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On LP-Sasakian Manifolds admitting a Semi-symmetric Non-metric Connection

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Abstract. In this paper, the object is to study a semi-symmetric non-metric connection on an LP-Sasakian manifold whose concircular curvature tensor satisfies certain curvature conditions.

1. Introduction

In 1924, Friedmann and Schouten [4] introduced the idea of semi-symmetric connection on a differentiable manifold. A linear connection $\widetilde{\nabla}$ on a differentiable manifold M is said to be a semi-symmetric connection if the torsion tensor T of the connection $\widetilde{\nabla}$ satisfies $T(X,Y) = u(Y)X - u(X)Y$, where u is a 1-form and ξ_1 is a vector field defined by $u(X) = g(X, \xi_1)$, for all vector fields $X \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M.

In 1932, Hayden [9] introduced the idea of semi-symmetric metric connections on a differential manifold (M, g). A semi-symmetric connection $\widetilde{\nabla}$ is said to be a semi-symmetric metric connection if $\widetilde{\nabla} g = 0$.

After a long gap the study of a semi-symmetric connection $\bar{\nabla}$ satisfying

$$\bar{\nabla} g \neq 0. \tag{1.1}$$

was initiated by Prvanović [14] with the name pseudo-metric semi-symmetric connection and was just followed by Andonie [16].

A semi-symmetric connection $\bar{\nabla}$ is said to be a *semi-symmetric non-metric connection* if it satisfies the condition (1.1).

In 1992, Agashe and Chafle [15] studied a semi-symmetric non-metric connection $\bar{\nabla}$, whose torsion tensor $\bar{T}$ satisfies $\bar{T}(X,Y) = u(Y)X - u(X)Y$ and $(\bar{\nabla}_X g)(Y,Z) = -u(Y)g(X,Z) - u(Z)g(X,Y) \neq 0$. They proved that the projective curvature tensor

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of the manifold with respect to these two connections are equal to each other. In 1992, Barua and Mukhopadhyay [5] studied a type of semi-symmetric connection $\bar{\nabla}$ which satisfies $(\bar{\nabla}_X g)(Y,Z) = 2u(X)g(Y,Z) - u(Y)g(X,Z) - u(Z)g(X,Y)$. Since $\bar{\nabla} g \neq 0$, this is another type of semi-symmetric non-metric connection. However, the authors preferred the name semi-symmetric semimetric connection.

In 1994, Liang [24] studied another type of semi-symmetric non-metric connection $\bar{\nabla}$ for which we have $(\bar{\nabla}_X g)(Y,Z) = 2u(X)g(Y,Z)$, where u is a non-zero 1-form and he called this a semi-symmetric recurrent metric connection.

The semi-symmetric non-metric connections was further developed by several authors such as De and Biswas [21], De and Kamilya [22], Liang [24], Singh et al. ([17, 18, 19]), Smaranda [7], Smaranda and Andonie [8], Barman ([1, 2, 3]) and many others.

A transformation of an n-dimensional differential manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation ([12, 23]). A concircular transformation is always a conformal transformation [23]. Here geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus the geometry of concircular transformations, i.e., the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism (see also [6]). An interesing invariant of a concircular transformation is the concircular curvature tensor $\mathbb{W}$ with respect to the Levi-Civita connection. It is defined by ([12, 13])

$$\mathbb{W}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.2}$$

where $X, Y, Z, U \in \chi(M)$, R and r are the curvature tensor and the scalar curvature with respect to the Levi-Civita connection.

The concircular curvature tensor $\overline{\mathbb{W}}$ with respect to the semi-symmetric non-metric connection is defined by

$$\mathbb{W}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.3}$$

where $\bar{R}$ and $\bar{r}$ are the curvature tensor and the scalar curvature with respect to the semi-symmetric non-metric connection. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

