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Generalized Incomplete Pochhammer Symbols and Their Applications to Hypergeometric Functions

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Abstract. In this paper, we present new generalized incomplete Pochhammer symbols and using this we introduce the extended generalized incomplete hypergeometric functions. We derive certain properties, generating functions and reduction formulas of these extended generalized incomplete hypergeometric functions. Special cases of this extended generalized incomplete hypergeometric functions are also discussed.

1. Introduction

In a recent paper, Srivastava *et al.* [10] have studied incomplete Pochhammer symbols and generalized incomplete hypergeometric functions associated with them. It was shown that these generalized incomplete hypergeometric functions have applications in areas such as communication theory, probability theory and groundwater pumping modeling. In the present paper, we introduce a new pair of generalized incomplete Pochhammer symbols and study the resulting extended generalized incomplete hypergeometric functions from the generating functions and reduction formulas point of view. We start with recalling the necessary definitions and results.

The incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$ are defined by [11]

$$\gamma(s,x)=\int_0^x t^{s-1}e^{-t}dt, \qquad \Re(s)\geq 0; x\geq 0 \tag{1.1}$$

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and

$$\Gamma(s,x)=\int_x^{\infty} t^{s-1}e^{-t}dt\,, \qquad \Re(s)\geq 0; x\geq 0 \tag{1.2}$$

respectively and satisfy the following decomposition formula:

$$\gamma(s,x)+\Gamma(s,x)=\Gamma(s), \qquad \Re(s)>0. \tag{1.3}$$

Srivastava *et al.* [10] introduced the generalized incomplete hypergeometric functions by

$${}_r\gamma_s\left[\begin{matrix}(a_1,x),a_2,\ldots,a_r;\\ b_1,\ldots,b_s;\end{matrix}\, z\right]=\sum_{n=0}^{\infty}\frac{(a_1;x)_n(a_2)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n}\frac{z^n}{n!} \tag{1.4}$$

and

$${}_r\Gamma_s\left[\begin{matrix}(a_1,x),a_2,\ldots,a_r;\\ b_1,\ldots,b_s;\end{matrix}\, z\right]=\sum_{n=0}^{\infty}\frac{[a_1;x]_n(a_2)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n}\frac{z^n}{n!}\,, \tag{1.5}$$

where $(a_1;x)_n=\frac{\gamma(a_1+n,x)}{\Gamma(a)}$ and $[a_1;x]_n=\frac{\Gamma(a_1+n,x)}{\Gamma(a)}$ are incomplete Pochhammer symbols. Note that ${}_r\gamma_s$ and ${}_r\Gamma_s$ satisfy the decomposition formula:

$$\begin{aligned}&{}_r\gamma_s\left[\begin{matrix}(a_1,x),a_2,\ldots,a_r;\\ b_1,\ldots,b_s;\end{matrix}\, z\right]+{}_r\Gamma_s\left[\begin{matrix}(a_1,x),a_2,\ldots,a_r;\\ b_1,\ldots,b_s;\end{matrix}\, z\right]\\ =\;& {}_rF_s\left[\begin{matrix}a_1,\ldots,a_r;\\ b_1,\ldots,b_s;\end{matrix}\, z\right]\end{aligned} \tag{1.6}$$

where ${}_rF_s$ is the generalized hypergeometric function.

The generalized incomplete gamma functions $\gamma(\lambda,x;\alpha)$ and $\Gamma(\lambda,x;\alpha)$ are defined as [2]:

$$\gamma(\lambda,x;\alpha)=\int_0^{x} t^{\lambda-1}\exp(-t-\frac{\alpha}{t})\,dt,\ \text{for } \alpha=0,\ \Re(\lambda)>0, \tag{1.7}$$

and

$$\Gamma(\lambda,x;\alpha)=\int_x^{\infty} t^{\lambda-1}\exp(-t-\frac{\alpha}{t})\,dt \tag{1.8}$$

respectively. These generalized incomplete gamma functions satisfy the following decomposition relation:

$$\gamma(\lambda,x;\alpha)+\Gamma(\lambda,x;\alpha)=2\alpha^{\frac{\lambda}{2}}K_\lambda(2\sqrt{\alpha}), \tag{1.9}$$

where $K_\lambda(z)$ is a modified Bessel function of third kind, also known as Macdonald function, [15].

R. Srivastava [12, 13] and Srivastava and Cho [14] investigated several properties of the incomplete hypereometric functions and some general classes of the incomplete hypergeometric polynomials associated with them.

