [11] introduced two classes of Riemannian manifolds determined by the covariant derivative of the Ricci tensor; the class A consisting of all Riemannian

* Corresponding Author. Received May 3, 2017; accepted March 14, 2018.
2010 Mathematics Subject Classification: 53C25, 53C35.
Key words and phrases: almost Kenmotsu manifold, generalized nullity distribution, Codazzi type of Ricci tensor, cyclic parallel Ricci tensor, div $C = 0$.

manifolds whose Ricci tensor S is of Codazzi type, that is,

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z),$$

for all smooth vector fields X, Y, Z.

The class B consisting all Riemannian manifolds whose Ricci tensor S is cyclic parallel, that is,

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(X,Z) + (\nabla_Z S)(X,Y) = 0,$$

for all smooth vector fields X, Y, Z.

A Riemannian manifold is said to be harmonic Weyl tensor if div $C = 0$, where C is the Weyl conformal curvature tensor of type $(1,3)$ defined by [22],

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{(2n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad -g(X,Z)QY] + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \end{aligned}$$

Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$, r is the scalar curvature of the manifold and 'div' denotes divergence. If div $C = 0$, then we get

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{4n}[X(r)g(Y,Z) - Y(r)g(X,Z)].$$

A Riemannian manifold is said to be harmonic if div $R = 0$, which is equivalent to

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z),$$

for all smooth vector fields X, Y, Z.

Recently Wang et al. [15] studied conformally flat almost Kenmotsu manifolds with ξ belonging to the generalized $(k,\mu)'$-nullity distribution. In 2016, Wang [20] studied cyclic parallel Ricci tensor in such a manifold . Moreover in [17] Wang et al. studied ϕ-recurrent almost Kenmotsu manifold with generalized $(k,\mu)'$-nullity distribution.

Motivated by the above studies in the present paper we study certain curvature conditions in generalized $(k,\mu)'$-almost Kenmotsu manifolds.

The present paper is organized as follows:

In Section 2, we first recall some basic formulas of almost Kenmotsu manifolds, while Section 3 contains some well-known results on almost Kenmotsu manifolds with generalized $(k,\mu)'$-nullity distribution. Section 4 is devoted to study Codazzi type of Ricci tensor in such a manifold. Next in Section 5 we study almost Kenmotsu manifolds with generalized $(k,\mu)'$-nullity distribution satisfying div $C = 0$. Finally, we study cyclic parallel Ricci tensor on an almost Kenmotsu manifold with generalized $(k,\mu)'$-nullity distribution.

2. Almost Kenmotsu Manifolds

A differentiable $(2n+1)$-dimensional manifold M is said to have a (ϕ,ξ,η)-*structure* or an *almost contact structure*, if it admits a $(1,1)$ tensor field ϕ, a characteristic vector field ξ and a 1-form η satisfying [1, 2],

$$\phi^2 = -I + \eta\otimes\xi,\ \eta(\xi) = 1, \tag{2.1}$$

where I denotes the identity endomorphism. Here also $\phi\xi = 0$ and $\eta\circ\phi = 0$; both can be derived from (2.1) easily.
If a manifold M with a (ϕ,ξ,η)-structure admits a Riemannian metric g such that