In this paper we study a type of semi-symmetric non-metric connection due to Agashe and Chafle [15] on LP-Sasakian manifolds. The paper is organized as follows: After introduction in section 2, we give a brief account of LP-Sasakian manifolds. Section 3 deals with the semi-symmetric non-metric connection. The relation between the curvature tensor of an LP-Sasakian manifold with respect to the semi-symmetric non-metric connection and Levi-Civita connection have been studied in section 4. Section 5 is devoted to obtain ξ-concircularly flat LP-Sasakian manifold with respect to the semi-symmetric non-metric connection. Next Section, we deals with the LP-Sasakian manifolds admitting semi-symmetric non-metric connection $\bar{\nabla}$ satisfying $\overline{\mathbb{W}} \cdot \bar{S} = 0$, where $\bar{S}$ denotes the Ricci tensor with respect to the semi-symmetric non-metric connection. Finally, we construct an example of a 5-dimensional LP-Sasakian manifold admitting the semi-symmetric non-metric connection to support the results obtained in Section 5.

2. LP-Sasakian Manifolds

An n-dimensional differentiable manifold M with structure (ϕ, ξ, η, g) is said to be a *Lorentzian almost Paracontact manifold* (briefly, *LAP-manifold*) ([10, 11]), if it admits a $(1,1)$- tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$g(X,\xi) = \eta(X); \eta(\xi) = -1; \phi(\xi) = 0; \eta(\phi) = 0, \tag{2.1}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$\nabla_X \xi = \phi X, \tag{2.4}$$

$$(\nabla_X \phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.5}$$

$$rank(\phi) = n - 1, \tag{2.6}$$

$$\Phi(X,Y) = \Phi(Y,X) = g(\phi X, Y), \tag{2.7}$$

where ∇ denotes the covariant differentiation with respect to Lorentzian metric g and for any vector field X and $Y \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M and the tensor field $\Phi(X,Y)$ is a symmetric $(0,2)$-tensor field [10].

An LAP-manifold with structure (ϕ, ξ, η, g) satisfying the relation [10]

$$(\nabla_Z \Omega)(X,Y) = g(Y,Z)\eta(X) + g(X,Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)$$

is called a *normal Lorentzian paracontact manifold* or *Lorentzian para-Sasakian manifold* (briefly *LP-Sasakian manifold*). Also, since the vector field η is closed in an LP-Sasakian manifold, we have ([10, 11])

$$(\nabla_X \eta)(Y) = \Phi(X,Y) = g(\phi X, Y), \Phi(X,\xi) = 0. \tag{2.8}$$

Also in an LP-Sasakian manifold, the following relations holds [10] :

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.9}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.10}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.11}$$

$$R(\xi,Y)\xi = X + \eta(X)\xi, \tag{2.12}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{2.13}$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{2.14}$$

for any vector field X, Y, $Z \in \chi(M)$, S denotes the Ricci tensor of M with respect to the Levi-Civita connection.

3. Semi-symmetric Non-metric Connection

Let M be an n-dimensional differential manifold with Lorentzian metric g. If $\bar{\nabla}$ is the semi-symmetric non-metric connection on a differential manifold M, a linear connection $\bar{\nabla}$ is given by [15]

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X. \tag{3.1}$$

Then $\bar{R}$ and R are related by [15]

$$\bar{R}(X,Y)Z = R(X,Y)Z + \beta(X,Z)Y - \beta(Y,Z)X, \tag{3.2}$$

for all vector fields $X, Y, Z \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M, where β is a $(0,2)$ tensor field denoted by

$$\beta(X,Z) = (\nabla_X \eta)(Z) - \eta(X)\eta(Z). \tag{3.3}$$

From (3.1) yields

$$(\bar{\nabla}_W g)(X,Y) = -\eta(X)g(Y,W) - \eta(Y)g(X,W) \neq 0. \tag{3.4}$$

4. Curvature Tensor of an LP-Sasakian Manifold with respect to the Semi-symmetric Non-metric Connection

In this section we obtain the expressions of the curvature tensor, Ricci tensor and scalar curvature of M with respect to the semi-symmetric non-metric connection defined by (3.1).

Analogous to the definitions of the curvature tensor R of M with respect to the Levi-Civita connection ∇, we define the curvature tensor $\bar{R}$ of M with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ given by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z,$$

where $X, Y, Z \in \chi(M)$, the set of all differentiable vector fields on M.