In [7], the authors have studied the following generalization of the Pochhammer symbol given by

$$(1.10)\quad (\lambda;p,\alpha,\beta)_\nu = \frac{\Gamma_p^{(\alpha,\beta)}(\lambda+\nu)}{\Gamma(\lambda)}, \quad \Re(\alpha)>0,\ \Re(\beta)>0,\ \Re(p)>0,\ \lambda,\nu\in\mathbb{C},$$

where $\Gamma_p^{(\alpha,\beta)}(x)$ is a generalization of gamma function given by [5]

$$(1.11)\qquad \Gamma_p^{(\alpha,\beta)}(x) = \int_0^\infty t^{x-1}\,{}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right)dt.$$

Note that for $\alpha=\beta$, $\Gamma_p^{(\alpha,\alpha)}(x)=\Gamma_p(x)$, [1, 2], and $(\lambda;p,\alpha,\alpha)_\nu=(\lambda;p)_\nu=\frac{\Gamma_p(\lambda+\nu)}{\Gamma_p(\nu)}$, [9]. Clearly for $p=0$, $\Gamma_p(x)=\Gamma(x)$, the gamma function.

Section-wise treatment is as follows. In Section 2, we define a new pair of generalized incomplete Pochhammer symbols and study the resulting extended generalized incomplete hypergeometric functions. In Section 3, we obtain generating functions for the families of extended generalized incomplete hypergeometric functions. Also, we present a theorem that gives reduction formulas for extended generalized incomplete hypergeometric functions.

2. Generalized Incomplete Pochhammer Symbols

We now introduce a pair of new extended incomplete gamma functions denoted by $\gamma(\lambda,x;p,\alpha,\beta)$ and $\Gamma(\lambda,x;p,\alpha,\beta)$ and defined as follows:

$$(2.1)\qquad \gamma(\lambda,x;p,\alpha,\beta) = \int_0^x t^{\lambda-1}\,{}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right)dt$$

and

$$(2.2)\qquad \Gamma(\lambda,x;p,\alpha,\beta) = \int_x^\infty t^{\lambda-1}\,{}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right)dt,$$

where $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(p)>0$, $\Re(\lambda)>0$. These extended incomplete gamma functions satisfy the decomposition relation:

$$(2.3)\qquad \gamma(\lambda,x;p,\alpha,\beta)+\Gamma(\lambda,x;p,\alpha,\beta)=\Gamma_p^{(\alpha,\beta)}(\lambda).$$

Using these extended incomplete gamma functions $\gamma(\lambda, x; p, \alpha, \beta)$ and $\Gamma(\lambda, x; p, \alpha, \beta)$, we introduce a pair of new generalized incomplete Pochhammer symbols $(\lambda, x; p, \alpha, \beta)_\nu$ and $[\lambda, x; p, \alpha, \beta]_\nu$, $\lambda, \nu \in \mathbb{C}$, $x \geq 0$, defined by:

$$(\lambda, x; p, \alpha, \beta)_\nu = \frac{\gamma(\lambda + \nu, x; p, \alpha, \beta)}{\Gamma(\lambda)}, \qquad \lambda, \nu \in \mathbb{C},\ x \geq 0, \tag{2.4}$$

and

$$[\lambda, x; p, \alpha, \beta]_\nu = \frac{\Gamma(\lambda + \nu, x; p, \alpha, \beta)}{\Gamma(\lambda)}, \qquad \lambda, \nu \in \mathbb{C},\ x \geq 0. \tag{2.5}$$

It is evident that the generalized incomplete Pochhammer symbols (2.4) and (2.5) satisfy the following decomposition formula

$$(\lambda, x; p, \alpha, \beta)_\nu + [\lambda, x; p, \alpha, \beta]_\nu = (\lambda; p, \alpha, \beta)_\nu, \qquad \lambda, \nu \in \mathbb{C}, \quad x \geq 0, \tag{2.6}$$

In view of (2.6), it is sufficient to study the properties of the $[\lambda, x; p, \alpha, \beta]_\nu$.

Motivated by the generalization of the incomplete Pochhammer symbols (2.4) and (2.5), we introduce the following extended generalized incomplete hypegeometric functions

$$ {}_r\gamma_s(p, \alpha, \beta) \left[\begin{array}{r} (a_1, x, p, \alpha, \beta),\ a_2, \ldots, a_r; \\ b_1, \ldots, b_s; \end{array} z \right] = \sum_{n=0}^{\infty} \frac{(a_1, x; p, \alpha, \beta)_n (a_2)_n \ldots (a_r)_n}{(b_1)_n \ldots (b_s)_n} \frac{z^n}{n!} \tag{2.7}$$

and

$$ {}_r\Gamma_s(p, \alpha, \beta) \left[\begin{array}{r} (a_1, x, p, \alpha, \beta),\ a_2, \ldots, a_r; \\ b_1, \ldots, b_s; \end{array} z \right] = \sum_{n=0}^{\infty} \frac{[a_1, x; p, \alpha, \beta]_n (a_2)_n \ldots (a_r)_n}{(b_1)_n \ldots (b_s)_n} \frac{z^n}{n!}. \tag{2.8}$$

From (2.7) and (2.8), we get the following decomposition formula

$$\begin{aligned} &{}_r\gamma_s(p, \alpha, \beta) \left[\begin{array}{r} (a_1, x, p, \alpha, \beta), a_2, \ldots, a_r; \\ b_1, \ldots, b_s; \end{array} z \right] + {}_r\Gamma_s(p, \alpha, \beta) \left[\begin{array}{r} (a_1, x, p, \alpha, \beta), a_2, \ldots, a_r; \\ b_1, \ldots, b_s; \end{array} z \right] \\ &= {}_rF_s \left[\begin{array}{r} (a_1, p, \alpha, \beta), a_2, \ldots, a_r; \\ b_1, \ldots, b_s; \end{array} z \right] \end{aligned} \tag{2.9}$$

where the right hand side is defined in [7]. Decomposition formula (2.9) is based on the decomposition formula (2.6), and generalizes (1.3).

