KYUNGPOOK Math. J. 58(2018), 117-135
https://doi.org/10.5666/KMJ.2018.58.1.117
pISSN 1225-6951 eISSN 0454-8124

A Characterization of Involutes and Evolutes of a Given Curve in $\mathbb{E}^n$

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Abstract. The orthogonal trajectories of the first tangents of the curve are called the involutes of x. The hyperspheres which have higher order contact with a curve x are known osculating hyperspheres of x. The centers of osculating hyperspheres form a curve which is called generalized evolute of the given curve x in n-dimensional Euclidean space $\mathbb{E}^n$. In the present study, we give a characterization of involute curves of order k (resp. evolute curves) of the given curve x in n-dimensional Euclidean space $\mathbb{E}^n$. Further, we obtain some results on these type of curves in $\mathbb{E}^3$ and $\mathbb{E}^4$, respectively.

1. Introduction

The notions of evolutes and involutes were studied by C. Huygens in his work [7] and studied in differential geometry and singularity theory of planar curves [1]. The evolute of a regular curve in the Euclidean plane is given by not only the locus of all its centres of the curvature, but also the envelope of normal lines of the regular curve, namely, the locus of singular loci of parallel curves. On the other hand, the involute of a regular curve is to replace the taut string by a line segment that is tangent to the curve on one end, while the other end traces out the involute. In ([3], [4]) T. Fukunaga and M. Takahashi defined the evolutes and the involutes of fronts in the plane without inflection points and gave properties of them. Meanwhile, E. Özyılmaz and S. Yılmaz studied the involute-evolute of W-curves in Euclidean 4-space $\mathbb{E}^4$ [13], see also, [16]. Recently, B. Divjak and Ž. M. Šipuš, considered the isotropic involutes (of order k) and the isotropic evolutes in n-dimensional isotropic

* Corresponding Author.
Received July 14, 2017; accepted February 8, 2018.
2010 Mathematics Subject Classification: 53A04, 53A05.
Key words and phrases: Frenet curve, involutes, evolutes.

space $\mathbb{I}_n^{(1)}$ [2, 10].

This paper is organized as follows: Section 2 gives some basic concepts of Frenet curves in Euclidean spaces. Section 3 gives some basic concepts of the involute curves of order k in $\mathbb{E}^n$. Section 4 explains some geometric properties about the involute curves of order k in $\mathbb{E}^3$, where $k = 1, 2$. Section 5 tells about the involute curves of order k in $\mathbb{E}^4$, where $k = 1, 2, 3$. Further these sections provides some properties and results of these type of curves. In the final section we consider generalized evolute curves in $\mathbb{E}^n$. Moreover, we present some results of generalized evolute curves in $\mathbb{E}^3$ and $\mathbb{E}^4$, respectively.

2. Basic Concepts

Let $x = x(t) : I \subset \mathbb{R} \to \mathbb{E}^n$ be a regular curve in $\mathbb{E}^n$, (i.e., $\|x'(t)\| \neq 0$). Then x is called a *Frenet curve of osculating order* d, $(2 \leq d \leq n)$ if $x'(t)$, $x''(t)$,...,$x^{(d)}(t)$ are linearly independent and $x'(t)$, $x''(t)$,...,$x^{(d+1)}(t)$ linearly dependent for all t in I. For the case $d = n$, the Frenet curve x is called generic curve in $\mathbb{E}^n$ [17]. From now on we assume that x is a generic curve in $\mathbb{E}^n$. To each generic curve x one can associates an orthonormal d-frame $V_1 = \frac{x'(t)}{\|x'(t)\|}, V_2, V_3 \ldots, V_n$ along x, the Frenet n-frame, and $n-1$ functions $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$:$I \longrightarrow \mathbb{R}$, the Frenet curvature, such that

$$
\begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ \ldots \\ V_n' \end{bmatrix} = v \begin{bmatrix} 0 & \kappa_1 & 0 & \ldots & 0 \\ -\kappa_1 & 0 & \kappa_2 & \ldots & 0 \\ 0 & -\kappa_2 & 0 & \ldots & 0 \\ \ldots & & & & \kappa_{n-1} \\ 0 & 0 & \ldots & -\kappa_{n-1} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \ldots \\ V_n \end{bmatrix} \tag{2.1}
$$

where, $v = \|x'(t)\|$ is the speed of the curve x. In fact, to obtain $V_1, V_2, V_3 \ldots, V_n$, it is sufficient to apply the Gram-Schmidt orthonormalization process to $x'(t)$, $x''(t), \ldots, x^{(n)}(t)$. Moreover, the functions $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$ are easily obtained as by-product during this calculation.

More precisely, $V_1, V_2, V_3 \ldots, V_n$ and $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$ are determined by the following formulas:

$$
\begin{aligned}
E_1(t) \ &: = x'(t), \\
E_\alpha(t) \ &: = x^{(\alpha)}(t) - \sum_{i=1}^{\alpha-1} < x^{(\alpha)}(t), E_i(t) > \frac{E_i(t)}{\|E_i(t)\|^2}, \\
V_\alpha \ &: = \frac{E_\alpha(t)}{\|E_\alpha(t)\|}, \quad 1 \leq \alpha \leq n
\end{aligned} \tag{2.2}
$$

and

$$
\kappa_\delta(t) := \frac{\|E_{\delta+1}(t)\|}{\|E_\delta(t)\| \, \|E_1(t)\|}, \tag{2.3}
$$

respectively, where $\delta \in \{1, 2, 3, \ldots, n-1\}$ (see, [5]).

The osculating hyperplane of a generic curve x at t is the subspace generated by $\{V_1, V_2, V_3 \ldots, V_n\}$ that passes through $x(t)$. The unit vector $V_n(t)$ is called *binormal vector* of x at t. The *normal hyperplane* of x at t is defined to be the one generated by $\{V_2, V_3 \ldots, V_n\}$ passing through $x(t)$ [14].

A Frenet curve of rank d for which the first Frenet curvature κ_1 is constant is called a *Salkowski curve* [15] (or T.C-curve [8]). Further, a Frenet curve for which $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$ are constant is called (*circular*) *helix* or W-*curve* [9]. Meanwhile, a Frenet curve with constant curvature ratios $\frac{\kappa_2}{\kappa_1}, \frac{\kappa_3}{\kappa_2}, \frac{\kappa_4}{\kappa_3}, \ldots, \frac{\kappa_{n-1}}{\kappa_{n-2}}$ is called a *ccr-curve* (see, [12], [11]). A ccr-curve in $\mathbb{E}^3$ is known as generalized helix.

