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On Deferred $f-$statistical Convergence

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Abstract. In this paper, we generalize the concept of deferred density to that of deferred $f-$density, where f is an unbounded modulus and introduce a new non-matrix convergence method, namely deferred $f-$statistical convergence or $S^f_{p,q}-$convergence. Apart from studying the Köthe-Toeplitz duals of $S^f_{p,q}$, the space of deferred $f-$statistically convergent sequences, a decomposition theorem is also established. We also introduce a notion of strongly deferred Cesàro summable sequences defined by modulus f and investigate the relationship between deferred $f-$statistical convergence and strongly deferred Cesàro summable sequences defined by f.

1. Introduction and Preliminaries

The idea of statistical convergence which is, in fact, a generalization of the usual notion of convergence was introduced by Fast [17] and Steinhaus [31] independently in the same year 1951 and since then several generalizations and applications of this concept have been investigated by various authors namely Bhardwaj *et al.* [4, 5, 6, 7], Connor [13, 14], Et and Şengül [16], Fridy [18], Işık [21], Işık and Akbaş [22], Mursaleen [26], Rath and Tripathy [28], Salat [30], Temizsu [32] *et al.* and many others.

The idea of statistical convergence depends upon the density of subsets of the set $\mathbb{N}$ of natural numbers. The natural density $\delta(K)$ of a subset K of the set $\mathbb{N}$ of natural numbers is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{ k \leq n : k \in K\}|$$

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where $|\{ k \leq n : k \in K\}|$ denotes the number of elements of K not exceeding n. Obviously, we have $\delta(K) = 0$ provided that K is a finite set.

A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$,

$$\delta\left(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}\right) = 0,$$

$$i.e., \lim_{n\to\infty} \frac{1}{n} |\{ k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write $S - \lim x_k = L$. Since $\lim x_k = L$ implies $S - \lim x_k = L$, statistical convergence may be considered as a regular summability method. The set of all statistically convergent sequences is denoted by S.

Connor [14], Ghosh and Srivastava [20], Bhardwaj and Singh [8, 9, 10], Çolak [12], Altin and Et [3] and some others have used a modulus function to extend the theory of statistical convergence and construct some new sequence spaces.

The idea of a modulus function was structured by Nakano [27] in 1953. Following Ruckle [29] and Maddox [25], we recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that (i) $f(x) = 0$ if and only if $x = 0$, (ii) $f(x + y) \leq f(x) + f(y)$ for $x \geq 0,\ \ y \geq 0$, (iii) f is increasing, (iv) f is continuous from the right at 0. Hence f must be continuous everywhere on $[0, \infty)$. A modulus may be unbounded or bounded. For example, $f(x) = x^p$ where $0 < p \leq 1$, is unbounded, but $f(x) = \frac{x}{1+x}$ is bounded.

In the year 2014, Aizpuru *et al*. [2] have defined a new concept of density with the help of an unbounded modulus function and as a consequence they obtained a new concept of non-matrix convergence which is intermediate between the ordinary convergence and the statistical convergence, and agrees with the statistical convergence when the modulus function is the identity mapping.

Contributing in this direction, Bhardwaj *et al*. [6] introduced and studied a new concept of f-statistical boundedness by using the approach of Aizpuru *et al*. [2]. It is shown that the concept of $f-$statistical boundedness is intermediate between the ordinary boundedness and the statistical boundedness. It is also proved that bounded sequences are precisely those sequences which are $f-$statistically bounded for every unbounded modulus f.

We now recall some definitions that will be needed in the sequel

Definition 1.1.([2]) Let $f-$be an unbounded modulus function. The $f-$*density* of a set $K \subset \mathbb{N}$ is defined by

$$\delta^f(K) = \lim_{n\to\infty} \frac{f(|\{k \leq n : k \in K\}|)}{f(n)}$$

provided the limit exists.

Remark 1.2. The concept of f-density reduces to that of natural density when $f(x) = x$. The equality $\delta^f(K) + \delta^f(\mathbb{N} - K) = 1$ does not hold, in general, where f is an unbounded modulus. However we can assert that if $\delta^f(K) = 0$, then $\delta^f(\mathbb{N} - K) = 1$.

