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Further Results on Chebyshev and Steffensen Inequalities

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Abstract. By making use of the Riemann-Liouville fractional integrals, we establish further results on Chebyshev inequality. Other Steffensen integral results of the weighted Chebyshev functional are also proved. Some classical results of the paper:[Steffensen's generalization of Chebyshev inequality. J. Math. Inequal., 9(1), (2015).] can be deduced as some special cases.

1. Introduction

Let us consider the functional, which is well known in the literature as Chebyshev functional, defined by [4]:

$$(1.1)\quad T(f,g):=\frac{1}{b-a}\left(\int_a^b f(x)g(x)dx\right)-\frac{1}{b-a}\left(\int_a^b f(x)dx\right)\frac{1}{b-a}\left(\int_a^b g(x)\right).$$

We know that if f and g are two monotonic functions having the same direction on $[a,b]$, the inequality $T(f,g)\geq 0$ is valid (see [4]).
In the case where $m\leq f\leq M$ and g is absolutely continuous, such that $g'\in L^\infty[a,b]$, it has been proved by A. M. Ostrowski [14] that:

$$|T(f,g)|\leq \frac{b-a}{8}(M-m)||g'||_\infty.$$

In 2014, P. Cerone and S. S. Dragomir [3] proved that if f and g are absolutely continuous on $[a,b]$; with $f',g'\in L^\infty[a,b]$, the inequality

$$T(f,g)\leq \frac{1}{12}||f'||_\infty||g'||_\infty(b-a)^2$$

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is valid.
Then in 2015, by considering the quantity:

$$T(f,g,p) := \int_a^b p(x) \int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx, \tag{1.2}$$

K. M. Awan et al. [1] proved that if ϕ is an absolutely continuous function on $[a,b]$ and p is a positive and integrable function on $[a,b]\,; (\phi')^2 \in L^1\,[a,b]$, the inequality

$$T(\phi,\phi,p) \le \frac{1}{P^2(b)} \int_a^b \tilde{P}(x)\left(\phi'\right)^2(x)dx \tag{1.3}$$

holds, with $P(x) := \int_a^x p(x)dx$ and $\tilde{P}(x) = P(x)\int_a^b tp(t)dt - P(b)\int_a^x tp(t)dt$.
In the literature, we find that the functionals $T(f,g)$ and $T(f,g,p)$ have attracted many researchers attention, for more details, we refer the interested reader to [2, 5, 6, 7, 9, 10, 12, 13, 15, 16, 17, 18].
The main aim of this work is to establish new integral inequalities for (1.1) and (1.2) by using the Riemann-Liouville fractional integration approach. We generalize some results related to the weighted Chebyshev functional. Other classes of the Chebyshev inequalities are also obtained as special cases. Our results have some relationships with those obtained in [1]. To prove our main results, we use some techniques that are more general than those derived in [9]. Some fractional results of this reference can be deduced as some special or equivalent cases while taking the variable x as a constant. Also, some other results in [1] are obtained as particular cases.

2. Preliminaries

We present the Riemann-Liouville integral definition and two of its properties. For more details, we refer the reader to [11].

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha \ge 0$, for a continuous function h on $[a,b]$ is defined as

$$\begin{gathered} J_a^\alpha h(t) = \tfrac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}h(\tau)d\tau; \quad \alpha>0, a<t\le b, \\ J_a^0 h(t) = h(t), \end{gathered} \tag{2.1}$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u}u^{\alpha-1}du$.

For $t=b$, we put:

$$J_a^\alpha h(b) = \frac{1}{\Gamma(\alpha)}\int_a^b (b-\tau)^{\alpha-1}h(\tau)d\tau.$$

We give the following property:

$$J_a^\alpha J^\beta h(t) = J_a^{\alpha+\beta}h(t), \alpha \ge 0, \beta \ge 0, \tag{2.2}$$

3. Main Results

To prove our main results, we need the following auxiliary result:

