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The Geometry of $\mathcal{L}(^2 l_\infty^2)$

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Abstract. We classify the extreme, exposed and smooth bilinear forms of the unit ball of the space of bilinear forms on l_∞^2.

1. Introduction

We write B_E for the closed unit ball of a real Banach space E and the dual space of E is denoted by E^*. An element $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y + z)$ implies $x = y = z$. An element $x \in B_E$ is called an *exposed point* of B_E if there is a $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. An element $x \in B_E$ is called a *smooth point* of B_E if there is a unique $f \in E^*$ so that $f(x) = 1 = \|f\|$. It is easy to see that every exposed point of B_E is an extreme point. We denote by $extB_E, expB_E$ and smB_E the sets of extreme, exposed and smooth points of B_E, respectively. A mapping $P : E \to \mathbb{R}$ is a continuous 2-homogeneous polynomial if there exists a continuous bilinear form L on the product $E \times E$ such that $P(x) = L(x, x)$ for every $x \in E$. We denote by $\mathcal{L}(^2E)$ the Banach space of all continuous bilinear forms on E endowed with the norm $\|L\| = \sup_{\|x\|=\|y\|=1} |L(x, y)|$. $\mathcal{P}(^2E)$ denotes the Banach space of all continuous 2-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [8].

In 1998, Choi *et al.* ([3], [4]) characterized the extreme points of the unit ball of $\mathcal{P}(^2 l_1^2)$ and $\mathcal{P}(^2 l_2^2)$. In 2007, the author [11] classified the exposed 2-homogeneous polynomials on $\mathcal{P}(^2 l_p^2)$ $(1 \le p \le \infty)$. Recently, the author ([14], [16], [20]) classified the extreme, exposed and smooth points of the unit ball of $\mathcal{P}(^2 d_*(1, w)^2)$, where $d_*(1, w)^2 = \mathbb{R}^2$ with the octagonal norm of weight w.

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In 2009, the author [13] classified the extreme, exposed and smooth points of the unit ball of $\mathcal{L}_s({}^2 l_\infty^2)$. Recently, the author ([15], [17]–[19]) classified the extreme, exposed and smooth points of the unit balls of $\mathcal{L}_s({}^2 d_*(1,w)^2)$ and $\mathcal{L}({}^2 d_*(1,w)^2)$.

We refer to ([1]–[7], [9]–[25] and references therein) for some recent work about extremal properties of multilinear mappings and homogeneous polynomials on some classical Banach spaces. In this paper, we classify the extreme, exposed and smooth bilinear forms of the unit ball of $\mathcal{L}({}^2 l_\infty^2)$.

2. The extreme points of the unit ball of $\mathcal{L}({}^2 l_\infty^2)$

Let $T((x_1,y_1),\ (x_2,y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}({}^2 l_\infty^2)$ for some real numbers a,b,c,d. For simplicity we will write $T((x_1,y_1),\ (x_2,y_2)) = (a,b,c,d)$. Let

$$
\begin{aligned}
T_1((x_1,y_1),\ (x_2,y_2)) &:= T((y_1,x_1),\ (y_2,x_2)) = (b,a,d,c),\\
T_2((x_1,y_1),\ (x_2,y_2)) &:= T((x_1,y_1),\ (x_2,-y_2)) = (a,-b,-c,d),\\
T_3((x_1,y_1),\ (x_2,y_2)) &:= T((x_1,y_1),\ (y_2,x_2)) = (c,d,a,b)\\
T_4((x_1,y_1),\ (x_2,y_2)) &:= T((y_2,x_2),\ (x_1,y_1)) = (c,d,b,a).
\end{aligned}
$$

Then $\|T_i\| = \|T\|$ $(i = 1,\ldots,4)$.

Theorem 2.1. *Let $T((x_1,y_1),\ (x_2,y_2)) = (a,b,c,d) \in \mathcal{L}({}^2 l_\infty^2)$. Then,*

$$
\|T\| = \max\{|a+b| + |c+d|, |a-b| + |c-d|\}.
$$

Proof. Since $\{(1,1),(1,-1),(-1,1),(-1,-1)\}$ is the set of all extreme points of the unit ball of l_∞^2 and T is bilinear,

$$
\begin{aligned}
\|T\| &= \max\{|T((1,1),(1,1))|, |T((1,-1),(-1,1))|,\\
&\qquad |T((1,1),(1,-1))|, |T((-1,1),(1,1))|\}\\
&= \max\{|a+b| + |c+d|, |a-b| + |c-d|\}.
\end{aligned}
$$

□

Note that if $\|T\| = 1$, then $|a| \le 1, |b| \le 1, |c| \le 1$ and $|d| \le 1$.