Given a generic curve x in $\mathbb{E}^4$, the Frenet 4-frame, V_1, V_2, V_3, V_4 and the Frenet curvatures $\kappa_1, \kappa_2, \kappa_3$ are given by

$$\begin{aligned}
V_1(t) &= \frac{x'(t)}{\|x'(t)\|} \\
V_4(t) &= \frac{x'(t) \wedge x''(t) \wedge x'''(t)}{\|x'(t) \wedge x''(t) \wedge x'''(t)\|} \\
V_3(t) &= \frac{V_4(t) \wedge x'(t) \wedge x''(t)}{\|V_4(t) \wedge x'(t) \wedge x''(t)\|} \\
V_2(t) &= \frac{V_3(t) \wedge V_4(t) \wedge x'(t)}{\|V_3(t) \wedge V_4(t) \wedge x'(t)\|}
\end{aligned} \tag{2.4}$$

and

$$\kappa_1(t) = \frac{\langle V_2(t), x''(t)\rangle}{\|x'(t)\|^2}, \ \kappa_2(t) = \frac{\langle V_3(t), x'''(t)\rangle}{\|x'(t)\|^3 \kappa_1(t)}, \kappa_3(t) = \frac{\langle V_4(t), x''''(t)\rangle}{\|x'(t)\|^4 \kappa_1(t)\kappa_2(t)}. \tag{2.5}$$

respectively, where $\wedge$ is the exterior product in $\mathbb{E}^4$ [5].

3. Involute Curves of Order k in $\mathbb{E}^n$

Definition 3.1. Let $x = x(s)$ be a regular generic curve in $\mathbb{E}^n$ given with the arclength parameter s (*i.e.*, $\|x'(s)\| = 1$). Then the curves which are orthogonal to the system of k-dimensional osculating hyperplanes of x, are called the *involutes of order* k [2] (or, k^{th} involute [6]) of the curve x. For simplicity, we call the *involutes of order* 1, simply the involutes of the given curve.

In order to find the parametrization of involutes $\overline{x}$ of order k of the curve x, we put

$$\overline{x}(s) = x(s) + \sum_{\alpha=1}^{k} \lambda_\alpha(s) V_\alpha(s), \ k \leq n-1 \tag{3.1}$$

where λ_α is a differentiable function and s is the parameter of $\overline{x}$ which is not necessarily an arclength parameter. The differentiation of the equation (3.1) and the Frenet formulae (2.1) are given in the following equation

$$\begin{aligned}\overline{x}'(s) &= (1+\lambda_1'-\kappa_1\lambda_2)(s)V_1(s)\\ &\quad+\sum_{\alpha=2}^{k-1}(\lambda_\alpha'-\lambda_{\alpha+1}\kappa_\alpha+\lambda_{\alpha-1}\kappa_{\alpha-1})(s)V_\alpha(s)\\ &\quad+(\lambda_k'+\lambda_{k-1}\kappa_{k-1})(s)V_k(s)+\kappa_k(s)\lambda_k(s)V_{k+1}(s).\end{aligned} \tag{3.2}$$

Furthermore, the involutes $\overline{x}$ of order k of the curve x are determined by

$$\langle\overline{x}'(s),V_j(s)\rangle=0,\quad 1\le j\le k,\quad k\le n-1.$$

This condition is satisfied if and only if

$$\begin{aligned}1+\lambda_1'-\kappa_1\lambda_2 &= 0,\\ \lambda_\alpha'-\lambda_{\alpha+1}\kappa_\alpha+\lambda_{\alpha-1}\kappa_{\alpha-1} &= 0,\\ \lambda_k'+\lambda_{k-1}\kappa_{k-1} &= 0,\end{aligned} \tag{3.3}$$

where $2\le\alpha\le n-1$ [2].

In the sequel we characterize the involutes of generic curves in $\mathbb{E}^3$ and $\mathbb{E}^4$.

4. Involutes in $\mathbb{E}^3$

In the present section we consider involutes of order 1 and of order 2 of curves in Euclidean 3-space $\mathbb{E}^3$, respectively.

4.1. Involutes of Order 1 in $\mathbb{E}^3$

Proposition 4.1.1. *Let $x=x(s)$ be a regular curve in $\mathbb{E}^3$ given with nonzero Frenet curvatures κ_1 and κ_2. Then Frenet curvatures $\overline{\kappa}_1$ and $\overline{\kappa}_2$ of the involute $\overline{x}$ of the curve x are given by*

$$\overline{\kappa}_1=\frac{\sqrt{\kappa_1^2+\kappa_2^2}}{|\kappa_1|\,|s-c|},\quad \overline{\kappa}_2=\frac{\left(\frac{\kappa_2}{\kappa_1}\right)'\kappa_1^2}{(\kappa_1^2+\kappa_2^2)\,(c-s)}. \tag{4.1}$$

Proof. Let $\overline{x}=\overline{x}(s)$ be the involute of the curve x in $\mathbb{E}^3$. Then by the use of (3.2) with (3.3) we get $1+\lambda_1'(s)=0$, and furthermore $\lambda(s)=(c-s)$ for some integral constant c. So, we get the following parametrization

$$\overline{x}(s)=x(s)+(c-s)V_1(s). \tag{4.2}$$

Further, the differentiation of (4.2) implies that

$$\begin{aligned}\overline{x}'(s) &= \varphi V_2,\quad \varphi(s):=\lambda(s)\kappa_1(s)\\ \overline{x}''(s) &= -\varphi\kappa_1V_1+\varphi'V_2+\varphi\kappa_2V_3,\\ \overline{x}'''(s) &= -\left\{(\kappa_1\varphi)'+\kappa_1\varphi'\right\}V_1+\left\{\varphi''-\kappa_1^2\varphi-\kappa_2^2\varphi\right\}V_2+\left\{(\kappa_2\varphi)'+\kappa_2\varphi'\right\}V_3.\end{aligned}$$

Now, an easy calculation gives

$$
\begin{aligned}
\|\overline{x}'(s)\| &= |\varphi| = |(c-s)\kappa_1|\,, \\
\|\overline{x}'(s)\times\overline{x}''(s)\| &= \varphi^2\sqrt{\kappa_1^2+\kappa_2^2}, \\
\langle \overline{x}'(s)\times\overline{x}''(s), \overline{x}'''(s)\rangle &= \varphi^3\left(\kappa_1\kappa_2'-\kappa_2\kappa_1'\right).
\end{aligned} \tag{4.3}
$$

The parameter s is not the arclength parameter of $\overline{x}$, so, as is shown in [2], we have

$$
\overline{\kappa}_1 = \frac{\|\overline{x}'(s)\times\overline{x}''(s)\|}{\|\overline{x}'(s)\|^3}, \quad \overline{\kappa}_2 = \frac{\langle \overline{x}'(s)\times\overline{x}''(s), \overline{x}'''(s)\rangle}{\|\overline{x}'(s)\times\overline{x}''(s)\|^2} \tag{4.4}
$$

Hence, from the relations (4.3) and (4.4) we deduce (4.1). □

By the use of (4.1) one can get the following result.