Lemma 3.1. *Let $f, g : [a,b] \to \mathbb{R}$ be two integrable functions, $p : [a,b] \to \mathbb{R}^+$ a positive integrable function and $g' \in L^1[a,b]$. Then for all $\alpha > 0, a < x \leq b$, we have*

$$\begin{aligned} &\frac{1}{J_a^\alpha p(x)} J_a^\alpha(pfg)(x) - \frac{1}{[J_a^\alpha p(x)]^2} J_a^\alpha(pf)(x) J_a^\alpha(pg)(x) \\ &= \frac{1}{[J_a^\alpha p(x)]^2} \int_a^x g'(t) \left\{ \int_a^t (x-y)^{\alpha-1} H_x(y) p(y) dy \right\} dt, \end{aligned} \tag{3.1}$$

with

$$H_x(y) = \frac{1}{\Gamma(\alpha)} \left[J_a^\alpha(pf)(x) - f(y) J_a^\alpha p(x) \right]. \tag{3.2}$$

Proof. Without considering the term $[J_a^\alpha p(x)]^{-2}$), we rewrite the right hand side of (3.1) as follows:

$$\begin{aligned} \int_a^x g'(t)\{\int_a^t (x-y)^{\alpha-1} H_x(y) p(y) dy\} dt &= g(t) \int_a^t (x-y)^{\alpha-1} H_x(y) p(y) dy|_{t=a}^{t=x} \\ &\quad - \int_a^x g(t)(x-t)^{\alpha-1} H_x(t) p(t) dt \\ &= g(x) \int_a^x (x-y)^{\alpha-1} H_x(y) p(y) dy \\ &\quad - \int_a^x g(t)(x-t)^{\alpha-1} H_x(t) p(t) dt. \end{aligned}$$

By using (3.2), we obtain:

$$\begin{aligned} \int_a^x g'(t)\{\int_a^t (x-y)^{\alpha-1} H_x(y) p(y) dy\} dt &= g(x) J_a^\alpha(pf)(x) \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} p(y) dy \\ &\quad - g(x) J_a^\alpha p(x) \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) p(y) dy \end{aligned}$$

$$
\begin{aligned}
&-J_a^\alpha(pf)(x)\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}g(t)p(t)dt\\
&+J_a^\alpha p(x)\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}g(t)f(t)p(t)dt\\
=\;& g(x)\Big[J_a^\alpha(pf)(x)J_a^\alpha p(x)-J_a^\alpha p(x)J_a^\alpha(pf)(x)\Big]\\
&+J_a^\alpha p(x)J_a^\alpha(pfg)(x)-J_a^\alpha(pf)(x)J_a^\alpha(pg)(x)\\
=\;& J_a^\alpha p(x)J_a^\alpha(pfg)(x)-J_a^\alpha(pf)(x)J_a^\alpha(pg)(x). \qquad (3.3)
\end{aligned}
$$

Multiplying both sides of (3.3) by $[J_a^\alpha p(x)]^{-2}$, we end the proof. □

Let us now prove the first main result:

Theorem 3.2. *Let $\phi:[a,b]\to\mathbb{R}$ be an absolutely continuous function, $p:[a,b]\to\mathbb{R}^+$ an integrable function and $(\phi')^2\in L^1[a,b]$. Then, for all $\alpha>0$ and $x\in]a,b]$, we have:*

$$\frac{1}{J_a^\alpha p(x)}J_a^\alpha(p\phi^2)(x)-\Big[\frac{1}{J_a^\alpha p(x)}J_a^\alpha(p\phi)(x)\Big]^2\leq\frac{1}{[J_a^\alpha p(x)]^2}\int_a^x\widetilde{P}_x(t)[\phi'(t)]^2dt, \qquad (3.4)$$

with

$$\widetilde{P}_x(t)=\frac{1}{\Gamma(\alpha)}\left[J_a^\alpha(xp(x))\int_a^t(x-y)^{\alpha-1}p(y)dy-J_a^\alpha p(x)\int_a^t(x-y)^{\alpha-1}yp(y)dy\right]. \qquad (3.5)$$

Proof. Thanks to Lemma 3.1, we have the following identity:

$$
\begin{aligned}
&\frac{1}{J_a^\alpha p(x)}J_a^\alpha[x(pg)(x)]-\frac{1}{J_a^\alpha p(x)}J_a^\alpha[xp(x)]\frac{1}{J_a^\alpha p(x)}J_a^\alpha(pg)(x)\\
=\;&\frac{1}{[J_a^\alpha p(x)]^2}\int_a^x g'(t)\{\int_a^t(x-y)^{\alpha-1}H_x^1(y)p(y)dy\}dt, \qquad (3.6)
\end{aligned}
$$

where

$$H_x^1(y):=\frac{1}{\Gamma(\alpha)}\Big[J_a^\alpha(xp(x))-yJ_a^\alpha p(x)\Big]. \qquad (3.7)$$