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y of T_pM^{2n+1}, then M is said to have an almost contact metric structure (ϕ,ξ,η,g). The fundamental 2-form Φ on an almost contact metric manifold is defined by $\Phi(X,Y) = g(X,\phi Y)$ for any X, Y of T_pM^{2n+1}. The condition for an almost contact metric manifold being normal is equivalent to vanishing of the $(1,2)$-type torsion tensor N_ϕ, defined by $N_\phi = [\phi,\phi] + 2d\eta\otimes\xi$, where $[\phi,\phi]$ is the Nijenhuis torsion of ϕ. Recently in [9, 10, 20], almost contact metric manifold such that η is closed and $d\Phi = 2\eta\wedge\Phi$ are studied and they are called *almost Kenmotsu manifolds*. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. Also Kenmotsu manifolds can be characterized by $(\nabla_X\phi)Y = g(\phi X,Y)\xi - \eta(Y)\phi X$, for any vector fields X, Y. It is well known [13] that a Kenmotsu manifold M^{2n+1} is locally a warped product $I\times_f N^{2n}$, where N^{2n} is a Kähler manifold, I is an open interval with coordinate t and the warping function f, defined by $f = ce^t$ for some positive constant c. Let us denote the distribution orthogonal to ξ by $\mathcal{D}$ and defined by $\mathcal{D} = Ker(\eta) = Im(\phi)$. In an almost Kenmotsu manifold, since η is closed, $\mathcal{D}$ is an integrable distribution. Let M^{2n+1} be an almost Kenmotsu manifold. We denote by $h = \frac{1}{2}\pounds_\xi\phi$ and $l = R(\cdot,\xi)\xi$ on M^{2n+1}. The tensor fields l and h are symmetric operators and satisfy the following relations [10]:

$$h\xi = 0,\ l\xi = 0,\ tr(h) = 0,\ tr(h\phi) = 0,\ h\phi + \phi h = 0, \tag{2.2}$$

$$\nabla_X\xi = -\phi^2X - \phi hX (\Rightarrow \nabla_\xi\xi = 0), \tag{2.3}$$

$$\phi l\phi - l = 2(h^2 - \phi^2), \tag{2.4}$$

$$\begin{aligned} R(X,Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y\phi h)X \\ -(\nabla_X\phi h)Y, \end{aligned} \tag{2.5}$$

for any vector fields X, Y. The $(1,1)$-type symmetric tensor field $h' = h\circ\phi$ is anti-commuting with ϕ and $h'\xi = 0$. Also it is clear that [3, 10, 21]

$$h = 0 \Leftrightarrow h' = 0,\ \ h'^2 = (k+1)\phi^2 (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{2.6}$$

3. ξ belongs to the Generalized $(k,\mu)'$-nullity Distribution

This section is devoted to study almost Kenmotsu manifolds with ξ belonging to the generalized $(k,\mu)'$-nullity distribution. Let $M^{2n+1}(\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold with ξ belonging to the generalized $(k,\mu)'$-nullity distribution, then according to Pastore and Saltarelli [14] we have

$$R(X,Y)\xi = k[\eta(Y)X-\eta(X)Y]+\mu[\eta(Y)h'X-\eta(X)h'Y], \tag{3.1}$$

where k, μ are smooth functions on M^{2n+1} and $h'=h\circ\phi$. Let $X\in\mathcal{D}$ be the eigenvector of h' corresponding to the eigenvalue λ. Then from (2.6) it is clear that $\lambda^2=-(k+1)$. Therefore $k\leq -1$ and $\lambda=\pm\sqrt{-k-1}$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigenspaces related to the non-zero eigen value λ and $-\lambda$ of h', respectively. In [14] Pastore and Saltarelli cited some examples of almost Kenmotsu manifold with ξ belonging to the generalized $(k,\mu)'$-nullity distribution. Before presenting our main theorems we recall some results:

Lemma 3.1.(Theorem 5.1 of [14]) *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be a generalized $(k,\mu)'$-almost Kenmotsu manifold such that $h'\neq 0$ and $n>1$. Then for any $X_\lambda,Y_\lambda,Z_\lambda\in[\lambda]'$ and $X_{-\lambda},Y_{-\lambda},Z_{-\lambda}\in[-\lambda]'$, the Riemannian curvature tensor satisfies*