Remark 1.3. *For any unbounded modulus f and $K \subset \mathbb{N}$, $\delta^f(K) = 0$ implies that $\delta(K) = 0$. But converse need not be true in the sense that a set having zero natural density may have non-zero $f-$density with respect to some unbounded modulus f.*

Definition 1.4.([2]) Let f be an unbounded modulus function. A number sequence $x = (x_k)$ is said to be *$f-$statistically convergent* to ℓ, or *S^f-convergent* to ℓ, if for each $\varepsilon > 0$,

$$\delta^f(\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\}) = 0,$$

$$i.e., \lim_{n\to\infty} \frac{f(|\{k \leq n : |x_k - \ell| \geq \varepsilon\}|)}{f(n)} = 0$$

and we write it as $S^f - \lim x_k = \ell$. The set of all $f-$statistically convergent sequences is denoted by S^f.

In view of Remark 1.3, it follows that every $f-$statistically convergent sequence is statistically convergent, but a statistically convergent sequence need not be $f-$statistically convergent for every unbounded modulus f.

In 1932, R.P. Agnew [1] introduced the concept of deferred Cesàro mean of real (or complex) valued sequences $x = (x_k)$ defined by

$$(D_{p,q}x)_n = \frac{1}{(q(n) - p(n))} \sum_{k=p(n)+1}^{q(n)} x_k,\ n = 1, 2, 3, \ldots, \tag{1}$$

where $p = \{p(n) : n \in \mathbb{N}\}$ and $q = \{q(n) : n \in \mathbb{N}\}$ are the sequences of non-negative integers satisfying

$$p(n) < q(n) \text{ and } \lim_{n\to\infty} q(n) = \infty$$

Agnew also showed that the method given by (1) has many more important properties besides being regular.

Motivating from the work of Agnew, the concepts of deferred density and deferred statistical convergence were given by Küçükaslan and Yılmaztürk [24, 33] as follows:

Let K be a subset of $\mathbb{N}$ and denote the set $\{k : p(n) < k \leq q(n), k \in K\}$ by $K_{p,q}(n)$.

Definition 1.5. The *deferred density* of K is defined by

$$\delta_{p,q}(K) = \lim_{n\to\infty} \frac{1}{(q(n) - p(n))} |K_{p,q}(n)|, \text{ provided the limit exists.}$$

The vertical bars indicate the cardinality of the enclosed set $K_{p,q}(n)$. If $q(n) = n$, $p(n) = 0$, then deferred density coincides with natural density of K.

Definition 1.6. A real valued sequence $x = (x_k)$ is said to be *deferred statistically convergent* to L, if for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{(q(n) - p(n))} |\{p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write $S_{p,q}-\lim x_k = L$. The set of all deferred statistically convergent sequences will be denoted by $S_{p,q}$. If $q\,(n) = n$, $p\,(n) = 0$, then deferred statistical convergence coincides with usual statistical convergence.

Quite recently, Et et al. [15] introduced and examined the concept of deferred statistical boundedness of order α and established the realtion between statistical boundedness and deferred statistical boundedness of order α.

In the present paper we extend the notion of deferred-density to that of deferred $f-$density in the same way as natural density was extended to $f-$density by Aizpuru et al. [2] and then introduce a new and more general non-matrix summability method, namely deferred $f-$statistical convergence where f is an unbounded modulus. It is shown that the terms of a deferred $f-$statistical convergent sequence (x_k) coincide to that of a convergent sequence for almost all k deferred with respect to f. Apart from studying various inclusion relations, a decomposition theorem is also estblished. Finally we conclude the paper by the introduction of the concept of strongly deferred Ces$\grave{a}$ro summable sequences defined by modulus f and it is shown that a bounded sequence which is deferred $f-$ statistical convergent to ℓ is strongly deferred Ces$\grave{a}$ro summable with respect to f to ℓ.

Throughout the paper, we consider the sequences of non negative integers $p = \{p(n) : n \in \mathbb{N}\}$and $q = \{q(n) : n \in \mathbb{N}\}$ satisfying $p(n) < q(n)$ and $\lim_{n\to\infty} q(n) = \infty$. Any other restriction (if needed) on $p(n)$ and $q(n)$ will be mentioned in the related theorems.

2. Deferred $f-$statistical Convergence

We begin this section by introducing a new concept of deferred $f-$density of a subset of $\mathbb{N}$.

Definition 2.1. The *deferred $f-$density* of a subset K of $\mathbb{N}$ is defined as

$$\begin{aligned}\delta^f_{p,q}(K) &= \lim_{n\to\infty} \frac{1}{f(q(n)-p(n))} f(|K_{p,q}(n)|)\\ &= \lim_{n\to\infty} \frac{1}{f(q(n)-p(n))} f(|\{k \in K : p(n)+1 \le k \le q(n)\}|\end{aligned}$$

provided the limit exists. The verticle bars indicate the cardinality of the enclosed set. It is obvious that finite sets have zero deferred $f-$density for any unbounded modulus f.