Corollary 4.1.2. *If $x = x(s)$ is a cylindrical helix in $\mathbb{E}^3$, then the involute $\overline{x}$ of x is a planar curve.*

4.2. Involutes of Order 2 in $\mathbb{E}^3$

An involute of order 2 of a regular curve x in $\mathbb{E}^3$ has the parametrization

$$
\overline{x}(s) = x(s) + \lambda_1(s)V_1(s) + \lambda_2(s)V_2(s) \tag{4.5}
$$

where V_1, V_2 are tangent and normal vectors of x in $\mathbb{E}^3$ and λ_1, λ_2 are differentiable functions satisfying

$$
\begin{aligned}
\lambda_1'(s) &= \kappa_1(s)\lambda_2(s) - 1, \\
\lambda_2'(s) &= -\lambda_1(s)\kappa_1(s).
\end{aligned} \tag{4.6}
$$

We obtain the following result.

Proposition 4.2.1. *Let $x = x(s)$ be a regular curve in $\mathbb{E}^3$ with nonzero Frenet curvatures κ_1 and κ_2. Then*

$$
\overline{\kappa}_1 = \frac{sgn(\kappa_2)}{|\lambda_2|}, \quad \overline{\kappa}_2 = \frac{\frac{\kappa_2}{\kappa_1}}{\lambda_2} \tag{4.7}
$$

holds, where $\overline{\kappa}_1$ and $\overline{\kappa}_2$ are Frenet curvatures of $\overline{x}$.

Proof. Let $\overline{x} = \overline{x}(s)$ be the involute of order 2 of the curve x in $\mathbb{E}^3$. Then by the use of (3.2) with (3.3) we get

$$
\overline{x}'(s) = \lambda_2(s)\kappa_2(s)V_3(s). \tag{4.8}
$$

Further, the differentiation of (4.8) implies that

$$
\begin{aligned}
\overline{x}'(s) &= \psi(s)V_3(s), \quad \psi(s) := \lambda_2(s)\kappa_2(s) \\
\overline{x}''(s) &= -\psi(s)\kappa_2(s)V_2(s) + \psi'(s)V_3(s), \\
\overline{x}'''(s) &= -\psi(s)\kappa_1(s)\kappa_2(s)V_1(s) - \left\{(\psi(s)\kappa_2(s))' + \kappa_2(s)\psi'(s)\right\}V_2(s) \\
&\quad + \left\{\psi''(s) + \psi(s)\kappa_2^2(s)\right\}V_3(s).
\end{aligned}
$$

Now, an easy calculation gives

$$
\begin{aligned}
\|\overline{x}'(s)\| &= |\psi(s)| = |\lambda_2(s)\kappa_2(s)|, \\
\|\overline{x}'(s) \times \overline{x}''(s)\| &= \psi(s)^2\kappa_2(s), \\
\langle \overline{x}'(s) \times \overline{x}''(s), \overline{x}'''(s)\rangle &= \psi(s)^3\kappa_1(s)\kappa_2^2(s).
\end{aligned} \tag{4.9}
$$

Hence, from the relations (4.4) and (4.9) we deduce (4.7). □

Corollary 4.2.2. *The involute $\overline{x}$ of order 2 of a generalized helix in $\mathbb{E}^3$ is also a generalized helix in $\mathbb{E}^3$.*

Solving the system of differential equations (4.6) we get the following result.

Corollary 4.2.3. *Let $x = x(s)$ be a unit speed Salkowski curve in $\mathbb{E}^3$. Then the involute $\overline{x}$ of order 2 of the curve x has the parametrization (4.5) given with the coefficient functions*

$$
\begin{aligned}
\lambda_1(s) &= c_1 \sin(\kappa_1 s) + c_2 \cos(\kappa_1 s), \\
\lambda_2(s) &= c_1 \cos(\kappa_1 s) - c_2 \sin(\kappa_1 s) - \frac{1}{\kappa_1}.
\end{aligned} \tag{4.10}
$$

where c_1 and c_2 are real constants.

5. Involutes in $\mathbb{E}^4$

In the present section we consider involutes of order k, $1 \leq k \leq 3$ of a given curve x in Euclidean 4-space $\mathbb{E}^4$.

5.1. Involutes of Order 1 in $\mathbb{E}^4$

The following result gives a simple representation of Theorem 1 in [16].

Proposition 5.1.1. *Let $x = x(s)$ be a regular curve in $\mathbb{E}^4$ given with the Frenet curvatures κ_1, κ_2 and κ_3. Then Frenet 4-frame, $\overline{V}_1, \overline{V}_2, \overline{V}_3$ and $\overline{V}_4$ and Frenet curvatures $\overline{\kappa}_1$, $\overline{\kappa}_2$ and $\overline{\kappa}_3$ of the involute $\overline{x}$ of the curve x are given by*

$$
\begin{aligned}
\overline{V}_1(s) &= V_2, \\
\overline{V}_2(s) &= \frac{-\kappa_1 V_1 + \kappa_2 V_3}{\sqrt{\kappa_1^2 + \kappa_2^2}}, \\
\overline{V}_3(s) &= \frac{-\left(\kappa_2 A - \kappa_1 C\right)\kappa_2 V_1 - \left(\kappa_2 A - \kappa_1 C\right)\kappa_1 V_3 + D\left(\kappa_1^2 + \kappa_2^2\right) V_4}{W\sqrt{\kappa_1^2 + \kappa_2^2}}, \\
\overline{V}_4(s) &= \frac{D\kappa_2 V_1 + D\kappa_1 V_3 - \left(\kappa_2 A - \kappa_1 C\right) V_4}{W},
\end{aligned} \tag{5.1}
$$

and

$$(5.2)\qquad \begin{aligned} \overline{\kappa}_1 &= \frac{\sqrt{\kappa_1^2+\kappa_2^2}}{|\varphi|}; \quad \varphi := (c-s)\kappa_1, \\ \overline{\kappa}_2 &= \frac{W}{\varphi^2\left(\kappa_1^2+\kappa_2^2\right)}, \\ \overline{\kappa}_3 &= -\frac{\left(\kappa_2 A-\kappa_1 C\right)\left(\kappa_3 C+D'\right)+D\left(\kappa_2 A'-\kappa_1 C'\right)+D^2\kappa_1\kappa_3}{W\varphi^4\overline{\kappa}_1\overline{\kappa}_2}, \end{aligned}$$

respectively, where

$$\begin{aligned} A &= \kappa_1'\varphi+2\kappa_1\varphi' \\ C &= \kappa_2'\varphi+2\kappa_2\varphi' \\ D &= \kappa_2\kappa_3\varphi \end{aligned}$$

and

$$(5.3)\qquad \begin{aligned} W &= \sqrt{D^2\left(\kappa_1^2+\kappa_2^2\right)+\left(\kappa_1 C-\kappa_2 A\right)^2} \\ &= |\varphi|\sqrt{\kappa_2^2\kappa_3^2\left(\kappa_1^2+\kappa_2^2\right)+\left(\kappa_1\kappa_2'-\kappa_2\kappa_1'\right)^2}. \end{aligned}$$

Proof. As in the proof of Proposition 4.1.1, the involute $\overline{x}=\overline{x}(s)$ of the curve x in $\mathbb{E}^4$ has the parametrization

$$\overline{x}(s)=x(s)+(c-s)V_1(s),$$

where V_1 is the unit tangent vector of x.