Combining (2.8) and (3.3), we get

$$\beta(X,Z) = g(\phi X, Z) - \eta(X)\eta(Z). \tag{4.1}$$

Using (4.1) in (3.2) [20], we have

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z &+ g(\phi X,Z)Y - \eta(X)\eta(Z)Y \\ &- g(\phi Y,Z)X + \eta(Y)\eta(Z)X.\end{aligned} \tag{4.2}$$

From (4.2), implies that

$$\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z. \tag{4.3}$$

Putting $X = \xi$ in (4.2) and using (2.1) and (2.11) [20], we obtain

$$\bar{R}(\xi,Y)Z = g(Y,Z)\xi - g(\phi Y,Z)\xi + \eta(Y)\eta(Z)\xi. \tag{4.4}$$

Again putting $Z = \xi$ in (4.2) and using (2.1) and (2.10) [20], we get

$$\bar{R}(X,Y)\xi = 0. \tag{4.5}$$

Combining (2.9) and (4.2), we have

$$\begin{aligned}g(\bar{R}(X,Y)Z,\xi) = \eta(\bar{R}(X,Y)Z) = g(Y,Z)\eta(X) &- g(X,Z)\eta(Y) \\ &+ g(\phi X,Z)\eta(Y) - g(\phi Y,Z)\eta(X).\end{aligned} \tag{4.6}$$

Taking a frame field from (4.2) [20], we obtain

$$\bar{S}(Y,Z) = S(Y,Z) - (n-1)g(\phi Y,Z) + (n-1)\eta(Y)\eta(Z), \tag{4.7}$$

where $\bar{S}$ denotes the Ricci tensor with respect to the semi-symmetric non-metric connection.

From (4.7), implies that

$$\bar{S}(Y,Z) = \bar{S}(Z,Y). \tag{4.8}$$

Putting $Z = \xi$ in (4.8) and using (2.13) [20], we get

$$\bar{S}(Y, \xi) = 0. \tag{4.9}$$

Combining (2.8) and (3.1), we have

$$(\bar{\nabla}_X \eta)(Y) = g(\phi X, Y) - \eta(X)\eta(Y). \tag{4.10}$$

Again taking a frame field from (4.7) [20], we obtain

$$\bar{r} = r - (n-1)(\alpha + 1), \tag{4.11}$$

where $\alpha = trace\ of\ \phi$ and $\bar{r}$ denotes the scalar curvature with respect to the semi-symmetric non-metric connection.

From [20] and the above discussions we can state the following:

Proposition 4.1. *For an LP-Sasakian manifold M with respect to the semi-symmetric non-metric connection $\bar{\nabla}$,*

(1) *the curvature tensor $\bar{R}$ is given by $\bar{R}(X,Y)Z = R(X,Y)Z + g(\phi X, Z)Y - \eta(X)\eta(Z)Y - g(\phi Y, Z)X + \eta(Y)\eta(Z)X$,*

(2) *the Ricci tensor $\bar{S}$ is given by $\bar{S}(Y,Z) = S(Y,Z) - (n-1)g(\phi Y, Z) + (n-1)\eta(Y)\eta(Z)$,*

(3) *the scalar curvature $\bar{r}$ is given by $\bar{r} = r - (n-1)(\alpha+1)$,*

(4) *$\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z$,*

(5) *the Ricci tensor $\bar{S}$ is symmetric.*

5. ξ-concircularly flat LP-Sasakian manifolds with respect to the semi-symmetric non-metric connection

Definition 5.1. *A LP-Sasakian manifold M with respect to the semi-symmetric non-metric connection is said to be ξ-concircularly flat if*

$$\overline{\mathbb{W}}(X,Y)\xi = 0,$$

for all vector fields $X, Y \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M.