Remark 2.2. For $q(n) = n \;\; p(n) = 0$ the deferred $f-$density reduces to $f-$density and when $f(x) = x$, the deferred $f-$density coincides with deferred density. If $q(n) = n \;\; p(n) = 0$ and $f(x) = x$, the deferred $f-$density turns out to be natural density.

The equality $\delta_{p,q}(K) + \delta_{p,q}(\mathbb{N} - K) = 1$ remains no longer true, if deferred density is replaced by deferred $f-$density, i.e., $\delta^f_{p,q}(K) + \delta^f_{p,q}(\mathbb{N} - K) = 1$ does not hold, in general where f is an unbounded modulus. Let us demonstrate it with the help of following example.

Example 2.3. Take $K = (2, 4, 6, \ldots)$, $q(n) = 4n$, $p(n) = 2n$ and $f(x) = log(x+1)$. Then

$$\begin{aligned}\delta^f_{p,q}(K) &= \lim_{n\to\infty} \frac{1}{f(q(n)-p(n))} f(|\{k \in K : p(n)+1 \le k \le q(n)\}| \\ &= \lim_{n\to\infty} \frac{f(n)}{f(2n)} = \lim_{n\to\infty} \frac{log(n+1)}{log(2n+1)} \\ &= 1.\end{aligned}$$

Also, $\delta^f_{p,q}(\mathbb{N} - K) = 1$.

Proposition 2.4. *If for any unbounded modulus f, $\delta^f_{p,q}(K) = 0$ then $\delta_{p,q}(K) = 0$.*

Proof. As $\lim_{n\to\infty} \frac{1}{f(q(n)-p(n))} f(|\{k \in K : p(n)+1 \le k \le q(n)\}| = 0$, so for every $p \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$ we have

$$\begin{aligned}f(|\{k \in K : p(n)+1 \le k \le q(n)\}| &\le \frac{1}{p} f(q(n)-p(n)) \\ &= \frac{1}{p} f(p.\frac{q(n)-p(n)}{p}) \le \frac{1}{p} p.f(\frac{q(n)-p(n)}{p}).\end{aligned}$$

Since f is increasing, so we have

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} |\{k \in K : p(n)+1 \le k \le q(n)\}| \le \frac{1}{p}$$

which results $\delta_{p,q}(K) = 0$. □

Remark 2.5. Converse of above proposition need not be true in the sense that a set having zero deferred density may have non-zero deferred $f-$density. This is illustrated by the following example.

Example 2.6. Let $f(x) = log(x+1)$ and $K = (1, 4, 9, 16, \ldots)$. Take $q(n) = n^2$ and $p(n) = n$. Then

$$\frac{1}{q(n)-p(n)} \{k \in K : p(n) < k \le q(n)\} = \frac{1}{q(n)-p(n)} \{k \in K : n < k \le n^2\} \le \frac{n}{n^2-n}$$

and so $\delta_{p,q}(K) = 0$ and

$$\begin{aligned}\delta^f_{p,q} &\ge \frac{\log(n-[\sqrt{n}]+1)}{log(n^2-n+1)} \\ &\ge \frac{log(n-[\sqrt{n}])}{log(n^2+1)} \\ &\to \frac{1}{2} \; as \quad n \to \infty.\end{aligned}$$

This implies that $\delta_{p,q}^{f} \neq 0$.

Before proceeding further we first introduce the following notation

For an unbounded modulus f, if $x = (x_k)$ is a sequence such that x_k satisfies property P for all k, except a set of deferred $f-$density zero, then we say $x = (x_k)$ satisfies P for "almost all k deferred with respect to f" and we abbreviate this by " a.a. k deferred w.r.t. f".

Definition 2.7. Let f be an unbounded modulus. A sequence $x = (x_k)$ is said to be *deferred $f-$statistically convergent* to ℓ if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{f(q(n)-p(n))} f(|\{p(n)+1 \leq k \leq q(n) : |x_k - \ell| > \varepsilon\}|) = 0,$$

$$i.e., \delta_{p,q}^{f}(\{k \in \mathbb{N} : |x_k - \ell| > \varepsilon\}) = 0,$$

$$i.e., |x_k - \ell| \leq \varepsilon \,\text{a.a.}k \,\text{deferred w.r.t.} f$$

In this case we write $S_{p,q}^{f} - \lim x_k = \ell$.