We now discuss the case $\alpha = \beta$ of (2.4) and (2.5). This leads to the pair of generalized incomplete Pochhammer symbols $(\lambda, x; p)_\nu$ and $[\lambda, x; p]_\nu$ defined as follows:

$$(\lambda, x; p)_\nu = \frac{\gamma(\lambda + \nu, x; p)}{\Gamma(\lambda)}, \qquad \lambda, \nu \in \mathbb{C}\,; x \geq 0, \tag{2.10}$$

and

$$[\lambda, x; p]_\nu = \frac{\Gamma(\lambda+\nu, x; p)}{\Gamma(\lambda)}, \qquad \lambda, \nu \in \mathbb{C}\,; x \geq 0. \tag{2.11}$$

Clearly $(\lambda, x; p)_\nu + [\lambda, x; p]_\nu = (\lambda; p)_\nu$. These incomplete Pochhammer symbols (2.10) and (2.11) lead to the following extended generalized incomplete hypegeometric functions

$$ {}_r\gamma_s(p)\left[\begin{matrix}(a_1, x, p), a_2, \ldots, a_r; \\ b_1, \ldots, b_s;\end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{(a_1; x, p)_n (a_2)_n \ldots (a_r)_n}{(b_1)_n \ldots (b_s)_n} \frac{z^n}{n!} \tag{2.12}$$

and

$$ {}_r\Gamma_s(p)\left[\begin{matrix}(a_1, x, p), a_2, \ldots, a_r; \\ b_1, \ldots, b_s;\end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{[a_1; x, p]_n (a_2)_n \ldots (a_r)_n}{(b_1)_n \ldots (b_s)_n} \frac{z^n}{n!}, \tag{2.13}$$

respectively. From (2.12) and (2.13), we get the decomposition formula

$$\begin{aligned} & {}_r\gamma_s(p)\left[\begin{matrix}(a_1, x, p), a_2, \ldots, a_r; \\ b_1, \ldots, b_s;\end{matrix} z\right] + {}_r\Gamma_s(p)\left[\begin{matrix}(a_1, x, p), a_2, \ldots, a_r; \\ b_1, \ldots, b_s;\end{matrix} z\right] \\ = \ & {}_rF_s\left[\begin{matrix}(a_1, p), a_2, \ldots, a_r; \\ b_1, \ldots, b_s;\end{matrix} z\right] \end{aligned} \tag{2.14}$$

where the right hand side is defined in [9]. Note that the decomposition formula (2.14) generalizes (1.6). Since $|(\lambda; x, p)_n| \leq |(\lambda)_n|$ and $|[\lambda; x, p]_n| \leq |(\lambda)_n|$, $n \in \mathbb{N}_0$; $\lambda \in \mathbb{C}$; $x \geq 0$ the infinite series in (2.12) and (2.13) converges absolutely. This can be verified from comparing these series with the case of the generalized hypergeometric function ${}_rF_s$, [6].

We now give the following results for the extended generalized incomplete hypergeometric function ${}_r\Gamma_s(p, \alpha, \beta)$, $r, s \in \mathbb{N}_0$. Analogous properties for the generalized incomplete hypergeometric function ${}_r\gamma_s(p, \alpha, \beta)$, $r, s \in \mathbb{N}_0$ can be derived using (2.9).

Theorem 2.1. *The following integral representation holds true:*

$$\begin{aligned} & {}_r\Gamma_s(p, \alpha, \beta)\left[\begin{matrix}(a_1, x, p, \alpha, \beta), a_2, \ldots, a_r; \\ b_1, \ldots, b_s;\end{matrix} z\right] \\ = \ & \frac{1}{\Gamma(a_1)} \int_x^\infty t^{a_1-1}\, {}_1F_1\left(\alpha; \beta; -t - \frac{p}{t}\right) {}_{r-1}F_s\left[\begin{matrix}a_2, \ldots, a_r; \\ b_1, \ldots, b_s;\end{matrix} zt\right] dt, \end{aligned} \tag{2.15}$$

$$\Re(\alpha) > 0,\ \Re(\beta) > 0,\ \Re(p) > 0;\ \Re(a_1) > 0,\ \alpha = \beta \text{ when } p = 0.$$

Proof. Putting the generalized incomplete Pochhammer symbol $[a_1, x; p, \alpha, \beta]_n$ in (2.8) by its integral representation (2.2), we get the desired result . □