Further, the differentiation of the position vector $\overline{x}(s)$ implies that

$$(5.4)\qquad \begin{aligned} \overline{x}'(s) &= \varphi V_2, \\ \overline{x}''(s) &= -\varphi\kappa_1 V_1+\varphi' V_2+\varphi\kappa_2 V_3, \\ \overline{x}'''(s) &= -\left\{(\kappa_1\varphi)'+\kappa_1\varphi'\right\}V_1+\left\{\varphi''-\kappa_1^2\varphi-\kappa_2^2\varphi\right\}V_2 \\ &\quad +\left\{(\kappa_2\varphi)'+\kappa_2\varphi'\right\}V_3+\varphi\kappa_2\kappa_3 V_4, \end{aligned}$$

where $\varphi=(c-s)\kappa_1$ is a differentiable function. Consequently, substituting

$$(5.5)\qquad \begin{aligned} A &= \kappa_1'\varphi+2\kappa_1\varphi' \\ B &= \varphi''-\kappa_1^2\varphi-\kappa_2^2\varphi \\ C &= \kappa_2'\varphi+2\kappa_2\varphi' \\ D &= \varphi\kappa_2\kappa_3, \end{aligned}$$

the last vector becomes

$$(5.6)\qquad \overline{x}'''=-AV_1+BV_2+CV_3+DV_4.$$

Furthermore, differentiating $\overline{x}'''$ with respect to s, we get

$$\begin{aligned} \overline{x}'''' &= -\{A' + \kappa_1 B\} V_1 + \{-\kappa_1 A - \kappa_2 C + B'\} V_2 \\ &\quad + \{\kappa_2 B - \kappa_3 D + C'\} V_3 + \{D' + \kappa_3 C\} V_4. \end{aligned} \tag{5.7}$$

Now, by the use of (5.4), we can compute the vector $\overline{x}'(s) \wedge \overline{x}''(s) \wedge \overline{x}'''(s)$ and second principal normal of $\overline{x}$ as the following;

$$\overline{x}'(s) \wedge \overline{x}''(s) \wedge \overline{x}'''(s) = \varphi^2 \{D\kappa_2 V_1 + D\kappa_1 V_3 + (\kappa_1 C - \kappa_2 A) V_4\}$$

and

$$\overline{V}_4(s) = \frac{x'(s) \wedge x''(s) \wedge x'''(s)}{\|x'(s) \wedge x''(s) \wedge x'''(s)\|} = \frac{D\kappa_2 V_1 + D\kappa_1 V_3 - (\kappa_2 A - \kappa_1 C) V_4}{W}, \tag{5.8}$$

where

$$W = \sqrt{D^2 (\kappa_1^2 + \kappa_2^2) + (\kappa_2 A - \kappa_1 C)^2}. \tag{5.9}$$

Similarly,

$$\overline{V}_4(s) \wedge \overline{x}'(s) \wedge \overline{x}''(s) = \frac{\varphi^2}{W} \left\{ -(\kappa_2 A - \kappa_1 C) \kappa_2 V_1 - (\kappa_2 A - \kappa_1 C) \kappa_1 V_3 + D \left(\kappa_1^2 + \kappa_2^2\right) V_4 \right\}$$

and

$$\begin{aligned} \overline{V}_3(s) &= \frac{\overline{V}_4(s) \wedge \overline{x}'(s) \wedge \overline{x}''(s)}{\left\|\overline{V}_4(s) \wedge \overline{x}'(s) \wedge \overline{x}''(s)\right\|} \\ &= \frac{-(\kappa_2 A - \kappa_1 C) \kappa_2 V_1 - (\kappa_2 A - \kappa_1 C) \kappa_1 V_3 + D \left(\kappa_1^2 + \kappa_2^2\right) V_4}{W \sqrt{\kappa_1^2 + \kappa_2^2}}. \end{aligned} \tag{5.10}$$

Finally, the vectors $\overline{V}_3(s) \wedge \overline{V}_4(s) \wedge \overline{x}'(s)$ and $\overline{V}_2(s)$ are

$$\overline{V}_3(s) \wedge \overline{V}_4(s) \wedge \overline{x}'(s) = \varphi \left\{ D^2 \left(\kappa_1^2 + \kappa_2^2\right) - (\kappa_2 A - \kappa_1 C)^2 \right\} (-\kappa_1 V_1 + \kappa_2 V_3)$$

and

$$\overline{V}_2(s) = \frac{\overline{V}_3(s) \wedge \overline{V}_4(s) \wedge \overline{x}'(s)}{\left\|\overline{V}_3(s) \wedge \overline{V}_4(s) \wedge \overline{x}'(s)\right\|} = \frac{-\kappa_1 V_1 + \kappa_2 V_3}{\sqrt{\kappa_1^2 + \kappa_2^2}}. \tag{5.11}$$

Consequently, an easy calculation gives

$$\begin{aligned} \left\langle \overline{V}_2(s), \overline{x}''(s) \right\rangle &= \varphi \sqrt{\kappa_1^2 + \kappa_2^2} \\ \left\langle \overline{V}_3(s), \overline{x}'''(s) \right\rangle &= \frac{W}{\sqrt{\kappa_1^2 + \kappa_2^2}} \\ \left\langle \overline{V}_4(s), \overline{x}''''(s) \right\rangle &= -\frac{(\kappa_2 A - \kappa_1 C)(\kappa_3 C + D') + D(\kappa_2 A' - \kappa_1 C') + D^2 \kappa_1 \kappa_3}{W}. \end{aligned} \tag{5.12}$$

Hence, from the relations (5.12) and (4.4) we deduce (5.2). This completes the proof of the proposition. □

If x is a W-curve we find the following results.

Corollary 5.1.2. *Let $\overline{x}$ be an involute of a generic x curve in $\mathbb{E}^4$ given with the Frenet curvatures $\overline{\kappa}_1, \overline{\kappa}_2$ and $\overline{\kappa}_3$. If x is a W-curve then the Frenet 4-frame, $\overline{V}_1, \overline{V}_2, \overline{V}_3$ and $\overline{V}_4$ and the Frenet curvatures $\overline{\kappa}_1$, $\overline{\kappa}_2$ and $\overline{\kappa}_3$ of the involute $\overline{x}$ of the curve x are given by*

$$\begin{aligned} \overline{V}_1(s) &= V_2, \\ \overline{V}_2(s) &= \frac{-\kappa_1 V_1 + \kappa_2 V_3}{\sqrt{\kappa_1^2+\kappa_2^2}} \\ \overline{V}_3(s) &= V_4 \\ \overline{V}_4(s) &= \frac{\kappa_2 V_1 + \kappa_1 V_3}{\sqrt{\kappa_1^2+\kappa_2^2}}, \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} \overline{\kappa}_1 &= \frac{\sqrt{\kappa_1^2+\kappa_2^2}}{|\varphi|}, \\ \overline{\kappa}_2 &= \frac{\kappa_2\kappa_3}{|\varphi|\sqrt{\kappa_1^2+\kappa_2^2}}, \\ \overline{\kappa}_3 &= \frac{-\kappa_1\kappa_3}{|\varphi|\sqrt{\kappa_1^2+\kappa_2^2}} \end{aligned} \tag{5.14}$$

respectively, where $\varphi = (c-s)\kappa_1$.