Replacing H_x^1 by its corresponding quantity in (3.6), we can write

$$
\begin{aligned}
&\frac{1}{J_a^\alpha p(x)} J_a^\alpha[x(pg)(x)] - \frac{1}{J_a^\alpha p(x)} J_a^\alpha[xp(x)] \frac{1}{J_a^\alpha p(x)} J_a^\alpha(pg)(x) \\
= \quad & \frac{1}{[J_a^\alpha p(x)]^2} \int_a^x g'(t)\{\int_a^t (x-y)^{\alpha-1} \frac{1}{\Gamma(\alpha)}\Big[J_a^\alpha(xp(x)) - yJ_a^\alpha p(x)\Big] p(y)dy\}dt \\
= \quad & \frac{1}{[J_a^\alpha p(x)]^2} \int_a^x g'(t) \times \\
& \frac{1}{\Gamma(\alpha)}\Big[J_a^\alpha(xp(x)) \int_a^t (x-y)^{\alpha-1}p(y)dy - J_a^\alpha p(x) \int_a^t (x-y)^{\alpha-1} yp(y)dy\Big]dt \\
(3.8) = \quad & \frac{1}{[J_a^\alpha p(x)]^2} \int_a^x g'(t)\widetilde{P}_x(t)dt.
\end{aligned}
$$

By the fractional Korkine identity, (as it is established in [8]), we obtain:

$$
\begin{aligned}
&\frac{1}{J_a^\alpha p(x)} J_a^\alpha(p\phi^2)(x) - \Big[\frac{1}{J_a^\alpha p(x)} J_a^\alpha(p\phi)(x)\Big]^2 \\
= \quad & \frac{1}{2[\Gamma(\alpha)J_a^\alpha p(x)]^2} \int_a^x \int_a^x (x-t)^{\alpha-1}(x-s)^{\alpha-1}[\phi(t)-\phi(s)]^2 dtds.
\end{aligned}
$$

On the other hand, we have

$$
\phi(t) - \phi(s) = \int_s^t \phi'(u)du.
$$

Therefore, we can write

$$
\Big[\int_s^t \phi'(u)du\Big]^2 \quad \leq \quad (t-s)\Big[\int_a^t (\phi'(u))^2du - \int_a^s (\phi'(u))^2du\Big].
$$

Defining $\Psi(v) := \int_a^v (\phi'(u))^2du$, it yields that:

$$
\begin{aligned}
&\frac{1}{J_a^\alpha p(x)} J_a^\alpha(p\phi^2)(x) - \Big[\frac{1}{J_a^\alpha p(x)} J_a^\alpha(p\phi)(x)\Big]^2 \\
\leq \quad & \frac{1}{2[\Gamma(\alpha)J_a^\alpha p(x)]^2} \int_a^x \int_a^x (x-t)^{\alpha-1}(x-s)^{\alpha-1}(t-s)(\Psi(t)-\Psi(s))dtds \\
(3.9) \qquad = \quad & \frac{1}{J_a^\alpha p(x)} J_a^\alpha[x(p\Psi)(x)] - \frac{1}{J_a^\alpha p(x)} J_a^\alpha[xp(x)] \frac{1}{J_a^\alpha p(x)} J_a^\alpha(p\Psi)(x).
\end{aligned}
$$

Thanks to (3.8) and (3.9), we obtain

$$\frac{1}{J_a^\alpha p(x)} J_a^\alpha(p\phi^2)(x) - \left[\frac{1}{J_a^\alpha p(x)} J_a^\alpha(p\phi)(x)\right]^2 \leq \frac{1}{[J_a^\alpha p(x)]^2} \int_a^x \Psi'(t)\widetilde{P}_x(t)dt. \tag{3.10}$$

Replacing $\Psi'(t)$ by $(\phi'(t))^2$ in the above inequality, we end the proof. □

Remark 3.3. 1 : The constant 1 is the best possible in (3.4).
2 : If we take $x = b$ and $\alpha = 1$ in Theorem 3.2, we get Lemma 2.1 of [1].