Theorem 5.1. *An n-dimensional LP-Sasakian manifold with respect to the semi-symmetric non-metric connection is ξ-concircularly flat if and only if the manifold with respect to the Levi-Civita connection is also ξ-concircularly flat provided trace of ϕ $=\alpha= n-1$.*

Proof. Combining (1.2), (1.3), (4.2) and (4.11), we get

$$\begin{aligned}\overline{\mathbb{W}}(X,Y)Z = \mathbb{W}(X,Y)Z + \frac{\alpha+1}{n}[g(Y,Z)X - g(X,Z)Y] + g(\phi X, Z)Y \\ -\eta(X)\eta(Z)Y - g(\phi Y, Z)X + \eta(Y)\eta(Z)X.\end{aligned} \tag{5.1}$$

Putting $Z = \xi$ in (5.1) and using (2.1), we have

$$\overline{\mathbb{W}}(X,Y)\xi = \mathbb{W}(X,Y)\xi + \frac{\alpha + 1 - n}{n}[\eta(Y)X - \eta(X)Y].$$

Hence the proof of theorem is completed. □

Theorem 5.2. *If a LP-Sasakian manifold $(n > 1)$ is ξ-concircularly flat with respect to the semi-symmetric non-metric connection if and only if the scalar curvature with respect to the semi-symmetric non-metric connection vanishes.*

Proof. Putting $Z = \xi$ in (1.2) and using (2.1) and (4.5), we have

$$\overline{\mathbb{W}}(X,Y)\xi = -\frac{\bar{r}}{n(n-1)}[\eta(Y)X - \eta(X)Y]. \tag{5.2}$$

Thus the theorem is proved. □

6. LP-Sasakian Manifold admitting Semi-symmetric Non-metric Connection $\bar{\nabla}$ satisfying $\overline{\mathbb{W}} \cdot \bar{S} = 0$

Theorem 6.1. *If an LP-Sasakian manifold with respect to the semi-symmetric non-metric connection satisfies $\overline{\mathbb{W}} \cdot \bar{S} = 0$, then the scalar curvature with respect to the Levi-Civita connection is $(n-1)(\alpha+1)$, where $\alpha = g(\phi e_i, e_i)$.*

Proof. We suppose that the manifold under consideration is the semi-symmetric non-metric connection M^n, that is,

$$(\overline{\mathbb{W}}(X,Y) \cdot \bar{S})(U,V) = 0,$$

where $X, Y, U, V \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M.

Then we have

$$\bar{S}(\overline{\mathbb{W}}(X,Y)U, V) + \bar{S}(U, \overline{\mathbb{W}}(X,Y)V) = 0. \tag{6.1}$$

Putting $U = \xi$ in (6.1) and using (4.9), it follows that

$$\bar{S}(\overline{\mathbb{W}}(X,Y)\xi, V) = 0. \tag{6.2}$$

In view of (1.2), (4.5) and (6.2), we get

$$\eta(Y)\bar{S}(X,V) - \eta(X)\bar{S}(Y,V) = 0. \tag{6.3}$$

Again putting $X = \xi$ in (6.3) and using (4.9) and (4.7), we have

$$S(Y,V) = (n-1)g(\phi Y, V) - (n-1)\eta(Y)\eta(V). \tag{6.4}$$

Taking a frame field from (6.4), we obtain

$$r = (n-1)(\alpha+1),$$

where $\alpha = g(\phi e_i, e_i)$.

This completes the proof of the theorem. □

7. Example

In this section we construct an example on LP-Sasakian manifold with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ which verify the result of section 5.

We consider the 5-dimensional manifold $M = \{(x, y, z, u, v) \in R^5\}$, where (x, y, z, u, v) are the standard coordinate in R^5.

We choose the vector fields

$$e_1 = -2\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}, \;\; e_2 = \frac{\partial}{\partial y}, \;\; e_3 = \frac{\partial}{\partial z}, \;\; e_4 = -2\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial z}, \;\; e_5 = \frac{\partial}{\partial v},$$

which are linearly independent at each point of M.

Let g be the Lorentzian metric defined by

$$g(e_i, e_j) = 0, \;\; i \neq j, \;\; i, j = 1, 2, 3, 4, 5$$

and

$$g(e_1, e_1) = g(e_2, e_2) = g(e_4, e_4) = g(e_5, e_5) = 1, g(e_3, e_3) = -1.$$

Let η be the 1-form defined by

$$\eta(Z) = g(Z, e_3),$$

for any $Z \in \chi(M)$.