Corollary 2.2. *The following integral representation holds true:*

(2.16)
$$ {}_2\Gamma_1(p,\alpha,\beta)\left[\begin{array}{r} (a,x,p,\alpha,\beta),b; \\ c; \end{array} z\right] = \frac{1}{\Gamma(a)}\int_x^{\infty} t^{a-1}\,{}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right){}_1F_1\left(b;c;zt\right)\,dt, $$

$$ \Re(\alpha)>0,\ \Re(\beta)>0,\ \Re(p)>0, x\geq 0;\ \Re(a)>0,\ \alpha=\beta \text{ when } p=0. $$

Theorem 2.3. *The following derivative formula holds true:*

$$ \frac{d^n}{dz^n}\left\{{}_r\Gamma_s(p,\alpha,\beta)\left[\begin{array}{r} (a_1,x,p,\alpha,\beta),a_2,\ldots,a_r; \\ b_1,\ldots,b_s; \end{array} z\right]\right\} $$
$$ = \frac{\prod_{i=1}^{r}(a_i)_n}{\prod_{j=1}^{s}(b_j)_n}\,{}_r\Gamma_s(p,\alpha,\beta)\left[\begin{array}{r} (a_1+n,x,p,\alpha,\beta),a_2+n,\ldots,a_r+n; \\ b_1+n,\ldots,b_s+n\,; \end{array} z\right],\quad n\in\mathbb{N}_0. $$
(2.17)

Proof. Obviously (2.17) is valid for $n=0$.

We first prove (2.17) for $n=1$:

Differentiating both sides of (2.15) with respect to z and simplifying, we get

$$ \frac{d}{dz}\left\{{}_r\Gamma_s(p,\alpha,\beta)\left[\begin{array}{r} (a_1,x,p,\alpha,\beta),a_2,\ldots,a_r; \\ b_1,\ldots,b_s; \end{array} z\right]\right\} $$
(2.18)
$$ = \frac{\prod_{i=1}^{r}(a_i)_1}{\prod_{j=1}^{s}(b_j)_1}\,{}_r\Gamma_s(p,\alpha,\beta)\left[\begin{array}{r} (a_1+1,x,p,\alpha,\beta),a_2+1,\ldots,a_r+1; \\ b_1+1,\ldots,b_s+1; \end{array} z\right]. $$

The general result is obtained by applying induction on $n\in\mathbb{N}_0$. □

Theorem 2.4. *The following integral representation holds true:*

$$ {}_r\Gamma_s(p,\alpha,\beta)\left[\begin{array}{r} (a_1,x,p,\alpha,\beta),a_2,\ldots,a_{r-1},b; \\ b_1,\ldots,b_{s-1},c; \end{array} z\right] $$
$$ = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{(c-b)-1}\,{}_{r-1}\Gamma_{s-1}(p,\alpha,\beta)\left[\begin{array}{r} (a_1,x,p,\alpha,\beta),a_2,\ldots,a_{r-1}; \\ b_1,\ldots,b_{s-1}; \end{array} zt\right]dt, $$
(2.19)

$$ \Re(\alpha)>0,\ \Re(\beta)>0,\ \Re(p)>0,\ \Re(c)>\Re(b)>0;\ x\geq 0. $$

Proof. The assertion of the theorem follows from:

(2.20)
$$ \frac{(b)_n}{(c)_n} = \frac{B(b+n,c-b)}{B(b,c-b)} = \frac{1}{B(b,c-b)}\int_0^1 t^{b+n-1}(1-t)^{c-b-1}dt\,, $$

$$ \Re(c)>\Re(b)>0;\ n\in\mathbb{N}_0. $$ □

We now present some theorems for extended incomplete Gauss hypergeometric function:

Theorem 2.5. *The following recurrence relation holds true:*

$$\begin{aligned}&[b-(c-1)]\,{}_2\Gamma_1(p,\alpha,\beta)\left[\begin{array}{r}(a,x,p,\alpha,\beta),b;\\ c;\end{array} z\right]\\ =\;& b\,{}_2\Gamma_1(p,\alpha,\beta)\left[\begin{array}{r}(a,x,p,\alpha,\beta),b+1;\\ c;\end{array} z\right]-(c-1)\,{}_2\Gamma_1(p,\alpha,\beta)\left[\begin{array}{r}(a,x,p,\alpha,\beta),b;\\ c-1;\end{array} z\right].\end{aligned} \tag{2.21}$$

Proof. As the confluent hypergeometric function ${}_1F_1$ satisfies the contiguous relation [6]:

$$(c-b-1)\,{}_1F_1\left(b;c;z\right)=(c-1){}_1F_1\left(b;c-1;z\right)-b\,{}_1F_1\left(b+1;c;z\right), \tag{2.22}$$

we have, using (2.16)

$$\begin{aligned}&(c-b-1)\,{}_2\Gamma_1(p,\alpha,\beta)\left[\begin{array}{r}(a,x,p,\alpha,\beta),b;\\ c;\end{array} z\right]\\ =\;&\frac{1}{\Gamma(a)}\int_x^\infty t^{a-1}\,{}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right)\left[(c-1)\,{}_1F_1\left(b;c-1;zt\right)-b\,{}_1F_1\left(b+1;c;zt\right)\right]dt,\end{aligned} \tag{2.23}$$