Proposition 2.8. *Let f be an unbounded modulus. Then every deferred f-statistically convergent sequence is deferred statistically convergent but converse need not be true.*

Proof. In view of Proposition 2.4, the result follows. In order to establish the strict inclusion, one may consider consider the sequence $x = (x_k)$ where

$$x_k = \begin{cases} k, & \text{if } k = n^2, \\ 0, & \text{if } k \neq n^2, \qquad n = 1, 2, 3, \ldots. \end{cases}$$

and $f(x) = \log(x+1)$ with $q(n) = n$ and $p(n) = 0$. □

Following the technique used in Aizpuru et al. [2], we prove the following

Proposition 2.9. *Let f be an unbounded modulus. Then $S_{p,q}^{f} - \lim x_k = \ell$ if and only if there exists $K \subset \mathbb{N}$ such that $\delta_{p,q}^{f}(K) = 0$ and $\lim_{k\in\mathbb{N}-K} x_k = \ell$.*

Proof. **Step 1**: For each $j \in \mathbb{N}$, let $K^j = \{k \in \mathbb{N} : |x_k - \ell| > \frac{1}{j}\}$. As $S_{p,q}^{f} - \lim x_k = \ell$, so $\delta_{p,q}^{f}(K^j) = 0$, i.e., $\lim_{n\to\infty} \frac{1}{f(q(n)-p(n))} f(|\{k \in K^j : p(n)+1 \leq k \leq q(n)\}|) = 0$. In otherwords $\lim_{n\to\infty} \frac{1}{f(q(n)-p(n))} f(|\{p(n)+1 \leq k \leq q(n) : |x_k - \ell| > \frac{1}{j}\}|) = 0$. It is to be noted that $K^j \subset K^{j+1}$. We only need to prove the case when some of the K^j's are non-empty. Without loss of generality we may assume that $K^1 \neq \phi$. Take any $n_1 \in K^1$.Now take $n_2 \in K^2$ with $n_2 > n_1$ and $\lim_{n\to\infty} \frac{1}{f(q(n)-p(n))} f(|\{p(n)+1 \leq k \leq q(n) : |x_k - \ell| > \frac{1}{2}\}|) < \frac{1}{2}$, for all $n \geq n_2$. Inductively we get $n_1 < n_2 < n_3 < \ldots$ such that $n_j \in K^j$ and $\lim_{n\to\infty} \frac{1}{f(q(n)-p(n))} f(|\{p(n)+1 \leq k \leq q(n) : |x_k - \ell| > \frac{1}{j}\}|) < \frac{1}{j}$, for all $n \geq n_j$. Now consider $K = \bigcup_{j\in\mathbb{N}} ([n_j, n_{j+1}) \cap K^j)$.

Step 2: Let $n \in \mathbb{N}$. We claim $K_{p,q}(n) \subset K^j_{p,q}(n)$ for some j.
Let $t \in K_{p,q}(n)$. Then $t \in K$ with $p(n)+1 < t \leq q(n)$. As $t \in K$ so $t \geq n_1$. Clearly, there exist some $j \in \mathbb{N}$ such that $n_j \leq t < n_{j+1}$ and so $t \in K^j$. As a result, we have $t \in K^j_{p,q}(n)$. This establishes the claim. Now

$$\begin{aligned}\frac{f(|K_{p,q}(n)|)}{f(q(n)-p(n))} &\leq \frac{f(|K^j_{p,q}(n)|)}{f(q(n)-p(n)} \\ &= \frac{1}{f(q(n)-p(n))} f(|\{p(n)+1 \leq k \leq q(n) : |x_k - \ell| > \frac{1}{j}\}|) \\ &\leq \frac{1}{j} \text{ for all } n \geq n_j\end{aligned}$$

and so $\delta^f_{p,q}(K) = 0$.

Step 3: Let $\varepsilon > 0$.Then there exists some $j \in \mathbb{N}$ such that $\frac{1}{j} < \varepsilon$. For $i \in \mathbb{N} - K$ and $i \geq n_j$, then there exists $p \geq j$ with $n_p \leq i \leq n_{p+1}$and this implies $i \notin K^p$, so $|x_i - \ell| < \frac{1}{p} \leq \frac{1}{j} < \varepsilon$. Thus $\lim_{i\in\mathbb{N}-K} x_i = \ell$.

Conversely, assume given condition holds. Let $\varepsilon > 0$ be given. Then there exists $k_0 \in \mathbb{N}$ such that $|x_k - \ell| \leq \varepsilon$ for all $k \in \mathbb{N} - K$ and $k \geq k_0$. Consequently, $\{k \in \mathbb{N} : |x_k - \ell| > \varepsilon\} \subset K \cup \{1, 2, \ldots, k_0\}$ which yields the result. □

Remark 2.10. The next proposition indicates that the terms of a deferred $f-$statistically convergent sequence (x_k) are coincident to that of a convergent sequence for almost all k deferred with respect to f.