Corollary 3.4. *If $\phi : [a, b] \to \mathbb{R}$ is a continuous function and $(\phi')^2 \in L^1[a, b]$, then we have*

$$\frac{1}{J_a^\alpha 1} J_a^\alpha \phi^2(x) - \left[\frac{1}{J_a^\alpha 1} J_a^\alpha \phi(x)\right]^2 \leq \frac{1}{(\alpha+1)\Gamma(\alpha)(J_a^\alpha 1)} \int_a^x (x-t)^\alpha (t-a)\left[\phi'(t)\right]^2 dt, \tag{3.11}$$

where $\alpha > 0, a < x \leq b$.

Proof. Taking $p(t) = 1$, in Theorem 3.2, we observe that

$$\widetilde{P}_x(t) = \frac{1}{\Gamma(\alpha)}\left[J_a^\alpha x \int_a^t (x-y)^{\alpha-1} dy - J_a^\alpha 1 \int_a^t (x-y)^{\alpha-1} y dy\right].$$

Taking into account that the following quantities

$$\begin{aligned}
J_a^\alpha x &= \frac{(\alpha a + x)(x-a)^\alpha}{\Gamma(\alpha+2)}, \\
\int_a^t (x-y)^{\alpha-1} dy &= \frac{1}{\alpha}\left[-(x-t)^\alpha + (x-a)^\alpha\right], \\
J_a^\alpha 1 &= \frac{(x-a)^\alpha}{\Gamma(\alpha+1)}, \\
\int_a^t (x-y)^{\alpha-1} y dy &= \frac{1}{\alpha(\alpha+1)}\left[-(x-t)^\alpha(\alpha t + x) + (x-a)^\alpha(\alpha a + x)\right],
\end{aligned}$$

we conclude that

$$\begin{aligned}
\widetilde{P}_x(t) &= \frac{1}{\alpha\Gamma(\alpha)\Gamma(\alpha+2)}\left\{-(\alpha a + x)(x-a)^\alpha(x-t)^\alpha + (x-a)^\alpha(x-t)^\alpha(\alpha t + x)\right\} \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\alpha+2)}(x-a)^\alpha(x-t)^\alpha(t-a) \\
&= \frac{J_a^\alpha 1}{(\alpha+1)\Gamma(\alpha)}(x-t)^\alpha(t-a).
\end{aligned}$$

This ends the proof. □

Remark 3.5. 1 : The constant $\frac{1}{\alpha+1}$ is the best possible in (3.11).
2 : If we take $x = b$ and $\alpha = 1$ in Corollary 3.4, we get Corollary 2.2 of [1].

Let us now prove the following result:

Theorem 3.6. *Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be two continuous functions, $(f')^2, (g')^2 \in L^1[a, b]$. If $p : [a, b] \longrightarrow \mathbb{R}^+$ is an integrable function, then for all $\alpha > 0$, $x \in]a, b]$, we have*

$$\frac{1}{J_a^\alpha p(x)} J_a^\alpha (pfg)(x) - \frac{1}{[J_a^\alpha p(x)]^2} J_a^\alpha (pf)(x) J_a^\alpha (pg)(x)$$

$$\leq \frac{1}{[J_a^\alpha p(x)]^2} \left(\int_a^x \widetilde{P}_x(t) \left[f'(t)\right]^2 dt \right)^{\frac{1}{2}} \left(\int_a^x \widetilde{P}_x(t) \left[g'(t)\right]^2 dt \right)^{\frac{1}{2}}, \tag{3.12}$$

where $\widetilde{P}_x$ is defined by (3.5).