Let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = e_2, \;\; \phi e_2 = e_1, \;\; \phi e_3 = 0, \;\; \phi e_4 = e_5, \;\; \phi e_5 = e_4.$$

Using the linearity of ϕ and g, we have

$$\eta(e_3) = -1,$$

$$\phi^2(Z) = Z + \eta(Z)e_3$$

and

$$g(\phi Z, \phi U) = g(Z, U) + \eta(Z)\eta(U),$$

for any $U, Z \in \chi(M)$. Hence $e_3 = \xi$ and $M(\phi, \xi, \eta, g)$ is a Lorentzian almost paracontact manifold.

Then we have

$$[e_1, e_2] = -2e_3, \;\; [e_1, e_3] = [e_1, e_4] = [e_1, e_5] = [e_2, e_3] = 0,$$

$$[e_4, e_5] = -2e_3, \;\; [e_2, e_4] = [e_2, e_5] = [e_3, e_4] = [e_3, e_5] = 0.$$

The Riemannian connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) - g(X,[Y,Z]) \\ +g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned} \tag{7.1}$$

Taking $e_3 = \xi$ and using Koszul's formula we get the following

$$\begin{array}{lllll} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = -e_3, & \nabla_{e_1} e_3 = e_2, & \nabla_{e_1} e_4 = 0, & \nabla_{e_1} e_5 = 0, \\ \nabla_{e_2} e_1 = e_3, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_3 = e_1, & \nabla_{e_2} e_4 = 0, & \nabla_{e_2} e_5 = 0, \\ \nabla_{e_3} e_1 = e_2, & \nabla_{e_3} e_2 = e_1, & \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_4 = e_5, & \nabla_{e_3} e_5 = e_4, \\ \nabla_{e_4} e_1 = 0, & \nabla_{e_4} e_2 = 0, & \nabla_{e_4} e_3 = e_5, & \nabla_{e_4} e_4 = 0, & \nabla_{e_4} e_5 = -e_3, \\ \nabla_{e_5} e_1 = 0, & \nabla_{e_5} e_2 = 0, & \nabla_{e_5} e_3 = e_4, & \nabla_{e_5} e_4 = e_3, & \nabla_{e_5} e_5 = 0. \end{array}$$

From the above calculations, the manifold under consideration satisfies $\eta(\xi) = -1$ and $\nabla_X \xi = \phi X$. Therefore, the manifold is an LP-Sasakian manifold.

Using (3.1) in above equations, we obtain

$$\begin{array}{lllll} \bar{\nabla}_{e_1} e_1 = 0, & \bar{\nabla}_{e_1} e_2 = -e_3, & \bar{\nabla}_{e_1} e_3 = e_2 - e_1, & \bar{\nabla}_{e_1} e_4 = 0, & \bar{\nabla}_{e_1} e_5 = 0, \\ \bar{\nabla}_{e_2} e_1 = e_3, & \bar{\nabla}_{e_2} e_2 = 0, & \bar{\nabla}_{e_2} e_3 = e_1 - e_2, & \bar{\nabla}_{e_2} e_4 = 0, & \bar{\nabla}_{e_2} e_5 = 0, \\ \bar{\nabla}_{e_3} e_1 = e_2, & \bar{\nabla}_{e_3} e_2 = e_1, & \bar{\nabla}_{e_3} e_3 = -e_3, & \bar{\nabla}_{e_3} e_4 = e_5, & \bar{\nabla}_{e_3} e_5 = e_4, \\ \bar{\nabla}_{e_4} e_1 = 0, & \bar{\nabla}_{e_4} e_2 = 0, & \bar{\nabla}_{e_4} e_3 = e_5 - e_4, & \bar{\nabla}_{e_4} e_4 = 0, & \bar{\nabla}_{e_4} e_5 = -e_3, \\ \bar{\nabla}_{e_5} e_1 = 0, & \bar{\nabla}_{e_5} e_2 = 0, & \bar{\nabla}_{e_5} e_3 = e_4 - e_5, & \bar{\nabla}_{e_5} e_4 = e_3, & \bar{\nabla}_{e_5} e_5 = 0. \end{array}$$