Corollary 5.1.3. *Let $\overline{x}$ be an involute of a generic curve x in $\mathbb{E}^4$ given with the Frenet curvatures $\overline{\kappa}_1, \overline{\kappa}_2$ and $\overline{\kappa}_3$. If x is a W-curve then $\overline{x}$ becomes a ccr-curve.*

Proof. Let x be a regular W-curve of $\mathbb{E}^4$. Since the ratios

$$\begin{aligned} \frac{\overline{\kappa}_2}{\overline{\kappa}_1} &= \frac{\kappa_2\kappa_3}{\kappa_1^2+\kappa_2^2} \\ \frac{\overline{\kappa}_3}{\overline{\kappa}_2} &= -\frac{\kappa_1}{\kappa_2} \end{aligned}$$

are constant functions then the involute curve $\overline{x}$ is a ccr-curve. □

5.2. Involutes of Order 2 in $\mathbb{E}^4$

An involute of order 2 of a regular curve x in $\mathbb{E}^4$ has the parametrization

$$\overline{x}(s) = x(s) + \lambda_1(s)V_1(s) + \lambda_2(s)V_2(s) \tag{5.15}$$

where V_1, V_2 are tangent and normal vectors of x in $\mathbb{E}^4$ and λ_1, λ_2 are differentiable functions satisfying

$$\begin{aligned} \lambda_1'(s) &= \kappa_1(s)\lambda_2(s) - 1, \\ \lambda_2'(s) &= -\lambda_1(s)\kappa_1(s). \end{aligned} \tag{5.16}$$

As in the previous subsection we get the following result.

Corollary 5.2.1. *Let $x = x(s)$ be a unit speed Salkowski curve in $\mathbb{E}^4$. Then the involute $\overline{x}$ of order 2 of the curve x has the parametrization (5.15) given with the coefficient functions*

$$\begin{aligned} \lambda_1(s) &= c_1 \sin(\kappa_1 s) + c_2 \cos(\kappa_1 s), \\ \lambda_2(s) &= c_1 \cos(\kappa_1 s) - c_2 \sin(\kappa_1 s) - \frac{1}{\kappa_1}. \end{aligned} \tag{5.17}$$

where c_1 and c_2 are real constants.

Proof. Assume that $x = x(s)$ is a unit speed Salkowski curve in $\mathbb{E}^4$ then $\kappa_1(s)$ is a constant function. So, differentiating first equation of (5.16) and using the second equation we get

$$\lambda_1''(s) = -\kappa_1^2 \lambda_1(s)$$

which has a solution $\lambda_1(s) = c_1 \sin(\kappa_1 s) + c_2 \cos(\kappa_1 s)$. And substituting this function into the first equation of (5.16) we obtain the second equation of (5.17). □

We obtain the following result.

Proposition 5.2.2. *Let $x = x(s)$ be a regular curve in $\mathbb{E}^4$ given with nonzero Frenet curvatures κ_1, κ_2 and κ_3. Then Frenet 4-frame, $\overline{V}_1, \overline{V}_2, \overline{V}_3$ and $\overline{V}_4$ and Frenet curvatures $\overline{\kappa}_1$, $\overline{\kappa}_2$ and $\overline{\kappa}_3$ of the involute $\overline{x}$ of order 2 of a regular curve x in $\mathbb{E}^4$ are given by*

$$\begin{aligned} \overline{V}_1(s) &= V_3, \\ \overline{V}_2(s) &= \frac{-\kappa_2 V_2 + \kappa_3 V_4}{\sqrt{\kappa_2^2 + \kappa_3^2}}, \\ \overline{V}_3(s) &= \frac{K\left(\kappa_2^2 + \kappa_3^2\right) V_1 + \left(\kappa_2 N - \kappa_3 L\right)\kappa_3 V_2 + \left(\kappa_2 N - \kappa_3 L\right)\kappa_2 V_4}{W\sqrt{\kappa_2^2 + \kappa_3^2}}, \\ \overline{V}_4(s) &= \frac{\left(\kappa_2 N - \kappa_3 L\right) V_1 + \kappa_3 K V_2 + \kappa_2 K V_4}{W}, \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} \overline{\kappa}_1 &= \frac{\sqrt{\kappa_2^2 + \kappa_3^2}}{|\phi|}; \quad \phi := \lambda_2(s)\kappa_2(s) \\ \overline{\kappa}_2 &= \frac{W}{\phi^2\left(\kappa_2^2 + \kappa_3^2\right)}, \\ \overline{\kappa}_3 &= \frac{\left(\kappa_2 N - \kappa_3 L\right)\left(\kappa_1 L + K'\right) + \left(\kappa_2 N' - \kappa_3 L'\right) K + \kappa_1 \kappa_3 K^2}{W \phi^4 \overline{\kappa}_1 \overline{\kappa}_2} \end{aligned} \tag{5.19}$$

where

$$\begin{aligned} K &= \kappa_1\kappa_2\phi \\ L &= 2\kappa_2\phi' + \kappa_2'\phi \\ N &= 2\kappa_3\phi' + \kappa_3'\phi \end{aligned}$$

and

$$\begin{aligned} W &= \sqrt{K^2\left(\kappa_2^2+\kappa_3^2\right)+\left(\kappa_2 N-\kappa_3 L\right)^2} \\ &= |\phi|\sqrt{\kappa_1^2\kappa_2^2\left(\kappa_2^2+\kappa_3^2\right)+\left(\kappa_2\kappa_3'-\kappa_3\kappa_2'\right)^2}. \end{aligned} \tag{5.20}$$

Proof. Let $\overline{x} = \overline{x}(s)$ be the involute of order 2 of the curve x in $\mathbb{E}^4$. Then by the use of (3.2), we get

$$\overline{x}'(s) = \phi V_3 \tag{5.21}$$

where $\phi = \lambda_2(s)\kappa_2(s)$ is a differentiable function. Further, the differentiation of (5.21) implies that

$$\begin{aligned} \overline{x}''(s) &= -\phi\kappa_2 V_2 + \phi' V_3 + \phi\kappa_3 V_4, \\ \overline{x}'''(s) &= \kappa_1\kappa_2\phi V_1 + \left\{2\kappa_2\phi' + \kappa_2'\phi\right\} V_2, \\ &\quad + \left\{\phi'' - \kappa_2^2\phi - \kappa_3^2\phi\right\} V_3 + \left\{2\kappa_3\phi' + \kappa_3'\phi\right\} V_4. \end{aligned} \tag{5.22}$$

Consequently, substituting

$$\begin{aligned} K &= \kappa_1\kappa_2\phi \\ L &= 2\kappa_2\phi' + \kappa_2'\phi \\ M &= \phi'' - \kappa_2^2\phi - \kappa_3^2\phi \\ N &= 2\kappa_3\phi' + \kappa_3'\phi \end{aligned} \tag{5.23}$$

the last vector becomes

$$\overline{x}''' = KV_1 - LV_2 + MV_3 + NV_4. \tag{5.24}$$

Furthermore, differentiating $\overline{x}'''$ with respect to s we get

$$\begin{aligned} \overline{x}'''' &= \left\{K' + \kappa_1 L\right\} V_1 + \left\{\kappa_1 K - \kappa_2 M - L'\right\} V_2 \\ &\quad + \left\{M' - \kappa_2 L - \kappa_3 N\right\} V_3 + \left\{N' + \kappa_3 M\right\} V_4. \end{aligned} \tag{5.25}$$

Hence, substituting (5.21)-(5.25) into (2.4) and (2.5), after some calculations as in the previous proposition, we get the result. □

If x is a W-curve then we find the following results.