Proof. By the fractional Korkine identity [8], we have

$$\frac{1}{J_a^\alpha p(x)} J_a^\alpha (pfg)(x) - \frac{1}{[J_a^\alpha p(x)]^2} J_a^\alpha (pf)(x) J_a^\alpha (pg)(x)$$

$$= \frac{1}{2[\Gamma(\alpha) J_a^\alpha p(x)]^2} \int_a^x \int_a^x (x-t)^{\alpha-1}(x-s)^{\alpha-1} p(t)p(s)[f(t)-f(s)]\,[g(t)-g(s)]\,dtds$$

$$= \frac{1}{2[\Gamma(\alpha) J_a^\alpha p(x)]^2} \int_a^x \int_a^x \left\{ \left[(x-t)^{\alpha-1}(x-s)^{\alpha-1} p(t)p(s)\right]^{\frac{1}{2}} (f(t)-f(s)) \right\} \times$$

$$\left\{ \left[(x-t)^{\alpha-1}(x-s)^{\alpha-1} p(t)p(s)\right]^{\frac{1}{2}} (g(t)-g(s)) \right\} dtds. \tag{3.13}$$

Then, using Cauchy Schwarz inequality for (3.13), we can write

$$\frac{1}{J_a^\alpha p(x)} J_a^\alpha (pfg)(x) - \frac{1}{[J_a^\alpha p(x)]^2} J_a^\alpha (pf)(x) J_a^\alpha (pg)(x)$$

$$\leq \frac{1}{2[\Gamma(\alpha) J_a^\alpha p(x)]^2} \left(\int_a^x \int_a^x \left[(x-t)^{\alpha-1}(x-s)^{\alpha-1} p(t)p(s)\right] (f(t)-f(s))^2 dtds \right)^{\frac{1}{2}} \times$$

$$\left(\int_a^x \int_a^x \left[(x-t)^{\alpha-1}(x-s)^{\alpha-1} p(t)p(s)\right] (g(t)-g(s))^2 dtds \right)^{\frac{1}{2}}.$$

$$= \left(\frac{1}{J_a^\alpha p(x)} J_a^\alpha (pf^2)(x) - \left[\frac{1}{J_a^\alpha p(x)} J_a^\alpha (pf)(x) \right]^2 \right)^{\frac{1}{2}} \times$$

$$\left(\frac{1}{J_a^\alpha p(x)} J_a^\alpha (pg^2)(x) - \left[\frac{1}{J_a^\alpha p(x)} J_a^\alpha (pg)(x) \right]^2 \right)^{\frac{1}{2}}.$$

We use (3.4) for

$$\left(\frac{1}{J_a^\alpha p(x)}J_a^\alpha(pf^2)(x)-\left[\frac{1}{J_a^\alpha p(x)}J_a^\alpha(pf)(x)\right]^2\right)$$

and

$$\left(\frac{1}{J_a^\alpha p(x)}J_a^\alpha(pg^2)(x)-\left[\frac{1}{J_a^\alpha p(x)}J_a^\alpha(pg)(x)\right]^2\right),$$

we obtain (3.12). □

Remark 3.7. If we take $x=b$ and $\alpha=1$ in Theorem 3.6, we get Theorem 2.3 of [1].

Corollary 3.8. *Let $f,g:[a,b]\longrightarrow\mathbb{R}$ be two absolutely continuous functions on $[a,b]$ and $(f')^2,(g')^2\in L^1[a,b]$. Then for all $\alpha>0$ and $a<x\leq b$, we have:*

$$\begin{aligned}&\frac{1}{J_a^\alpha 1}J_a^\alpha(fg)(x)-\frac{1}{[J_a^\alpha 1]^2}J_a^\alpha f(x)J_a^\alpha g(x)\\ \leq\ &\frac{\alpha}{(\alpha+1)J_a^\alpha 1}\sqrt{J_a^{\alpha+1}\left[(x-a)\left(f'(x)\right)^2\right]J_a^{\alpha+1}\left[(x-a)\left(g'(x)\right)^2\right]}\end{aligned}\tag{3.14}$$

Proof. Thanks to (3.12) and taking $p(t)=1$, we obtain

$$\begin{aligned}&\frac{1}{J_a^\alpha 1}J_a^\alpha(fg)(x)-\frac{1}{[J_a^\alpha 1]^2}J_a^\alpha f(x)J_a^\alpha g(x)\\ \leq\ &\sqrt{\left(\frac{1}{J_a^\alpha 1}J_a^\alpha f^2(x)-\left[\frac{1}{J_a^\alpha 1}J_a^\alpha f(x)\right]^2\right)\left(\frac{1}{J_a^\alpha 1}J_a^\alpha g^2(x)-\left[\frac{1}{J_a^\alpha 1}J_a^\alpha g(x)\right]^2\right)}.\end{aligned}\tag{3.15}$$