$$\begin{aligned}
R(X_\lambda,Y_\lambda)Z_{-\lambda} &= 0,\\
R(X_{-\lambda},Y_{-\lambda})Z_\lambda &= 0,\\
R(X_\lambda,Y_{-\lambda})Z_\lambda &= (k+2)g(X_\lambda,Z_\lambda)Y_{-\lambda},\\
R(X_\lambda,Y_{-\lambda})Z_{-\lambda} &= -(k+2)g(Y_{-\lambda},Z_{-\lambda})X_\lambda,\\
R(X_\lambda,Y_\lambda)Z_\lambda &= (k-2\lambda)[g(Y_\lambda,Z_\lambda)X_\lambda-g(X_\lambda,Z_\lambda)Y_\lambda],\\
R(X_{-\lambda},Y_{-\lambda})Z_{-\lambda} &= (k+2\lambda)[g(Y_{-\lambda},Z_{-\lambda})X_{-\lambda}-g(X_{-\lambda},Z_{-\lambda})Y_{-\lambda}].
\end{aligned}$$

If $n>1$, then the Ricci operator Q of M^{2n+1} defined by $g(QX,Y)=S(X,Y)$ is given by [20]

$$Q=-2nid+2n(k+1)\eta\otimes\xi+[\mu-2(n-1)]h'. \tag{3.2}$$

Moreover, the scalar curvature of M^{2n+1} is $2n(k-2n)$.

Also for an almost Kenmotsu manifold with generalized $(k,\mu)'$-nullity distribution [14]

$$\begin{aligned}
(\nabla_X h')Y &= -g(h'X+h'^2X,Y)-\eta(Y)(h'X+h'^2X)\\
&\quad -(\mu+2)\eta(X)h'Y,
\end{aligned} \tag{3.3}$$

for all smooth vectors fields X,Y.

From (3.1) it follows that

$$R(\xi,X)Y=k[g(X,Y)\xi-\eta(Y)X]+\mu[g(h'X,Y)\xi-\eta(Y)h'X]. \tag{3.4}$$

Contracting X in (3.1) we have

$$S(Y,\xi) = 2nk\eta(Y). \tag{3.5}$$

4. Almost Kenmotsu Manifolds with ξ belonging to the Generalized $(k,\mu)'$-nullity Distribution satisfying Codazzi Type of Ricci Tensor

In this section we characterize an almost Kenmotsu manifold with ξ belonging to the generalized $(k,\mu)'$-nullity distribution whose Ricci tensor is of Codazzi type.

Taking covariant differentiation of (3.2) we obtain

$$\begin{aligned}(\nabla_Y Q)X = \; & 2n(k+1)[(\nabla_Y\eta)X + \eta(X)\nabla_Y\xi] \\ & + [\mu - 2(n-1)](\nabla_Y h')X.\end{aligned} \tag{4.1}$$

Using (2.1), (2.3) and (3.3) in (4.1) yields

$$\begin{aligned}(\nabla_Y Q)X = \; & 2n(k+1)[g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + g(Y,h'X)\xi + \eta(X)Y + \eta(X)h'Y] \\ & - [\mu - 2(n-1)][g(h'Y,X)\xi + g(h'^2Y,X)\xi + \eta(X)h'Y + \eta(X)h'^2Y + \\ & (\mu+2)\eta(Y)h'X].\end{aligned} \tag{4.2}$$

Suppose the Ricci tensor of the manifold M^{2n+1} is of Codazzi type. Then

$$(\nabla_Y Q)X = (\nabla_X Q)Y, \tag{4.3}$$

for all smooth vector fields X, Y.

Making use of (4.2) in (4.3) implies

$$\begin{aligned}& 2n(k+1)[g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + g(Y,h'X)\xi + \eta(X)Y \\ & +\eta(X)h'Y] - [\mu - 2(n-1)][g(h'Y,X)\xi + g(h'^2Y,X)\xi + \eta(X)h'Y + \\ & \eta(X)h'^2Y + (\mu+2)\eta(Y)h'X] = 2n(k+1)[g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + \\ & g(Y,h'X)\xi + \eta(Y)X + \eta(Y)h'X] - [\mu - 2(n-1)][g(h'Y,X)\xi + \\ & g(h'^2X,Y)\xi + \eta(Y)h'X + \eta(Y)h'^2X + (\mu+2)\eta(X)h'Y].\end{aligned} \tag{4.4}$$