Proposition 2.11. *A sequence $x = (x_k)$ is deferred $f-$statistically convergent if and only if there exists a convergent sequence $y = (y_k)$ such that $x_k = y_k$ a.a. k deferred w.r.t. f.*

Proof. Let $x = (x_k)$ is deferred $f-$statistically convergent sequence. Then there exists ℓ such that for each $\varepsilon > 0$, $\delta^f_{p,q}(B) = 0$ where $B = \{k \in \mathbb{N} : |x_k - \ell| > \varepsilon\}$. Consider

$$y_k = \begin{cases} x_k, \text{if}\, k \in \mathbb{N} - B; \\ \ell, \text{if}\, k \in B. \end{cases}$$

Then $y = (y_k) \in c$ and $y_k = x_k$ a.a. k deferred w.r.t. f.
Conversely, let $\lim y_k = \ell$. Then for $\varepsilon > 0$, there exists a positive integer k_0 such that $|y_k - \ell| < \epsilon$ for all $k \geq k_0$ Let $A = \{k \in \mathbb{N} : x_k \neq y_k\}$. Now $\{k \in \mathbb{N} : |x_k - \ell| > \varepsilon\} \subset A \cup \{1, 2, 3, \ldots, k_0\}$ yields the result. □

Proposition 2.12.(Decomposition Theorem) *If $x = (x_k)$ is a deferred $f-$statistically convergent sequence, then there exists a convergent sequence $y = (y_k)$ and a deferred $f-$statistically null sequence $z = (z_k)$ such that $x = y + z$. However, this decomposition is not unique.*

Proof. Let $x = (x_k)$ is a deferred $f-$statistically convergent sequence. Then there exists ℓ such that for each $\varepsilon > 0$, $\delta^f_{p,q}(A) = 0$ where $A = \{k \in \mathbb{N} : |x_k - \ell| > \varepsilon\}$. Define sequences $y = (y_k)$ and $z = (z_k)$ as follows :

$$y_k = \begin{cases} x_k, \text{if } k \in \mathbb{N} - A; \\ \ell, \text{if } k \in A. \end{cases}$$

$$z_k = \begin{cases} 0, \text{if } k \in \mathbb{N} - A; \\ x_k - \ell, \text{if } k \in A. \end{cases}$$

Clearly $x = y+z$ where y is a convergent sequence and z is a deferred $f-$statistically null sequence, i.e., $S^f_{p,q} \subset c+S^f_{p,q;0}$ where c and $S^f_{p,q;0}$ denote the spaces of convergent and deferred $f-$statistically null sequences. As c, $S^f_{p,q;0} \subset S^f_{p,q}$, so $c + S^f_{p,q;0} \subset S^f_{p,q}$. Consequently, we have $S^f_{p,q} = c + S^f_{p,q;0}$. Using the fact that $\phi \subset c \cap S^f_{p,q;0}$ where ϕ is the space of finitely non-zero scalar sequences, we have $S^f_{p,q} \neq c \oplus S^f_{p,q;0}$, i.e., decomposition is not unique. □

Before proceeding to the computation of duals we recall

The idea of dual sequence spaces was introduced by Köthe and Toeplitz [23] whose main results concerned α-duals; the α-dual of sequence space X being defined as

$$X^\alpha = \{a = (a_k) \in s : \sum_k |a_k x_k| < \infty \quad \textit{for all} \quad x = (x_k) \in X\}$$

where s denotes the space of scalar sequences.

In the same paper [23], they also introduced another kind of dual, namely, the β-dual (see [11] also, where it is called the g-dual by Chillingworth) defined as

$$X^\beta = \{a = (a_k) \in s : \sum_k a_k x_k \quad \textit{converges for all} \quad x = (x_k) \in X\}.$$

Proposition 2.13. $[S^f_{p,q}]^\alpha = [S^f_{p,q}]^\beta = \phi$, *the space of finitely non-zero scalar sequences.*