Now (2.23) can be written as

$$\begin{aligned}&[b-(c-1)]{}_2\Gamma_1(p,\alpha,\beta)\left[\begin{array}{r}(a,x,p,\alpha,\beta),b;\\ c;\end{array} z\right]\\ =\;& b{}_2\Gamma_1(p,\alpha,\beta)\left[\begin{array}{r}(a,x,p,\alpha,\beta),b+1;\\ c;\end{array} z\right]-(c-1){}_2\Gamma_1(p,\alpha,\beta)\left[\begin{array}{r}(a,x,p,\alpha,\beta),b;\\ c-1;\end{array} z\right].\end{aligned} \tag{2.24}$$

□

Theorem 2.6. *The following recurrence relation holds true:*

$$\begin{aligned}&\frac{az}{c}\,{}_2\Gamma_1(p,\alpha,\beta)\left[\begin{array}{r}(a+1,x,p,\alpha,\beta),b+1;\\ c+1;\end{array} z\right]\\ =\;&{}_2\Gamma_1(p,\alpha,\beta)\left[\begin{array}{r}(a,x,p,\alpha,\beta),b+1;\\ c;\end{array} z\right]-\,{}_2\Gamma_1(p,\alpha,\beta)\left[\begin{array}{r}(a,x,p,\alpha,\beta),b;\\ c;\end{array} z\right].\end{aligned} \tag{2.25}$$

Proof. Using (2.16), we can write the l.h.s. of (2.25)

$$\begin{aligned}&az\,{}_2\Gamma_1(p,\alpha,\beta)\left[\begin{array}{r}(a+1,x,p,\alpha,\beta),b+1;\\ c+1;\end{array} z\right]\\ =\;&\frac{a}{\Gamma(a+1)}\int_x^\infty t^{a-1}\,{}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right)\left[zt\,{}_1F_1\left(b+1;c+1;zt\right)\right]dt.\end{aligned} \tag{2.26}$$

Using the contiguous relation [6]:

$$c\,{}_1F_1\left(b;c;z\right)-c\,{}_1F_1\left(b-1;c;z\right)=z\;{}_1F_1\left(b;c+1;z\right), \tag{2.27}$$

eq (2.26) can be written as

$$az\,{}_2\Gamma_1(p,\alpha,\beta)\left[\begin{array}{r}(a+1,x,p,\alpha,\beta),b+1;\\ c+1;\end{array} z\right]$$

$$= \frac{c}{\Gamma(a)}\int_x^\infty t^{a-1}\,{}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right)\left[{}_1F_1\left(b+1;c;zt\right)-\,{}_1F_1\left(b;c;zt\right)\right]dt. \tag{2.28}$$

This gives

$$\frac{az}{c}\,{}_2\Gamma_1(p,\alpha,\beta)\left[\begin{array}{r}(a+1,x,p,\alpha,\beta),b+1;\\ c+1;\end{array} z\right]$$

$$= {}_2\Gamma_1(p,\alpha,\beta)\left[\begin{array}{r}(a,x,p,\alpha,\beta),b+1;\\ c;\end{array} z\right]-\,{}_2\Gamma_1(p,\alpha,\beta)\left[\begin{array}{r}(a,x,p,\alpha,\beta),b;\\ c;\end{array} z\right]. \tag{2.29}$$

□

Following theorem gives a relation between ${}_2\Gamma_1(p,\alpha,\beta)$ and the incomplete gamma function $\gamma(\lambda,x)$.

Theorem 2.7. *The following integral representation holds true:*

$$ {}_2\Gamma_1(p,\alpha,\beta)\left[\begin{array}{r}(a,x,p,\alpha,\beta),b;\\ b+1;\end{array} -z\right]=\frac{bz^{-a}}{\Gamma(a)}\int_x^\infty t^{(a-b)-1}\,{}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right)\gamma(b,zt)\,dt, \tag{2.30}$$

$$\Re(\alpha)>0,\ \Re(\beta)>0,\ \Re(p)>0;\ \Re(a)>0 \text{ when } \alpha=\beta,\ p=0.$$

Proof: Putting $c=b+1$ and replacing z by $-z$ in (2.16), we get

$$ {}_2\Gamma_1(p,\alpha,\beta)\left[\begin{array}{r}(a,x,p,\alpha,\beta),b;\\ b+1;\end{array} -z\right]$$

$$=\frac{1}{\Gamma(a)}\int_x^\infty t^{a-1}\,{}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right){}_1F_1\left[\begin{array}{r}b;\\ b+1;\end{array} -zt\right]dt. \tag{2.31}$$

Using ${}_1F_1[s,s+1;-x]=s\,x^{-s}\gamma(s,x)$, we find that

$$ {}_2\Gamma_1(p,\alpha,\beta)\left[\begin{array}{r}(a,x,p,\alpha,\beta),b;\\ b+1;\end{array} -z\right]=\frac{bz^{-a}}{\Gamma(a)}\int_x^\infty t^{(a-b)-1}\,{}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right)\gamma(b,zt)dt. \tag{2.32}$$

This completes the proof.