Using the fact $h'^2 = (k+1)\phi^2$ in (4.4) we have

$$\begin{aligned}& 2n(k+1)[(\eta(X)Y - \eta(Y)X) + (\eta(X)h'Y - \eta(Y)h'X)] \\ & -[\mu - 2(n-1)][(\eta(X)h'Y - \eta(Y)h'X) - ((\eta(X)Y - \eta(Y)X)) \\ & +(\mu+2)(\eta(Y)h'X - \eta(X)h'Y)] = 0.\end{aligned} \tag{4.5}$$

Putting $Y = \xi$ in the foregoing equation yields

$$\begin{aligned}& 2n(k+1)[(\eta(X)\xi - X) - h'X)] - [\mu - 2(n-1)][-h'X \\ & -(\eta(X)\xi - X) + (\mu+2)h'X] = 0.\end{aligned} \tag{4.6}$$

Let $X \in [\lambda]'$. Then from (4.6) we get

$$2n(k+1)(\lambda+1) + [\mu - 2(n-1)][-\lambda - \lambda^2 + (\mu+2)\lambda] = 0. \tag{4.7}$$

Again assume that $X \in [-\lambda]'$. Then from (4.6) we obtain

$$2n(k+1)(-\lambda+1) + [\mu - 2(n-1)][\lambda - \lambda^2 - (\mu+2)\lambda] = 0. \tag{4.8}$$

Adding (4.7) and (4.8) we get

$$(k+1)(\mu+2) = 0,$$

that is, either $k = -1$ or $\mu = -2$. If $k = -1$, then $h' = 0$, which is a contradiction. Therefore, $\mu = -2$. Putting $\mu = -2$ in (4.7) yields

$$(\lambda+1)(k+1+\lambda) = 0. \tag{4.9}$$

Using the fact $\lambda^2 = -(k+1)$ in (4.9), we have

$$\lambda(\lambda^2 - 1) = 0, \tag{4.10}$$

that is, either $\lambda = 0$ or $\lambda^2 = 1$. If $\lambda = 0$, then $h' = 0$, which is a contradiction. Therefore, $\lambda^2 = 1$. Making use of $\lambda^2 = 1$ in $\lambda^2 = -(k+1)$ implies $k = -2$. Then we have from Lemma 3.1,

$$R(X_\lambda, Y_\lambda)Z_\lambda = -4[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda]$$

and

$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = 0,$$

for any vector field $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. Also noticing $\mu = -2$, it follows from Lemma 3.1 that $K(X, \xi) = -4$ for any $X \in [\lambda]'$ and $K(X, \xi) = 0$ for any $X \in [-\lambda]'$. Again from Lemma 3.1, we see that $K(X, Y) = -4$ for any $X, Y \in [\lambda]'$; $K(X, Y) = 0$ for any $X, Y \in [-\lambda]'$ and $K(X, Y) = 0$ for any $X \in [\lambda]'$, $Y \in [-\lambda]'$. Also the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $H = -(1-\lambda)\xi$, where H is the mean curvature vector field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Here $\lambda = 1$, then two orthogonal distributions $[\xi] \oplus [\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $H^{n+1}(-4) \times \mathbb{R}^n$.

Conversely, let M^{2n+1} is locally isometric to $H^{n+1}(-4) \times \mathbb{R}^n$. Then by Theorem 4.4 of [20] it follows that M^{2n+1} is locally symmetric. Then the manifold satisfies the condition of Codazzi type of Ricci tensor, that is, $(\nabla_Y Q)X = (\nabla_X Q)Y$, for all smooth vector fields X, Y.