In (3.15), applying Corollary 3.4 for

$$\frac{1}{J_a^\alpha 1}J_a^\alpha f^2(x)-\left[\frac{1}{J_a^\alpha 1}J_a^\alpha f(x)\right]^2$$

and

$$\frac{1}{J_a^\alpha 1}J_a^\alpha g^2(x)-\left[\frac{1}{J_a^\alpha 1}J_a^\alpha g(x)\right]^2,$$

we can write

$$
\begin{aligned}
&\frac{1}{J_a^{\alpha}1}J_a^{\alpha}(fg)(x)-\frac{1}{[J_a^{\alpha}1]^2}J_a^{\alpha}f(x)J_a^{\alpha}g(x)\\
\leq\ &\sqrt{\frac{1}{(\alpha+1)\Gamma(\alpha)(J_a^{\alpha}1)}\int_a^x (x-t)^{\alpha}(t-a)\left[f'(t)\right]^2 dt}\ \times\\
&\sqrt{\frac{1}{(\alpha+1)\Gamma(\alpha)(J_a^{\alpha}1)}\int_a^x (x-t)^{\alpha}(t-a)\left[g'(t)\right]^2 dt}\\
=\ &\sqrt{\frac{\alpha}{(\alpha+1)\Gamma(\alpha+1)(J_a^{\alpha}1)}\int_a^x (x-t)^{\alpha}(t-a)\left[f'(t)\right]^2 dt}\ \times\\
&\sqrt{\frac{\alpha}{(\alpha+1)\Gamma(\alpha+1)(J_a^{\alpha}1)}\int_a^x (x-t)^{\alpha}(t-a)\left[g'(t)\right]^2 dt}\\
=\ &\frac{\alpha}{(\alpha+1)J_a^{\alpha}1}\sqrt{J_a^{\alpha+1}\left[(x-a)\left(f'(x)\right)^2\right].J_a^{\alpha+1}\left[(x-a)\left(g'(x)\right)^2\right]}.
\end{aligned}
$$

The proof is thus achieved. □

Remark 3.8. If we take $x=b$ and $\alpha=1$ in Corollary 3.8, we get Corollary 2.4 of [1].

We also present to the reader the following theorem:

Theorem 3.9. *Let $g:[a,b]\longrightarrow\mathbb{R}$ be an increasing function, $p:[a,b]\longrightarrow\mathbb{R}^+$ be an integrable function and f be a differentiable function, with $f'\in L^{\infty}[a,b]$. Then, for all $\alpha>0$ and $a<x\leq b$, we have*

$$
\begin{aligned}
&\frac{1}{J_a^{\alpha}p(x)}J_a^{\alpha}(pfg)(x)-\frac{1}{[J_a^{\alpha}p(x)]^2}J_a^{\alpha}(pf)(x)J_a^{\alpha}(pg)(x)\\
\leq\ &\frac{\parallel f'\parallel_{\infty}}{\left(J_a^{\alpha}p(x)\right)^2}\int_a^x g'(t)\widetilde{P}_x(t)dt,
\end{aligned}
\tag{3.16}
$$

where $\widetilde{P}_x$ is defined by (3.7).

Proof. We have

$$\begin{aligned}
&\left|\frac{1}{J_a^\alpha p(x)}J_a^\alpha(pfg)(x)-\frac{1}{[J_a^\alpha p(x)]^2}J_a^\alpha(pf)(x)J_a^\alpha(pg)(x)\right|\\
=\;&\frac{1}{2[\Gamma(\alpha)J_a^\alpha p(x)]^2}\left|\int_a^x\int_a^x(x-u)^{\alpha-1}(x-v)^{\alpha-1}p(u)p(v)[f(u)-f(v)][g(u)-g(v)]dudv\right|\\
\leq\;&\frac{1}{2[\Gamma(\alpha)J_a^\alpha p(x)]^2}\int_a^x\int_a^x(x-u)^{\alpha-1}(x-v)^{\alpha-1}p(u)p(v)\left|f(u)-f(v)\right|\left|g(u)-g(v)\right|dudv.
\end{aligned}$$