3. Generating Functions and Reduction Formulas

The main generating functions for the families of extended generalized incomplete hypergeometric functions are presented in the following theorem. For $\lambda \in \mathbb{C}$ and $N \in \mathbb{N}$, let $\Delta(N, \lambda)$ denotes the following array of N parameters:

$$\frac{\lambda}{N}, \frac{\lambda+1}{N}, \ldots, \frac{\lambda+N-1}{N},$$

the array $\Delta(N, \lambda)$ being assumed to be empty when $N = 0$.

Theorem 3.1. *The following generating function holds true for the families of extended generalized incomplete hypergeometric functions* ${}_r\gamma_s(p, \alpha, \beta)$ *and* ${}_r\Gamma_s(p, \alpha, \beta)$*:*

$$(1-t)^{-\lambda}{}_{r+N}\gamma_s(p,\alpha,\beta)\left[\begin{array}{r} \Delta(N,\lambda), (a_1, x, p, \alpha, \beta), a_2, \cdots, a_r; \\ b_1, \cdots, b_s; \end{array} \frac{z}{(1-t)^N}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}{}_{r+N}\gamma_s(p,\alpha,\beta)\left[\begin{array}{r} \Delta(N,\lambda+n), (a_1, x, p, \alpha, \beta), a_2, \cdots, a_r; \\ b_1, \cdots, b_s; \end{array} z\right] t^n, \tag{3.1}$$

$$|t| < 1,\ \lambda \in \mathbb{C},\ N \in \mathbb{N},$$

and

$$(1-t)^{-\lambda}{}_{r+N}\Gamma_s(p,\alpha,\beta)\left[\begin{array}{r} \Delta(N,\lambda), (a_1, x, p, \alpha, \beta), a_2, \cdots, a_r; \\ b_1, \cdots, b_s; \end{array} \frac{z}{(1-t)^N}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}{}_{r+N}\Gamma_s(p,\alpha,\beta)\left[\begin{array}{r} \Delta(N,\lambda+n), (a_1, x, p, \alpha, \beta), a_2, \cdots, a_r; \\ b_1, \cdots, b_s; \end{array} z\right] t^n, \tag{3.2}$$

$$|t| <,\ \lambda \in \mathbb{C},\ N \in \mathbb{N}.$$

provided each member of (3.1) and (3.2) exists.

Proof. Using (2.8) and the identity

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} z^n = (1-z)^{-\lambda}, \qquad |z| < 1; \lambda \in \mathbb{C}. \tag{3.3}$$

we get the generating function (3.1). Proof of (3.2) is similar. □

Theorem 3.2. *Each of the following generating function hold true for families of the extended generalized incomplete hypergeometric functions* ${}_r\gamma_s(p, \alpha, \beta)$ *and* ${}_r\Gamma_s(p, \alpha, \beta)$ *:*

$$(1-t)^{-\lambda}{}_{r+N}\gamma_s(p,\alpha,\beta)\left[\begin{array}{r} \Delta(N,\lambda), (a_1, x, p, \alpha, \beta), a_2, \cdots, a_r; \\ b_1, \cdots, b_s; \end{array} z\left(-\frac{t}{1-t}\right)^N\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}{}_{r+N}\gamma_s(p,\alpha,\beta)\left[\begin{array}{r} \Delta(N,-n), (a_1, x, p, \alpha, \beta), a_2, \cdots, a_r; \\ b_1, \cdots, b_s; \end{array} z\right] t^n, \tag{3.4}$$

$$|t|<1,\ \lambda\in\mathbb{C},\ N\in\mathbb{N};$$

$$\begin{aligned}&(1-t)^{-\lambda}{}_{r+N}\Gamma_s(p,\alpha,\beta)\left[\begin{array}{r}\Delta(N,\lambda),(a_1,x,p,\alpha,\beta),a_2,\cdots,a_r;\\ b_1,\cdots,b_s;\end{array} z\left(-\frac{t}{1-t}\right)^N\right]\\ &=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}{}_{r+N}\Gamma_s(p,\alpha,\beta)\left[\begin{array}{r}\Delta(N,-n),(a_1,x,p,\alpha,\beta),a_2,\cdots,a_r;\\ b_1,\cdots,b_s;\end{array} z\right]t^n,\end{aligned}\tag{3.5}$$

$$|t|<1,\ \lambda\in\mathbb{C},\ N\in\mathbb{N};$$

$$\begin{aligned}&(1-t)^{-\lambda}{}_{r+2N}\gamma_s(p,\alpha,\beta)\left[\begin{array}{r}\Delta(2N,\lambda),(a_1,x,p,\alpha,\beta),a_2,\cdots,a_r;\\ b_1,\cdots,b_s;\end{array} z\left(-\frac{4t}{(1-t)^2}\right)^N\right]\\ &=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}{}_{r+2N}\gamma_s(p,\alpha,\beta)\left[\begin{array}{r}\Delta(N,-n),\Delta(N;\lambda+n),(a_1,x,p,\alpha,\beta),a_2,\cdots,a_r;\\ b_1,\cdots,b_s;\end{array} z\right]t^n,\end{aligned}\tag{3.6}$$