Thus we have the following:

Proposition 4.1. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be an almost Kenmotsu manifolds with generalized $(k,\mu)'$-nullity distribution such that $h' \neq 0$ and $n > 1$. Then the Ricci tensor of the manifold is of Codazzi type if and only if the manifold is locally isometric to $H^{n+1}(-4) \times \mathbb{R}^n$.*

5. Almost Kenmotsu Manifolds with Generalized $(k,\mu)'$-nullity Distribution satisfying div $C = 0$

Let M^{2n+1} be an almost Kenmotsu manifold whose Weyl conformal curvature tensor is divergence free, that is, div $C = 0$. Then we have

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{4n}[X(r)g(Y,Z) - Y(r)g(X,Z)]. \tag{5.1}$$

Using (3.2) in (5.1) yields

$$\begin{aligned}&2n(k+1)[\eta(Y)g(X,Z) - \eta(X)g(Y,Z) + \eta(Y)g(h'X,Z) - \eta(X)g(h'Y,Z)]\\&-[\mu-2(n-1)][\eta(Y)g(h'X,Z) - \eta(X)g(h'Y,Z) + \eta(Y)g(h'^2X,Z)\\&-\eta(X)g(h'^2Y,Z) - (\mu+2)\eta(Y)g(h'X,Z)] = \frac{1}{4n}[X(r)g(Y,Z) - Y(r)g(X,Z)].\end{aligned} \tag{5.2}$$

Using (3.5) and the fact $h^2 = (k+1)\phi^2$ in (5.2) implies

$$\begin{aligned}&2n(k+1)[\eta(Y)g(X,Z) - \eta(X)g(Y,Z) + \eta(Y)g(h'X,Z) - \eta(X)g(h'Y,Z)]\\&-[\mu-2(n-1)][\eta(Y)g(h'X,Z) - \eta(X)g(h'Y,Z) - \eta(Y)g(X,Z) + \eta(X)g(Y,Z)\\&-(\mu+2)\eta(Y)g(h'X,Z)] = \frac{1}{2}[X(k)g(Y,Z) - Y(k)g(X,Z)].\end{aligned} \tag{5.3}$$

Replacing Y by ξ in (5.3) we have

$$\begin{aligned}&2n(k+1)[g(X,Z) - \eta(X)\eta(Z) + g(h'X,Z)] - [\mu-2(n-1)][g(h'X,Z)\\&-g(X,Z) + \eta(X)\eta(Z) - (\mu+2)g(h'X,Z)] = \frac{1}{2}[X(k)\eta(Z) - \xi(k)g(X,Z)].\end{aligned} \tag{5.4}$$

Let $X,Z \in D[\lambda]'$. Then from (5.4) we obtain

$$2n(k+1)[1+\lambda] - [\mu-2(n-1)][\lambda+\lambda^2-\lambda(\mu+2)] = 0. \tag{5.5}$$

Again we assume $X,Z \in [-\lambda]'$. Then (5.4) implies

$$2n(k+1)[1-\lambda] - [\mu-2(n-1)][-\lambda+\lambda^2+\lambda(\mu+2)] = 0. \tag{5.6}$$

Adding (5.5) and (5.6) we have

$$(k+1)(\mu+2) = 0,$$

that is, either $k = -1$ or $\mu = -2$. If $k = -1$, then $h' = 0$, which is a contradiction. Therefore, $\mu = -2$. Making use of $\mu = -2$ in (5.5) yields

$$(\lambda+1)(k+1+\lambda) = 0. \tag{5.7}$$

Using the fact $\lambda^2 = -(k+1)$ in (5.7), we have

$$\lambda(\lambda^2 - 1) = 0, \tag{5.8}$$

that is, either $\lambda = 0$ or $\lambda^2 = 1$. If $\lambda = 0$, then $h' = 0$, which is a contradiction. Therefore, $\lambda^2 = 1$. Making use of $\lambda^2 = 1$ in $\lambda^2 = -(k+1)$, implies $k = -2$. Then we have from Lemma 3.1,

$$R(X_\lambda, Y_\lambda)Z_\lambda = -4[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda]$$

and

$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = 0,$$

for any vector field $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. Also noticing $\mu = -2$, it follows from Lemma 3.1 that $K(X, \xi) = -4$ for any $X \in [\lambda]'$ and $K(X, \xi) = 0$ for any $X \in [-\lambda]'$. Again from Lemma 3.1, we see that $K(X, Y) = -4$ for any $X, Y \in [\lambda]'$; $K(X, Y) = 0$ for any $X, Y \in [-\lambda]'$ and $K(X, Y) = 0$ for any $X \in [\lambda]'$, $Y \in [-\lambda]'$. Also the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $H = -(1-\lambda)\xi$, where H is the mean curvature vector field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Here $\lambda = 1$, then two orthogonal distributions $[\xi] \oplus [\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $H^{n+1}(-4) \times \mathbb{R}^n$.