Corollary 5.2.3. *Let $\overline{x}$ be an involute of order 2 of a generic curve x in $\mathbb{E}^4$ given with the Frenet curvatures $\overline{\kappa}_1, \overline{\kappa}_2$ and $\overline{\kappa}_3$. If x is a W-curve then the Frenet 4-frame, $\overline{V}_1, \overline{V}_2, \overline{V}_3$ and $\overline{V}_4$ and Frenet curvatures $\overline{\kappa}_1$, $\overline{\kappa}_2$ and $\overline{\kappa}_3$ of the involute $\overline{x}$ of order 2 of a regular curve x in $\mathbb{E}^4$ are given by*

$$
\begin{aligned}
\overline{V}_1(s) &= V_3, \\
\overline{V}_2(s) &= \frac{-\kappa_2 V_2 + \kappa_3 V_4}{\sqrt{\kappa_2^2 + \kappa_3^2}} \\
\overline{V}_3(s) &= V_1 \\
\overline{V}_4(s) &= \frac{\kappa_3 V_2 + \kappa_2 V_4}{\sqrt{\kappa_2^2 + \kappa_3^2}},
\end{aligned} \tag{5.26}
$$

and

$$
\begin{aligned}
\overline{\kappa}_1 &= \frac{\sqrt{\kappa_2^2 + \kappa_3^2}}{|\phi|}, \\
\overline{\kappa}_2 &= \frac{\kappa_1 \kappa_2}{|\phi| \sqrt{\kappa_2^2 + \kappa_3^2}}, \\
\overline{\kappa}_3 &= \frac{\kappa_1 \kappa_3}{|\phi| \sqrt{\kappa_2^2 + \kappa_3^2}}
\end{aligned} \tag{5.27}
$$

where $\phi(s) = \lambda_2(s)\kappa_2(s)$.

Corollary 5.2.4. *Let $\overline{x}$ be an involute of order 2 of a generic curve x in $\mathbb{E}^4$ given with the Frenet curvatures $\overline{\kappa}_1$, $\overline{\kappa}_2$ and $\overline{\kappa}_3$. If x is a W-curve then $\overline{x}$ becomes a ccr-curve.*

Proof. Let x be a regular W-curve of $\mathbb{E}^4$. Since the ratios

$$
\begin{aligned}
\frac{\overline{\kappa}_2}{\overline{\kappa}_1} &= \frac{\kappa_1 \kappa_2}{\kappa_2^2 + \kappa_3^2} \\
\frac{\overline{\kappa}_3}{\overline{\kappa}_2} &= -\frac{\kappa_3}{\kappa_2}
\end{aligned}
$$

are constant functions then the involute curve $\overline{x}$ is a ccr-curve. □

5.3. Involutes of Order 3 in $\mathbb{E}^4$

An involute of order 3 of a regular curve x in $\mathbb{E}^4$ has the parametrization

$$\overline{x}(s) = x(s) + \lambda_1(s)V_1(s) + \lambda_2(s)V_2(s) + \lambda_3(s)V_3(s) \tag{5.28}$$

where

$$
\begin{gathered}
\lambda_1'(s) = \kappa_1(s)\lambda_2(s) - 1, \\
\lambda_2'(s) = \lambda_3\kappa_2 - \lambda_1\kappa_1 \\
\lambda_3'(s) = -\lambda_2(s)\kappa_2(s).
\end{gathered} \tag{5.29}
$$

By solving the system of differential equations in (5.29) we get the following result.

Corollary 5.3.1. *Let $x = x(s)$ be a unit speed W-curve in $\mathbb{E}^4$. Then the involute $\overline{x}$ of order 3 of the curve x has the parametrization (5.28) given with the coefficient functions*

$$\begin{aligned} \lambda_1(s) &= \frac{\kappa_1\left(c_1\sin(\sqrt{\kappa}s) - c_2\cos(\sqrt{\kappa}s)\right)}{\sqrt{\kappa}} - \frac{\kappa_2^2 s}{\kappa} + c_3, \\ \lambda_2(s) &= c_1\cos(\sqrt{\kappa}s) + c_2\sin(\sqrt{\kappa}s) + \frac{\kappa_1}{\kappa}, \\ \lambda_3(s) &= \frac{\kappa_2\left(c_2\cos(\sqrt{\kappa}s) - c_1\sin(\sqrt{\kappa}s)\right)}{\sqrt{\kappa}} - \frac{\kappa_1\kappa_2 s}{\kappa} + c_4, \end{aligned} \tag{5.30}$$

where $\kappa = \kappa_1^2 + \kappa_2^2$, c_1, c_2, c_3 and c_4 are real constants.

Proof. Suppose that x is a unit speed W-curve in $\mathbb{E}^4$ then the Frenet curvatures κ_1, κ_2 and κ_3 of x are constant functions. Consequently, if $\overline{x}$ is the involute of x which is an order 3 curve then (5.29) holds. Differentiating the second equation of (5.29) and using the others we get $\lambda_2(s) = c_1\cos(\sqrt{\kappa}s) + c_2\sin(\sqrt{\kappa}s) + \frac{\kappa_1}{\kappa}$. Further, substituting this function into (5.29) we get the result. □

We obtain the following result.

Proposition 5.3.2. *Let $x = x(s)$ be a regular curve in $\mathbb{E}^4$ given with nonzero Frenet curvatures κ_1, κ_2 and κ_3. Then Frenet 4-frame, $\overline{V}_1, \overline{V}_2, \overline{V}_3$ and $\overline{V}_4$ and Frenet curvatures $\overline{\kappa}_1$, $\overline{\kappa}_2$ and $\overline{\kappa}_3$ of the involute $\overline{x}$ of order 3 of a regular curve x in $\mathbb{E}^4$ are given by*

$$\begin{aligned} \overline{V}_1(s) &= V_4, \\ \overline{V}_2(s) &= -V_3, \\ \overline{V}_3(s) &= V_2, \\ \overline{V}_4(s) &= V_1, \end{aligned} \tag{5.31}$$

and

$$\begin{aligned} \overline{\kappa}_1 &= \frac{\kappa_3}{|\psi|}, \\ \overline{\kappa}_2 &= \frac{\kappa_2}{|\psi|}, \\ \overline{\kappa}_3 &= -\frac{\kappa_1}{|\psi|}, \end{aligned} \tag{5.32}$$

where $\psi(s) = \lambda_3(s)\kappa_3(s)$.