Proof. We here only prove $[S^f_{p,q}]^\alpha = \phi$, as the proof of part $[S^f_{p,q}]^\beta = \phi$, will be similar. It is sufficient to show that $[S^f_{p,q}]^\alpha \subset \phi$ since $\phi \subset [S^f_{p,q}]^\alpha$ obviously. Let $(a_k) \in [S^f_{p,q}]^\alpha$. Then $\sum_k |a_k x_k| < \infty$ for all $x = (x_k) \in S^f_{p,q}$. Suppose $(a_k) \notin \phi$, i.e., (a_k) has infinitely many non-zero terms. Following Lemma 5 of [19], for each $n \in \mathbb{N}$, if $(p(n), q(n)]$ contains a k such that $a_k \neq 0$, let m_n be the least such k; otherwise leave m_n undefined. Thus there are infinitely many m_n's and $m_n \in (p(n), q(n)]$. Now define $x_k = \frac{1}{|a_k|}$ if $k = m_n$ for some $n = 1, 2, 3 \ldots$ and $x_k = 0$ otherwise. For $\varepsilon > 0$, we have $\frac{1}{f(q(n)-p(n))} f(|\{p(n)+1 < k \leq q(n) : |x_k - 0| > \varepsilon\}|) \leq \frac{1}{f(q(n)-p(n))} f(1) \to 0$ as $n \to \infty$ and so $(x_k) \in S^f_{p,q}$. But $\sum\limits_{p(n)+1<k\leq q(n)} |a_k x_k| = 1$ for infinitely many n and so $\sum_k |a_k x_k| = \infty$ □

3. Strongly Deferred Cesàro Summable Sequences defined by Modulus

We begin this section by introducing the notion of strongly deferred Cesàro summable sequences with respect to modulus f which is a generalization of the spaces of strongly Cesàro summable sequences. It is shown that bounded deferred $f-$ statistical convergent sequences are strongly deferred Cesàro summable with respect to f.

Definition 3.1. Let f be a modulus. We define

$$w^f_{p,q;0} = \{x \in s : \lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} f(|x_k|) = 0\},$$

$$w^f_{p,q} = \{x \in s : \lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} f(|x_k - \ell|) = 0 \text{ for some number } \ell\},$$

$$w^f_{p,q;\infty} = \{x \in s : \sup_n \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} f(|x_k|) < \infty\}.$$

Some well-known spaces are obtained by specializing f and p,q. For example, if $q(n)=n$, $p(n)=0$ then the sequence space defined above becomes $w_0(f)$ $w(f)$ and $w_\infty(f)$ of Maddox [25], respectively. If we take $f(x)=x$ $q(n)=n$, $p(n)=0$, we obtain the familiar spaces w_0, w and w_∞ of strongly Cesàro summable sequences, respectively.

It is easy to see that $w^f_{p,q;0}$, $w^f_{p,q}$ and $w^f_{p,q;\infty}$ are linear spaces over the complex field $\mathbb{C}$. We now establish some inclusion relations between the spaces $w^f_{p,q;0}$, $w^f_{p,q}$ and $w^f_{p,q;\infty}$.

Proposition 3.2. *For any modulus f, $w^f_{p,q;0} \subset w^f_{p,q} \subset w^f_{p,q;\infty}$.*

Proof. We establish only the second inclusion, the first being obvious. Let $x=(x_k) \in w^f_{p,q}$. By definition of modulus function (iii) and (ii), we have

$$\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} f(|x_k|) \le \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} f(|x_k-\ell|) + \frac{1}{q(n)-p(n)} f(|\ell|) \sum_{k=p(n)+1}^{q(n)} 1. \qquad \square$$

Following the technique used in [5], we have the following

Proposition 3.3. *For any modulus f, we have $w_{p,q;0} \subset w^f_{p,q;0}$, $w_{p,q} \subset w^f_{p,q}$ and $w_{p,q;\infty} \subset w^f_{p,q;\infty}$.*

Proposition 3.4. *Let f be a modulus. If $\lim_{t\to\infty}\frac{f(t)}{t} > 0$, then*

(i) $w^f_{p,q;0} \subset w_{p,q;0}$.

(ii) $w^f_{p,q} \subset w_{p,q}$.

(iii) $w^f_{p,q;\infty} \subset w_{p,q;\infty}$.

In view of Proposition 3.3 and Proposition 3.4, we have the following

Proposition 3.5. *Let f be any modulus such that $\lim_{t\to\infty}\frac{f(t)}{t} > 0$, then*

(i) $w^f_{p,q;0} = w_{p,q;0}$.

(ii) $w^f_{p,q} = w_{p,q}$.

(iii) $w^f_{p,q;\infty} = w_{p,q;\infty}$.