This leads to the following:

Proposition 5.1. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with generalized $(k, \mu)'$-nullity distribution such that $h' \neq 0$ and $n > 1$. If the manifold is of harmonic Weyl conformal curvature tensor, that is, div $C = 0$, then the manifold is locally isometric to $H^{n+1}(-4) \times \mathbb{R}^n$.*

Since div $R = 0$ implies div $C = 0$, thus we have the following:

Corollary 5.1. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with generalized $(k, \mu)'$-nullity distribution such that $h' \neq 0$ and $n > 1$. If the manifold satisfies div $R = 0$, then the manifold is locally isometric to $H^{n+1}(-4) \times \mathbb{R}^n$.*

Again, since $\nabla C = 0$ (conformally symmetric) implies div $C = 0$, so we obtain the following:

Corollary 5.2. *Conformally symmetric almost Kenmotsu manifolds with generalized $(k, \mu)'$-nullity distribution such that $h' \neq 0$ and $n > 1$ is locally isometric to $H^{n+1}(-4) \times \mathbb{R}^n$.*

Remark 5.1. The above Corollary have been proved by De et al. [4].

Suppose the Ricci tensor of the manifold is of Codazzi type. Then the scalar curvature r is constant. Now if r is constant then from (5.1) it clear that div $C = 0$.

From Proposition 4.1, Proposition 5.1 and the above discussions we can state the following:

Theorem 5.1. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold with generalized $(k,\mu)'$-nullity distribution such that $h'\neq 0$ and $n>1$. Then the following statements are equivalent:*

(i) *The Ricci tensor of M^{2n+1} is of Coddazi type,*

(ii) *The manifold M^{2n+1} satisfies div $C=0$,*

(iii) *The manifold M^{2n+1} is locally isometric to $H^{n+1}(-4)\times\mathbb{R}^n$.*

6. Almost Kenmotsu Manifolds with Generalized $(k,\mu)'$-nullity Distribution satisfying Cyclic Parallel Ricci Tensor

In this section we study almost Kenmotsu manifolds with generalized $(k,\mu)'$-nullity distribution satisfying cyclic parallel Ricci tensor. Suppose the manifold $M^{(2n+1)}$ satisfies cyclic parallel Ricci tensor. Then

$$(\nabla_X S)(Y,Z)+(\nabla_Y S)(X,Z)+(\nabla_Z S)(X,Y)=0, \tag{6.1}$$

Taking inner product with Z in (4.2) yields

$$\begin{aligned}(\nabla_Y S)(X,Z) &= 2n(k+1)[g(X,Y)\eta(Z)-2\eta(X)\eta(Y)\eta(Z)+g(Y,h'X)\eta(Z)\\ &\quad +g(Y,Z)\eta(X)]-[\mu-2(n-1)][g(h'Y,X)\eta(Z)+g(h'^2Y,X)\eta(Z)\\ &\quad +g(h'Y,Z)\eta(X)+g(h'^2Y,Z)\eta(X)+(\mu+2)g(h'X,Z)\eta(Y)].\end{aligned} \tag{6.2}$$