By using the above results, we can easily obtain the components of the curvature tensors as follows:

$$\begin{array}{llll} R(e_1, e_2)e_4 = 2e_5, & R(e_1, e_2)e_5 = 2e_4, & R(e_4, e_5)e_1 = 2e_2, & R(e_4, e_5)e_2 = 2e_1, \\ R(e_1, e_2)e_2 = 3e_1, & R(e_1, e_3)e_3 = -e_1, & R(e_2, e_1)e_1 = -3e_2, & R(e_2, e_3)e_3 = -e_2, \\ R(e_3, e_1)e_1 = e_3, & R(e_3, e_2)e_2 = -e_3, & R(e_3, e_4)e_4 = e_3, & R(e_3, e_5)e_5 = -e_3, \\ R(e_4, e_3)e_3 = -e_4, & R(e_4, e_5)e_5 = 3e_4, & R(e_5, e_3)e_3 = -e_5, & R(e_5, e_4)e_4 = -3e_5, \\ R(e_1, e_4)e_2 = e_5, & R(e_1, e_4)e_5 = -e_2, & R(e_1, e_5)e_2 = e_4, & R(e_1, e_5)e_4 = e_2, \\ R(e_2, e_4)e_1 = -e_5, & R(e_2, e_4)e_5 = -e_1, & R(e_2, e_5)e_1 = -e_4, & R(e_2, e_5)e_4 = e_1 \end{array}$$

and

$$
\begin{array}{lll}
\bar{R}(e_1,e_2)e_4 = 2e_5, & \bar{R}(e_1,e_2)e_5 = 2e_4, & \bar{R}(e_2,e_3)e_1 = e_3, \\
\bar{R}(e_3,e_4)e_5 = e_3, & \bar{R}(e_4,e_5)e_1 = 2e_2, & \bar{R}(e_4,e_5)e_2 = 2e_1, \\
\bar{R}(e_1,e_2)e_2 = 3e_1 - e_2, & \bar{R}(e_2,e_1)e_1 = -3e_2 + e_1, & \bar{R}(e_3,e_1)e_1 = e_3, \\
\bar{R}(e_3,e_2)e_2 = -e_3, & \bar{R}(e_3,e_4)e_4 = e_3, & \bar{R}(e_3,e_5)e_5 = -e_3, \\
\bar{R}(e_4,e_5)e_5 = 3e_4 - e_5, & \bar{R}(e_5,e_4)e_4 = -3e_5 + e_4, & \bar{R}(e_1,e_3)e_2 = -e_1 + e_2, \\
\bar{R}(e_1,e_4)e_2 = e_5 - e_4, & \bar{R}(e_1,e_4)e_5 = -e_2 + e_1, & \bar{R}(e_1,e_5)e_2 = e_4 - e_5, \\
\bar{R}(e_1,e_5)e_4 = e_2 - e_1, & \bar{R}(e_2,e_4)e_1 = -e_5 + e_4, & \bar{R}(e_2,e_4)e_5 = -e_1 + e_2, \\
\bar{R}(e_2,e_5)e_1 = -e_4 + e_5 & \bar{R}(e_2,e_5)e_4 = e_1 - e_2, & \bar{R}(e_3,e_5)e_4 = -e_3
\end{array}
$$

and other curvature tensor $R(e_i,e_j)e_k = \bar{R}(e_i,e_j)e_k = 0; \forall i,j,k = 1,2,3,4,5$. From these curvature tensors, we can be calculated the Ricci tensors as follows:

$$S(e_1,e_1) = S(e_3,e_3) = S(e_4,e_4) = -4, \; S(e_2,e_2) = S(e_5,e_5) = 4$$

and

$$\bar{S}(e_1,e_1) = \bar{S}(e_4,e_4) = -4, \; \bar{S}(e_3,e_3) = 0 \; \bar{S}(e_2,e_2) = \bar{S}(e_5,e_5) = 4.$$

Therefore, the scalar curvature tensors $r = -4$ and $\bar{r} = 0$ with respect to the Levi-Civita connection and the semi-symmetric non-metric connection respectively.

Let X and Y are any two vector fields given by

$$X = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5$$

and

$$Y = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5.$$

Using the above relations of curvature tensors and scalar curvature tensor with respect to the semi-symmetric non-metric connection respectively, we get

$$\overline{\mathbb{W}}(X,Y)\xi = 0.$$

Hence the manifold under consideration satisfies the Theorem 5.2 of Section 5. □

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