$$|t|<1,\ \lambda\in\mathbb{C},\ N\in\mathbb{N};$$

$$\begin{aligned}&(1-t)^{-\lambda}{}_{r+2N}\Gamma_s(p,\alpha,\beta)\left[\begin{array}{r}\Delta(2N,\lambda),(a_1,x,p,\alpha,\beta),a_2,\cdots,a_r;\\ b_1,\cdots,b_s;\end{array} z\left(-\frac{4t}{(1-t)^2}\right)^N\right]\\ &=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}{}_{r+2N}\Gamma_s(p,\alpha,\beta)\left[\begin{array}{r}\Delta(N,-n),\Delta(N;\lambda+n),(a_1,x,p,\alpha,\beta),a_2,\cdots,a_r;\\ b_1,\cdots,b_s;\end{array} z\right]t^n,\end{aligned}\tag{3.7}$$

$$|t|<1,\ \lambda\in\mathbb{C},\ N\in\mathbb{N};$$

$$\begin{aligned}&(1-t)^{-\lambda}{}_{r}\gamma_s(p,\alpha,\beta)\left[\begin{array}{r}(a_1,x,p,\alpha,\beta),a_2,\cdots,a_r;\\ b_1,\cdots,b_s;\end{array} zt^N\right]\\ &=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}{}_{r+N}\gamma_{s+N}(p,\alpha,\beta)\left[\begin{array}{l}\Delta(N,-n),(a_1,x,p,\alpha,\beta),a_2,\cdots,a_r;\\ \Delta(N,1-\lambda-n),\qquad b_1,\cdots,b_s;\end{array} z\right]t^n,\end{aligned}\tag{3.8}$$

$$|t|<1,\ \lambda\in\mathbb{C},\ N\in\mathbb{N};$$

$$\begin{aligned}&(1-t)^{-\lambda}{}_{r}\Gamma_s(p,\alpha,\beta)\left[\begin{array}{r}(a_1,x,p,\alpha,\beta),a_2,\cdots,a_r;\\ b_1,\cdots,b_s;\end{array} zt^N\right]\\ &=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}{}_{r+N}\Gamma_{s+N}(p,\alpha,\beta)\left[\begin{array}{l}\Delta(N,-n),(a_1,x,p,\alpha,\beta),a_2,\cdots,a_r;\\ \Delta(N,1-\lambda-n),\qquad b_1,\cdots,b_s;\end{array} z\right]t^n,\end{aligned}\tag{3.9}$$

$$|t|<1,\ \lambda\in\mathbb{C},\ N\in\mathbb{N},$$

provided that each member of equations (3.4) *to* (3.9) *exists.*

Proof. The proof of Theorem (3.2) is similar to the Theorem (3.1) □

We remark that adding (3.1) and (3.2) and applying (2.9) leads to the result appearing in [7, Eq. 4.1], Similarly, adding (3.4) and (3.5) leads to [7, Eq. 4.3] and adding (3.6) and (3.7) leads to [7, Eq. 4.4] and adding (3.8) and (3.9) leads to [7, Eq. 4.5].

Theorem 3.3. *The following reduction formulas hold true for extended generalized incomplete hypergeometric functions:*

$$\begin{aligned} &{}_{r+1}\gamma_s(p,\alpha,\beta)\left[\begin{array}{r} (a_0,x,p,\alpha,\beta), b_1+m_1,\cdots,b_t+m_t,a_{t+1},\cdots,a_r;\\ b_1,\cdots,b_t,b_{t+1},\cdots,b_s; \end{array} z\right]\\ = &\sum_{j_1=0}^{m_1}\cdots\sum_{j_t=0}^{m_t}\wedge(j_1,\cdots,j_t)\,z^{J_t}\\ &\times{}_{r-t+1}\gamma_{s-t}(p,\alpha,\beta)\left[\begin{array}{r} (a_0+J_t,x,p,\alpha,\beta), a_{t+1}+J_t,\cdots,a_r+J_t;\\ b_{t+1}+J_t,\cdots,b_s+J_t; \end{array} z\right], \end{aligned} \tag{3.10}$$

$x \geq 0,\ t \leq \min(r,s),\ r,s \in \mathbb{N}_0,\ r < s$ *when* $z \in \mathbb{C}$, $r = s$ *when* $|z| < 1$,

and

$$\begin{aligned} &{}_{r+1}\Gamma_s(p,\alpha,\beta)\left[\begin{array}{r} (a_0,x,p,\alpha,\beta), b_1+m_1,\cdots,b_t+m_t,a_{t+1},\cdots,a_r;\\ b_1,\cdots,b_t,b_{t+1},\cdots,b_s; \end{array} z\right]\\ = &\sum_{j_1=0}^{m_1}\cdots\sum_{j_t=0}^{m_t}\wedge(j_1,\cdots,j_t)\,z^{J_t}\\ &\times{}_{r-t+1}\Gamma_{s-t}(p,\alpha,\beta)\left[\begin{array}{r} (a_0+J_t,x,p,\alpha,\beta), a_{t+1}+J_t,\cdots,a_r+J_t;\\ b_{t+1}+J_t,\cdots,b_s+J_t; \end{array} z\right], \end{aligned} \tag{3.11}$$