Using (6.2) in (6.1) yields

$$\begin{aligned}&2n(k+1)[g(X,Y)\eta(Z)-2\eta(X)\eta(Y)\eta(Z)+g(Y,h'X)\eta(Z)\\ &+g(X,Z)\eta(Y)]-[\mu-2(n-1)][g(h'Y,X)\eta(Z)+g(h'^2Y,X)\eta(Z)\\ &+g(h'X,Z)\eta(Y)+g(h'^2X,Z)\eta(Y)+(\mu+2)g(h'Y,Z)\eta(X)]\\ &+2n(k+1)[g(X,Y)\eta(Z)-2\eta(X)\eta(Y)\eta(Z)+g(Y,h'X)\eta(Z)\\ &+g(Y,Z)\eta(X)]-[\mu-2(n-1)][g(h'Y,X)\eta(Z)+g(h'^2Y,X)\eta(Z)\\ &+g(h'Y,Z)\eta(X)+g(h'^2Y,Z)\eta(X)+(\mu+2)g(h'X,Z)\eta(Y)]\\ &+2n(k+1)[g(X,Z)\eta(Y)-2\eta(X)\eta(Y)\eta(Z)+g(Z,h'X)\eta(Y)\\ &+g(Y,Z)\eta(X)]-[\mu-2(n-1)][g(h'Z,X)\eta(Y)+g(h'^2Z,X)\eta(Y)\\ &+g(h'Y,Z)\eta(X)+g(h'^2Y,Z)\eta(X)\\ &+(\mu+2)g(h'X,Y)\eta(Z)]=0.\end{aligned} \tag{6.3}$$

Replacing Z by ξ in (6.3) we obtain

$$\begin{aligned}&2n(k+1)[g(X,Y)-\eta(X)\eta(Y)+g(h'X,Y)]-[\mu-2(n-1)][g(h'Y,X)\\ &+g(h'^2Y,X)]=0.\end{aligned} \tag{6.4}$$

Let $X, Y \in [\lambda]'$. Then from (6.4) we have

$$2n(k+1)(1+\lambda) - [\mu - 2(n-1)](\lambda + \lambda^2) = 0. \tag{6.5}$$

Now we assume that $X, Y \in [-\lambda]'$. Then from (6.4) it follows that

$$2n(k+1)(1-\lambda) - [\mu - 2(n-1)](-\lambda + \lambda^2) = 0. \tag{6.6}$$

Using (6.5) and (6.6) implies

$$(k+1)(\mu+2) = 0,$$

that is, either $k = -1$ or $\mu = -2$. If $k = -1$, then $h' = 0$, which is a contradiction. Therefore, $\mu = -2$. Making use of $\mu = -2$ in (6.5) yields

$$(\lambda+1)(k+1+\lambda) = 0. \tag{6.7}$$

Using the fact $\lambda^2 = -(k+1)$ in (6.7), we have

$$\lambda(\lambda^2 - 1) = 0, \tag{6.8}$$

that is, either $\lambda = 0$ or $\lambda^2 = 1$. If $\lambda = 0$, then $h' = 0$, which is a contradiction. Therefore, $\lambda^2 = 1$. Making use of $\lambda^2 = 1$ in $\lambda^2 = -(k+1)$, implies $k = -2$. Then we have from Lemma 3.1,

$$R(X_\lambda, Y\lambda)Z_\lambda = -4[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda]$$

and

$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = 0,$$

for any vector field $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. Also noticing $\mu = -2$, it follows from Lemma 3.1 that $K(X, \xi) = -4$ for any $X \in [\lambda]'$ and $K(X, \xi) = 0$ for any $X \in [-\lambda]'$. Again from Lemma 3.1, we see that $K(X, Y) = -4$ for any $X, Y \in [\lambda]'$; $K(X, Y) = 0$ for any $X, Y \in [-\lambda]'$ and $K(X, Y) = 0$ for any $X \in [\lambda]'$, $Y \in [-\lambda]'$. Also the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $H = -(1-\lambda)\xi$, where H is the mean curvature vector field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Here $\lambda = 1$, then two orthogonal distributions $[\xi] \oplus [\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $H^{n+1}(-4) \times \mathbb{R}^n$.