Proof. Let $\overline{x} = \overline{x}(s)$ be the involute of order 3 of the curve x in $\mathbb{E}^4$. Then by the use of (3.2) with (3.3), we get

$$\overline{x}'(s) = \psi V_4 \tag{5.33}$$

where $\psi = \lambda_3(s)\kappa_3(s)$ is a differentiable function. Further, the differentiation of (5.33) implies that

$$\begin{aligned}\overline{x}''(s) &= -\psi\kappa_3 V_3 + \psi' V_4,\\ \overline{x}'''(s) &= \kappa_2\kappa_3\psi V_2 - \{2\kappa_3'\psi + \kappa_3'\phi\} V_3 + \{\psi'' - \kappa_3^2\psi\} V_4.\end{aligned}$$

Consequently, substituting

$$\begin{aligned}E &= \kappa_2\kappa_3\psi\\ F &= 2\kappa_3'\psi + \kappa_3'\phi\\ G &= \psi'' - \kappa_3^2\psi\end{aligned} \tag{5.34}$$

the last vector becomes

$$\overline{x}''' = EV_2 - FV_3 + GV_4. \tag{5.35}$$

Furthermore, differentiating $\overline{x}'''$ with respect to s we get

$$\begin{aligned}\overline{x}'''' &= -\kappa_1 E V_1 + \{\kappa_2 F + E'\} V_2\\ &\quad + \{\kappa_2 E - \kappa_3 G - F'\} V_3 + \{G' - \kappa_3 F\} V_4.\end{aligned} \tag{5.36}$$

Hence, substituting (5.33)-(5.36) into (2.4) and (2.5), after some calculations we get the result. □

Corollary 5.3.3. *The involute $\overline{x}$ of order 3 of a ccr-curve x in $\mathbb{E}^4$ is also a ccr-curve of $\mathbb{E}^4$.*

Proof. Let x be a regular ccr-curve of $\mathbb{E}^4$. Since the ratios

$$\begin{aligned}\frac{\overline{\kappa}_2}{\overline{\kappa}_1} &= \frac{\kappa_2}{\kappa_3}\\ \frac{\overline{\kappa}_3}{\overline{\kappa}_2} &= -\frac{\kappa_1}{\kappa_2}\end{aligned}$$

are constant functions then the involute curve $\overline{x}$ is also a ccr-curve. □

6. Generalized Evolute Curves in $\mathbb{E}^{m+1}$

Let $x = x(s)$ be a generic curve in $\mathbb{E}^n$ given with Frenet frame $V_1, V_2, V_3, \ldots, V_n$ and Frenet curvatures $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$. For simplicity, we can take $n = m + 1$, to construct the Frenet frame $V_1 = T, V_2 = N_1, V_3 = N_2, \ldots, V_n = N_m$ and Frenet curvatures $\kappa_1, \kappa_2, \ldots, \kappa_m$. The centre of the osculating hypersphere of x at a point lies in the hyperplane normal to the x at that point. The curve passing through the centers of the osculating hyperspheres of x defined by

$$\widetilde{x} = x + \sum_{i=1}^{m} c_i N_i, \tag{6.1}$$

which is called *generalized evolute* (or focal curve) of x, where $c_1, c_2, \ldots, c_m$ are smooth functions of the parameter of the curve x. The function c_i is called the i^{th} focal curvature of γ. Moreover, the function c_1 never vanishes and $c_1 = \frac{1}{k_1}$ [18].

The differentiation of the equation (6.1) and the Frenet formulae (2.1) give the following equation

$$\begin{aligned} \widetilde{x}'(s) &= (1-\kappa_1 c_1)T + (c_1' - \kappa_2 c_2)N_1 \\ &\quad + \sum_{i=2}^{m-1}(c_{i-1}\kappa_i + c_i' - c_{i+1}\kappa_{i+1})N_i + (c_{m-1}\kappa_m + c_m')N_m. \end{aligned} \tag{6.2}$$

Since, the osculating planes of $\widetilde{x}$ are the normal planes of x, and the points of $\widetilde{x}$ are the center of the osculating sphere of x then the generalized evolutes $\widetilde{x}$ of the curve x are determined by

$$\langle \widetilde{x}'(s), T(s)\rangle = \langle \widetilde{x}'(s), N_1(s)\rangle = \ldots = \langle \widetilde{x}'(s), N_{m-1}(s)\rangle = 0. \tag{6.3}$$

This condition is satisfied if and only if

$$\begin{aligned} 1-\kappa_1 c_1 &= 0 \\ c_1' - \kappa_2 c_2 &= 0 \\ &\vdots \\ c_{i-1}\kappa_i + c_i' - c_{i+1}\kappa_{i+1} &= 0, \quad 2 \le i \le m-1. \end{aligned} \tag{6.4}$$

hold. So, the focal curvatures of a curve parametrized by arclength s satisfy the following "scalar Frenet equation" for $c_m \neq 0$:

$$\frac{R_m^2}{2c_m} = c_{m-1}\kappa_m + c_m' \tag{6.5}$$

where

$$R_m = \|\widetilde{x} - x\| = \sqrt{c_1^2 + c_2^2 + \ldots + c_m^2}$$

is the radius of the osculating m-sphere [18]. Consequently, the generalized evolutes $\widetilde{x}$ of the curve x are represented by the formulas (6.1), and

$$\widetilde{x}'(s) = (c_{m-1}\kappa_m + c_m')N_m. \tag{6.6}$$

If $\widetilde{x}'(s) = 0$, then R_m is constant and the curve x is spherical.

From the equalities in (6.4) one can get (see, [18])

$$\kappa_i = \frac{c_1c_1' + c_2c_2' + \ldots + c_{i-1}c_{i-1}'}{c_{i-1}c_i}. \tag{6.7}$$

The following result gives the relations between the Frenet frames and Frenet curvatures of x and its evolute $\widetilde{x}$.

Theorem 6.1.([18]) *Let $x = x(s)$ be a generic curve in $\mathbb{E}^{m+1}$ given with Frenet frame $T, N_1, N_2, \ldots, N_m$ and Frenet curvatures $\kappa_1, \kappa_2, \ldots, \kappa_m$. Then Frenet frame $\widetilde{T}, \widetilde{N}_1, \widetilde{N}_2, \ldots, \widetilde{N}_m$ and Frenet curvatures $\widetilde{\kappa}_1, \widetilde{\kappa}_2, \ldots, \widetilde{\kappa}_m$ of the generalized evolute $\widetilde{x}$ of x in $\mathbb{E}^{m+1}$ are given by*

$$\begin{aligned} \widetilde{T} &= \epsilon N_m \\ \widetilde{N}_k &= \delta_k N_{m-k}; \quad 1 \leq k \leq m-1 \\ \widetilde{N}_m &= \pm T \end{aligned} \tag{6.8}$$

and

$$\frac{\widetilde{\kappa}_1}{|\kappa_m|} = \frac{\widetilde{\kappa}_2}{\kappa_{m-1}} = \ldots = \frac{|\widetilde{\kappa}_m|}{\kappa_1} = \frac{1}{|c_{m-1}\kappa_m + c'_m|} \tag{6.9}$$

where $\epsilon(s)$ is the sign of $(c_{m-1}\kappa_m + c'_m)(s)$ and δ_k the sign of $(-1)^k\epsilon(s)\kappa_m(s)$.