Proposition 3.6. *Let f be an unbounded modulus such that there is a positive constant c such that $f(xy) > cf(x)f(y)$ for all $x \geq 0$ $y \geq 0$ and $\lim_{t\to\infty}\frac{f(t)}{t} > 0$. If a sequence is strongly deferred Cesàro summable with respect f to ℓ, then it is deferred $f-$statistically convergent to ℓ.*

Proof. For any sequence $x = (x_k)$ and $\varepsilon > 0$, by the definition of modulus function (ii) and (iii) we have

$$\begin{aligned}\sum_{k=p(n)+1}^{q(n)} f(|x_k-\ell|) &\geq f(\sum_{k=p(n)+1}^{q(n)} |x_k-\ell|)\\ &\geq f(|\{p(n)+1 \leq k < q(n) : |x_k-\ell| \geq \varepsilon\}|\,\varepsilon)\\ &\geq cf(|p(n)+1 \leq k < q(n) : |x_k-\ell| \geq \varepsilon|)f(\varepsilon).\end{aligned}$$

As $x = (x_k) \in w^f_{p,q}$, so $x = (x_k) \in S^f_{p,q}$. □

Corollary 3.7. *If $x_k \to \ell$ then $S^f_{p,q} - \lim x_k = \ell$.*

Taking $f(x) = x$ in Proposition 3.6, we obtain the following result, which is Theorem 2.1.1 of Kücükasalan and Yilmaztürk [24].

Corollary 3.8.*If a sequence is strongly deferred Cesàro summable to ℓ, then it is deferred statistical convergent to ℓ.*

Taking $f(x) = x$ and $q(n) = n$, $p(n) = 0$ in Proposition 3.6, we obtain the following result, which is contained in Theorem 2.1 of Connor [13], for the case $q = 1$.

Corollary 3.9. *If a sequence is strongly Cesàro summable to ℓ, then it is statistically convergent to ℓ.*

Taking $q(n) = n$, $p(n) = 0$ in Proposition 3.6, we obtain the following result, which is particular case of part(a) of Theorem 8 of Connor [14].

Corollary 3.10 *Let f be an unbounded modulus such that $\lim_{t\to\infty} \frac{f(t)}{t} > 0$. If a sequence is strongly Cesàro summable with respect to f to ℓ then it is $f-$statistically convergent to ℓ.*

Let us recall that ℓ_∞ is the set of all bounded sequences.

Proposition 3.11. *If $x = (x_k) \in \ell_\infty$ and $(x_k) \in S^f_{p,q}$ with $S^f_{p,q} - \lim x_k = l$, then $(x_k) \in w^f_{p,q}$, i.e.,a bounded sequence which is deferred $f-$ statistical convergent to ℓ is strongly deferred Cesàro summable with respect to f to ℓ.*

Proof. Suppose that $(x_k) \in \ell_\infty$ and $S^f_{p,q} - \lim x_k = \ell$. There exists a positive real number M such that $|x_k - \ell| \leq M$ for all k.

$$
\begin{aligned}
&\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} f(|x_k - \ell|) \\
&= \frac{1}{q(n)-p(n)} \Big(\sum_{k=p(n)+1, |x_k-\ell|\geq\varepsilon}^{q(n)} + \sum_{k=p(n)+1, |x_k-\ell|<\varepsilon}^{q(n)} \Big) f(|x_k - \ell|) \\
&\leq \frac{1}{q(n)-p(n)} \Big(f(M) \sum_{k=p(n)+1, |x_k-\ell|\geq\varepsilon}^{q(n)} 1 + f(\varepsilon) \sum_{k=p(n)+1, |x_k-\ell|<\varepsilon}^{q(n)} \Big) \\
&\leq f(M) \frac{1}{q(n)-p(n)} |\{k : p(n)+1 \leq k \leq q(n), |x_k - \ell| \geq \varepsilon\}| \\
&+ f(\varepsilon) \frac{1}{q(n)-p(n)} |\{k : p(n)+1 \leq k \leq q(n), |x_k - \ell| < \varepsilon\}|
\end{aligned}
$$

which yields the proof. □

References

[1] R. P. Agnew, *On deferred Cesàro mean*, Ann. Math., **33**(1932), 413–421.

[2] A. Aizpuru, M. C. Listàn-Garcìa and F. Rambla-Barreno, *Density by moduli and statistical convergence*, Quest. Math., **37**(2014), 525–530.

[3] Y. Altin and M. Et, *Generalized difference sequence spaces defined by a modulus function in a locally convex space*, Soochow J. Math., **31(2)**(2005), 233–243.

[4] V. K. Bhardwaj and I. Bala, *On weak statistical convergence*, Int. J. Math. Math. Sci., Art. ID **38530**(2007), 9 pp.