Thus we can state:

Theorem 6.1 *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an almost Kenmotsu manifolds with generalized $(k, \mu)'$-nullity distribution such that $h' \neq 0$ and $n > 1$. If the manifold satisfies the cyclic parallel Ricci tensor, then the manifold is locally isometric to $H^{n+1}(-4) \times \mathbb{R}^n$.*

References

[1] D. E. Blair, *Contact manifolds in Riemannian geometry,* Lecture Notes on Mathematics **509**, Springer-Verlag, Berlin, 1976.

[2] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds,* Progress in Mathematics **203**, Birkhäuser, Boston, 2010.

[3] D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, *Contact metric manifolds satisfying a nullity condition,* Israel J. Math., **91**(1995), 189–214.

[4] U. C. De, G. Ghosh and P. Majhi, *Some semisymmetry conditions on almost Kenmotsu manifolds with generalized nullity distribution,* to be appeared in Int. Elect. J. Geom in 2018.

[5] U. C. De and K. Mandal, *Ricci solitons on almost Kenmotsu manifolds,* An. Univ. Oradea Fasc. Mat., **23**(2016), 109–116.

[6] U. C. De and K. Mandal, *On ϕ-Ricci recurrent almost Kenmotsu manifolds with nullity distributions,* Int. Electron. J. Geom., **9**(2016), 70–79.

[7] U. C. De and K. Mandal, *On a type of almost Kenmotsu manifolds with nullity distributions,* Arab J. Math. Sci., **23**(2017), 109–123.

[8] U. C. De and K. Mandal, *On locally ϕ-conformally symmetric almost Kenmotsu manifolds with nullity distributions,* Commun. Korean Math. Soc., **32**(2017), 401–416.

[9] G. Dileo and A. M. Pastore, *Almost Kenmotsu manifolds and local symmetry,* Bull. Belg. Math. Soc. Simon Stevin, **14**(2007), 343–354.

[10] G. Dileo and A. M. Pastore, *Almost Kenmotsu manifolds and nullity distributions,* J. Geom., **93**(2009), 46–61.

[11] A. Gray, *Einstein-like manifolds which are not Einstein,* Geom. Dedicata, **7**(1978), 259-280.

[12] D. Janssens and L. Vanhecke, *Almost contact structures and curvature tensors,*

[13] K. Kenmotsu, *A class of almost contact Riemannian manifolds,* Tohoku Math. J., **24**(1972), 93–103.

[14] A. M. Pastore and V. Saltarelli, *Generalized nullity distributions on almost Kenmotsu manifolds,* Int. Electron. J. Geom., **4**(2011), 168–183. Kodai Math J., **4**(1981), 1–27.

[15] Y. Wang, *Conformally flat CR-integrable almost Kenmotsu manifolds,* Bull. Math. Soc. Sci. Math. Roumanie, **59**(2016), 375–387.

[16] Y. Wang and X. Liu, *Riemannian semisymmetric almost Kenmotsu manifolds and nullity distributions,* Ann. Polon. Math., **112**(2014), 37–46.

[17] Y. Wang and X. Liu, *On ϕ-recurrent almost Kenmotsu manifolds,* Kuwait. J. Sci., **42**(2015), 65–77.

[18] Y. Wang and X. Liu, *Locally symmetric CR-integrable almost Kenmotsu manifolds,* Mediterr. J. Math., **12**(2015), 159–171.

[19] Y. Wang and X. Liu, *A Schur-type Theorem for CR-integrable almost Kenmotsu manifolds,* Math. Slovaca, **66**(2016), 1217–1226.

[20] Y. Wang and X. Liu, *On almost Kenmotsu manifolds satisfying some nullity distributions,* Proc. Nat. Acad. Sci. India Sect. A, **86**(2016), 347–353.

[21] Y. Wang and W. Wang, *Curvature properties of almost Kenmotsu manifolds with generalized nullity conditions,* Filomat, **30(14)**(2016), 3807–3816.

[22] K. Yano and M. Kon, *Structures on manifolds,* World Scientific Publishing Co., Singapore, 1984.