$x \geq 0,\ t \leq \min(r,s),\ r,s \in \mathbb{N}_0,\ r < s$ *when* $z \in \mathbb{C}$, $r = s$ *when* $|z| < 1$,

where, $J_t = j_1 + \cdots + j_t$ *and*

$$\wedge(j_1,\cdots,j_t) = \binom{m_1}{j_1}\cdots\binom{m_t}{j_t}\frac{(b_2+m_2)_{J_1}\cdots(b_t+m_t)_{J_{t-1}}(a_0)_{J_t}(a_{t+1})_{J_t}\cdots(a_r)_{J_t}}{(b_1)_{J_1}\cdots(b_t)_{J_t}(b_{t+1})_{J_t}\cdots(b_s)_{J_t}}.$$

Proof. The proof of (3.3) is based upon induction on $t \in \mathbb{N}$. We first prove (3.3) for $t = 1$, that is,

$$\begin{aligned} &{}_{r+1}\gamma_s(p,\alpha,\beta)\left[\begin{array}{r} (a_0,x,p,\alpha,\beta), b_1+m_1,a_2,\cdots,a_r;\\ b_1,\cdots,b_s; \end{array} z\right]\\ = &\sum_{j_1=0}^{m_1}\binom{m_1}{j_1}\frac{(a_0)_{j_1}(a_2)_{j_1}\cdots(a_r)_{j_1}}{(b_1)_{j_1}\cdots(b_s)_{j_1}}z^{j_1} \end{aligned}$$

$$\times_r\gamma_{s-1}(p,\alpha,\beta)\left[\begin{array}{r}(a_0+j_1,x,p,\alpha,\beta),a_2+j_1,\cdots,a_r+j_1;\\ b_2+j_1,\cdots,b_s+j_1;\end{array} z\right],$$

(3.12)

$$x\geq 0,\ m_1,r,s\in\mathbb{N}_0,\ r<s \text{ when } z\in\mathbb{C},\ r=s \text{ when } |z|<1.$$

Using the identity $\binom{m_1}{j_1}=\frac{(-1)^{j_1}(-m_1)_{j_1}}{j_1!}$, we write right hand side of (3.12) as

$$\sum_{j_1=0}^{m_1}\frac{(-1)^{j_1}(-m_1)_{j_1}(a_0)_{j_1}(a_2)_{j_1}\cdots(a_r)_{j_1}}{j_1!(b_1)_{j_1}\cdots(b_s)_{j_1}}z^{j_1}$$

$$\times\ {}_r\gamma_{s-1}(\alpha)\left[\begin{array}{r}(a_0+j_1,x,p,\alpha,\beta),a_2+j_1,\cdots,a_r+j_1;\\ b_2+j_1,\cdots,b_s+j_1;\end{array} z\right]$$

$$=\sum_{n=0}^{\infty}\sum_{j_1=0}^{m_1}\frac{(-1)^{j_1}(-m_1)_{j_1}(a_0)_{j_1}(a_2)_{n+j_1}\cdots(a_r)_{n+j_1}(a_0+j_1,x;p,\alpha,\beta)_n}{j_1!(b_1)_{j_1}(b_2)_{n+j_1}\cdots(b_s)_{n+j_1}}\frac{z^{n+j_1}}{n!}$$

$$=\sum_{n=0}^{\infty}\frac{(a_0,x;p,\alpha,\beta)_n(a_2)_n\cdots(a_r)_n}{(b_2)_n\cdots(b_s)_n}\frac{z^n}{n!}\sum_{j_1=0}^{min\{m_1,n\}}\frac{(-m_1)_{j_1}(-n)_{j_1}}{j_1!(b_1)_{j_1}}$$

(3.13)

$$=\sum_{n=0}^{\infty}\frac{(a_0,x;p,\alpha,\beta)_n(b_1+m_1)_n(a_2)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n}\frac{z^n}{n!},$$

using Chu-Vandermonde formula, [6]. This establishes (3.12). Inductively (3.10) is proved. The second reduction formula (3.11) can be proved in a similar manner. □

4. Conclusion

We remark that our results on extended generalized incomplete hypergeometric functions generalize the corresponding results on generalized incomplete hypergeometric functions when $\alpha=\beta$ and $p=0$, [10, 12, 13]. Further, for $\alpha=\beta$, $p=0$ and $x=0$, the results of extended generalized incomplete hypergeometric functions ${}_r\Gamma_s(p,\alpha,\beta)$ reduce to the generalized hypergeometric functions ${}_rF_s$, [3, 4, 8].

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