We have also

$$\left|f(u)-f(v)\right|\leq\left\|f'\right\|_\infty\left|u-v\right|.$$

Then, it yields that

$$\begin{aligned}
&\left|\frac{1}{J_a^\alpha p(x)}J_a^\alpha(pfg)(x)-\frac{1}{[J_a^\alpha p(x)]^2}J_a^\alpha(pf)(x)J_a^\alpha(pg)(x)\right|\\
\leq\;&\frac{\|f'\|_\infty}{2[\Gamma(\alpha)J_a^\alpha p(x)]^2}\int_a^x\int_a^x(x-u)^{\alpha-1}(x-v)^{\alpha-1}p(u)p(v)\left|u-v\right|\left|g(u)-g(v)\right|dudv.
\end{aligned}$$

Since g is an increasing function, then we can write

$$\forall u,v\in[a,b],(u-v)(g(u)-g(v))\geq 0.$$

Hence,

$$\begin{aligned}
&\left|\frac{1}{J_a^\alpha p(x)}J_a^\alpha(pfg)(x)-\frac{1}{[J_a^\alpha p(x)]^2}J_a^\alpha(pf)(x)J_a^\alpha(pg)(x)\right|\\
\leq\;&\frac{\|f'\|_\infty}{2[\Gamma(\alpha)J_a^\alpha p(x)]^2}\int_a^x\int_a^x(x-u)^{\alpha-1}(x-v)^{\alpha-1}p(u)p(v)(u-v)(g(u)-g(v))dudv\\
=\;&\|f'\|_\infty\left[\frac{1}{J_a^\alpha p(x)}J_a^\alpha[x(pg)(x)]-\frac{1}{[J_a^\alpha p(x)]^2}J_a^\alpha[xp(x)]J_a^\alpha(pg)(x)\right]
\end{aligned}$$

Using (3.8) for

$$\frac{1}{J_a^\alpha p(x)}J_a^\alpha[x(pg)(x)]-\frac{1}{[J_a^\alpha p(x)]^2}J_a^\alpha[xp(x)]J_a^\alpha(pg)(x),$$

we obtain (3.16). □

Remark 3.10. If we take $x=b$ and $\alpha=1$ in Theorem 3.9, we get Theorem 2.5 of [1].

Corollary 3.11. *Let $g : [a,b] \longrightarrow \mathbb{R}$ be an increasing function on $[a,b]$ and $f : [a,b] \longrightarrow \mathbb{R}$ be a differentiable function, with $f' \in L^{\infty}[a,b]$. Then, for all $\alpha > 0$, $x \in]a,b]$, we have*

$$\frac{1}{J_a^\alpha 1} J_a^\alpha (fg)(x) - \frac{1}{[J_a^\alpha 1]^2} J_a^\alpha f(x) J_a^\alpha g(x) \leq \frac{\alpha \left\| f' \right\|_\infty}{(\alpha+1)\, J_a^\alpha 1} J^{\alpha+1} \left[(x-a)\, g'(x) \right] \tag{3.17}$$

Proof. Taking $p(t) = 1$ in the expression of $\widetilde{P}_x$, we have

$$\widetilde{P}_x(t) = \frac{J_a^\alpha 1}{(\alpha+1)\Gamma(\alpha)} (x-t)^\alpha (t-a).$$

Hence, the inequality (3.16) can be written as:

$$\begin{aligned} & \frac{1}{J_a^\alpha 1} J_a^\alpha (fg)(x) - \frac{1}{[J_a^\alpha 1]^2} J_a^\alpha f(x) J_a^\alpha g(x) \\ \leq \; & \frac{\left\| f' \right\|_\infty}{(\alpha+1)\Gamma(\alpha) J_a^\alpha 1} \int_a^x (x-t)^\alpha (t-a) g'(t) dt \\ = \; & \frac{\alpha \left\| f' \right\|_\infty}{(\alpha+1)(J_a^\alpha 1)\Gamma(\alpha+1)} \int_a^x (x-t)^\alpha (t-a) g'(t) dt \\ = \; & \frac{\alpha \left\| f' \right\|_\infty}{(\alpha+1)\, J_a^\alpha 1} J^{\alpha+1} \left[(x-a)\, g'(x) \right]. \end{aligned}$$

□

Remark 3.12. If we take $x = b$ and $\alpha = 1$ in Corollary 3.11, we get Corollary 2.6 of [1].

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