[5] V. K. Bhardwaj and S. Dhawan, *$f-$statistical convergence of order α and strong Cesàro summability of order α with respect to a modulus*, J. Inequal. Appl., **2015:332**(2015), 14 pp.

[6] V. K. Bhardwaj, S. Dhawan and S. Gupta *Density by moduli and statistical boundedness*, Abstr. Appl. Anal., Art. ID **2143018**(2016), 6 pp.

[7] V. K. Bhardwaj and S. Gupta, *On some generalizations of statistical boundedness*, J. Inequal. Appl., **2014:12**(2014), 11 pp.

[8] V. K. Bhardwaj and N. Singh, *On some sequence spaces defined by a modulus*, Indian J. Pure Appl. Math., **30(8)**(1999), 809–817.

[9] V. K. Bhardwaj and N. Singh, *Some sequence spaces defined by $\left|\bar{N}, p_n\right|$ summability and a modulus function*, Indian J. Pure Appl. Math., **32(12)**(2001), 1789–1801.

[10] V. K. Bhardwaj and N. Singh, *Banach space valued sequence spaces defined by a modulus*, Indian J. Pure Appl. Math., **32(12)**(2001), 1869–1882.

[11] H. R. Chillingworth, *Generalized "dual" sequence spaces*, Nederal. Akad. Wetensch. Indag. Math. **20**(1958), 307–315.

[12] R. Çolak, *Lacunary strong convergence of difference sequences with respect to a modulus function*, Filomat, **17**(2003), 9–14.

[13] J. S. Connor, *The statistical and strong $p-$ Cesàro convergence of sequences*, Analysis, **8**(1988), 47–63.

[14] J. Connor, *On strong matrix summability with respect to a modulus and statistical convergence*, Canad. Math. Bull., **32(2)**(1989), 194–198.

[15] M. Et, V. K. Bhardwaj and S. Gupta, *On deferred statistical boundedness of order α*, (communicated).

[16] M. Et and H. Şengül, *Some Cesàro-type summability spaces of order α and lacunary statistical convergence of order α*, Filomat, **28(8)**(2014), 1593–1602.

[17] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2**(1951), 241–244.

[18] J. A. Fridy, *On statistical convergence*, Analysis, **5**(1985), 301–313.

[19] J. A. Fridy and C. Orhan, *Lacunary statistical summability*, J. Math. Anal. Appl., **173**(1993), 497–504.

[20] D. Ghosh and P. D. Srivastava, *On some vector valued sequence spaces defined using a modulus function*, Indian J. Pure Appl. Math., **30(8)**(1999), 819–826.

[21] M. Işık, *Generalized vector-valued sequence spaces defined by modulus functions*, J. Inequal. Appl., Art. ID **457892**(2010), 7 pp.

[22] M. Işık and K. E. Akbaş, *On $\lambda-$statistical convergence of order α in probability*, J. Inequal. Spec. Funct., **8(4)**(2017), 57–64.

[23] G. Köthe and O. Toeplitz, *Lineare Räume mit unendlich vielen Koordinaten und Ringe unendlicher Matrizen*, J. Reine Agnew. Math., **171**(1934), 193–226.

[24] M. Küçükaslan and M. Yılmaztürk, *On deferred statistical convergence of sequences*, Kyungpook Math. J. **56**(2016), 357–366.

[25] I. J. Maddox, *Inclusion between FK spaces and Kuttner's theorem*, Math. Proc. Camb. Philos. Soc., **101**(1987), 523–527.

[26] M. Mursaleen, *$\lambda-$statistical convergence*, Math. Slovaca, **50(1)**(2000), 111–115.

[27] H. Nakano, *Concave modulars*, J. Math. Soc. Japan, **5**(1953), 29–49.

[28] D. Rath and B. C. Tripathy, *On statistically convergent and statistically Cauchy sequences*, Indian J. Pure. Appl. Math., **25(4)**(1994), 381–386.

[29] W. H. Ruckle, *FK spaces in which the sequence of coordinate vectors is bounded*, Canad. J. Math., **25**(1973), 973–978.

[30] T. Salat, *On statistically convergent sequences of real numbers*, Math. Slovaca, **30**(1980), 139–150.

[31] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloquium Mathematicum, **2**(1951), 73–74.

[32] F. Temizsu, M. Et and M. Çınar, *Δ^m- deferred statistical convergence of order α*, Filomat, **30(3)**(2016), 667–673.

[33] M. Yılmaztürk and M. Küçükaslan, *On strongly deferred Cesàro summability and deferred statistical convergence of the sequences*, Bitlis Eren Univ. J. Sci. and Technol., **3**(2011), 22–25.