6.1. Evolutes in $\mathbb{E}^3$

A generalized evolute of a regular curve x in $\mathbb{E}^3$ has the parametrization

$$\widetilde{x}(s) = x(s) + c_1(s)N_1(s) + c_2(s)N_2(s) \tag{6.10}$$

where N_1 and N_2 are normal vectors of x in $\mathbb{E}^3$ and c_1, c_2 are focal curvatures satisfying

$$c_1(s) = \frac{1}{\kappa_1(s)}, \quad c_2(s) = \frac{\rho'(s)}{\kappa_2(s)}. \tag{6.11}$$

where $\rho = c_1 = \frac{1}{\kappa_1}$ is the radius of the curvature of x.

We obtain the following result.

Proposition 6.1.1. *Let $x = x(s)$ be a regular curve in $\mathbb{E}^3$ given with nonzero Frenet curvatures κ_1 and κ_2. Then Frenet curvatures $\widetilde{\kappa}_1$ and $\widetilde{\kappa}_2$ of the evolute $\widetilde{x}$ of the curve x are given by*

$$\widetilde{\kappa}_1 = \frac{\kappa_2^2}{|\rho\kappa_2^2 + \rho'|}, \quad \widetilde{\kappa}_2 = \frac{\kappa_1\kappa_2}{|\rho\kappa_2^2 + \rho'|}. \tag{6.12}$$

where $\rho = \frac{1}{\kappa_1}$ is the radius of the curvature of x.

Proof. As a consequence of (6.9) we get (6.12). □

Corollary 6.1.2. *The evolute $\widetilde{x}$ of a generalized helix in $\mathbb{E}^3$ is also a generalized helix in $\mathbb{E}^3$.*

By the use of (6.5) with (6.11) one can get the following result.

Corollary 6.1.3. *A regular curve with nonzero curvatures κ_1 and κ_2 lies on a sphere if and only if*

$$\left(\frac{\rho'}{\kappa_2}\right)' + \rho\kappa_2 = 0 \tag{6.13}$$

holds, where $\rho = \frac{1}{\kappa_1}$ is the radius of the curvature of x.

6.2. Evolutes in $\mathbb{E}^4$

A generalized evolute of a generic curve x in $\mathbb{E}^4$ has the parametrization

$$\widetilde{x}(s) = x(s) + c_1(s)N_1(s) + c_2(s)N_2(s) + c_3(s)N_3(s) \tag{6.14}$$

where N_1, N_2 and N_3 are normal vectors of x in $\mathbb{E}^4$ and c_1, c_2 and c_3 are focal curvatures satisfying

$$c_1(s) = \frac{1}{\kappa_1(s)}, \quad c_2(s) = \frac{\rho'(s)}{\kappa_2(s)}, c_3(s) = \frac{\rho(s)\kappa_2(s) + \left(\frac{\rho'(s)}{\kappa_2(s)}\right)'}{\kappa_3(s)}. \tag{6.15}$$

where $\rho = \frac{1}{\kappa_1}$ is the radius of the curvature of x.

We obtain the following result.

Proposition 6.2.1. *Let $x = x(s)$ be a regular curve in $\mathbb{E}^4$ given with nonzero Frenet curvatures κ_1, κ_2 and κ_3. Then Frenet 4-frame, $\widetilde{T}, \widetilde{N}_1, \widetilde{N}_2$ and $\widetilde{N}_3$ and Frenet curvatures $\widetilde{\kappa}_1$, $\widetilde{\kappa}_2$ and $\widetilde{\kappa}_3$ of the evolute $\widetilde{x}$ of a regular curve x in $\mathbb{E}^4$ are given by*

$$\begin{aligned} \widetilde{T}(s) &= N_3, \\ \widetilde{N}_1(s) &= -N_2, \\ \widetilde{N}_2(s) &= N_1, \\ \widetilde{N}_3(s) &= T, \end{aligned} \tag{6.16}$$

and

$$\begin{aligned} \widetilde{\kappa}_1 &= \frac{\kappa_3}{|\psi|}, \\ \widetilde{\kappa}_2 &= \frac{\kappa_2}{|\psi|}, \\ \widetilde{\kappa}_3 &= -\frac{\kappa_1}{|\psi|} \end{aligned} \tag{6.17}$$

where $\psi(s) = c_2(s)\kappa_3(s) + c_3'(s)$ is a smooth function.

Proof. As a consequence of (6.8) with (6.9) we get the result. □

Corollary 6.2.2. *The evolute $\widetilde{x}$ of a ccr-curve x in $\mathbb{E}^4$ is also a ccr-curve of $\mathbb{E}^4$.*

Proof. Let x be a regular ccr-curve of $\mathbb{E}^4$. Since the ratios

$$\begin{aligned}\frac{\widetilde{\kappa}_2}{\widetilde{\kappa}_1} &= \frac{\kappa_2}{\kappa_3}\\ \frac{\widetilde{\kappa}_3}{\widetilde{\kappa}_2} &= -\frac{\kappa_1}{\kappa_2}\end{aligned}$$

are constant functions then the evolute curve $\widetilde{x}$ is also a ccr-curve. □

By the use of (6.6) with (6.11) one can get the following result.

Corollary 6.2.3. *A regular curve with nonzero curvatures κ_1, κ_2 and κ_3 lies on a sphere if and only if*

$$\left(\frac{\rho(s)\kappa_2(s) + \left(\frac{\rho'(s)}{\kappa_2(s)}\right)'}{\kappa_3(s)}\right)' + \rho'(s)\frac{\kappa_3(s)}{\kappa_2(s)} = 0 \tag{6.18}$$

holds, where $\rho = \frac{1}{\kappa_1}$ is the radius of the curvature.

Proposition 6.2.4. *[11] A curve $x = x(s) : I \subset \mathbb{R} \to \mathbb{E}^4$ is a spherical, i.e., it is contained in a sphere of radius R, if and only if x can be decomposed as*

$$x(s) = m - \frac{R}{\kappa_1}N_1(s) + \frac{R\kappa_1'}{\kappa_2\kappa_1^2}N_2(s) + \frac{R}{\kappa_3}\left(\frac{\kappa_1'}{\kappa_2\kappa_1^2}\right)' N_3(s). \tag{6.19}$$

where m is the center of the sphere.

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