Theorem 2.2. *Let $T((x_1,y_1),\ (x_2,y_2)) = (a,b,c,d) \in \mathcal{L}({}^2 l_\infty^2)$. Then, the followings are equivalent:*

(1) *T is extreme;*

(2) *(b,a,d,c) is extreme;*

(3) *$(a,-b,-c,d)$ is extreme;*

(4) *(c,d,a,b) is extreme;*

(5) *(c,d,b,a) is extreme.*

Proof. It follows from Theorem 2.1 and the above remark of Theorem 2.1. □

Let

$$\begin{aligned} Norm(T) &= \{((x_1,y_1),(x_2,y_2)) \in \{((1,1),\ (1,1)),((1,-1),\ (1,-1)),((1,1),(1,-1)),\\ &\quad ((1,-1),\ (1,1))\} : |T((x_1,y_1),(x_2,y_2))| = \|T\|\}. \end{aligned}$$

We call $Norm(T)$ *the norming set* of T.

Theorem 2.3. *Let $T \in \mathcal{L}(^2 l^2_\infty)$ with $\|T\| = 1$. Then, $T \in extB_{\mathcal{L}(^2 l^2_\infty)}$ if and only if*

$$Norm(T) = \{((1,1),(1,1)),((1,-1),(1,-1)),((1,1),(1,-1)),((1,-1),(1,1))\}.$$

Proof. Suppose that $T \in \mathcal{L}(^2 l^2_\infty)$ with $\|T\| = 1$. By Theorem 2.2, we may assume that $a \geq |b|$ and $c \geq |d|$.

($\Leftarrow$): Obviously,

$$1 = |T((1,1),(1,1))| = |T((1,-1),(1,-1))| = |T((1,1),(1,-1))| = |T((1,-1),(1,1))|.$$

Hence, $1 = |a+b+c+d| = |a+b-c-d| = |a-b-c+d| = |a-b+c-d|$, so $1 = a+b+c+d = a-b+c-d$, so $c = 1-a, d = -b$. Since $1 = |a+b-c-d| = |a-b-c+d|$, $1 = |(2a-1)+2b| = |(2a-1)-2b|$. Hence, $(2a-1)2b = 0$, so $a = \frac{1}{2}$ or $b = 0$. If $a = \frac{1}{2}$, then $b = \pm\frac{1}{2}$ and $T = (\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$ or $T = (\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Note that $(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$ and $T = (\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$ are extreme. If $b = 0$, then $a = 0$ or $a = 1$, so $T = (0,0,1,0)$ or $T = (1,0,0,0)$. Note that $(0,0,1,0)$ and $(1,0,0,0)$ are extreme. By Theorem 2.2, $(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}),(0,1,0,0),(0,0,0,1)$ are extreme.

($\Rightarrow$): Suppose that $T \in extB_{\mathcal{L}(^2 l^2_\infty)}$ and the necessary condition is not true. Then

$$\begin{aligned} Norm(T) &\subset \{((1,1),(1,1)),((1,-1),(1,-1)),((1,1),(1,-1))\} \text{ or}\\ Norm(T) &\subset \{((1,1),(1,1)),((1,-1),(1,-1)),((1,-1),(1,1))\} \text{ or}\\ Norm(T) &\subset \{((1,1),(1,1)),((1,1),(1,-1)),((1,-1),(1,1))\} \text{ or}\\ Norm(T) &\subset \{((1,-1),(1,-1)),((1,1),(1,-1)),((1,-1),(1,1))\}. \end{aligned}$$

Without loss of generality we may assume that

$$\begin{aligned} Norm(T) &= \{((1,1),(1,1)),((1,-1),(1,-1)),((1,1),(1,-1))\} \text{ or}\\ Norm(T) &= \{((1,1),(1,1)),((1,-1),(1,-1)),((1,-1),(1,1))\} \text{ or}\\ Norm(T) &= \{((1,1),(1,1)),((1,1),(1,-1)),((1,-1),(1,1))\} \text{ or}\\ Norm(T) &= \{((1,-1),(1,-1)),((1,1),(1,-1)),((1,-1),(1,1))\}. \end{aligned}$$

Case 1: $Norm(T) = \{((1,1),(1,1)),((1,-1),(1,-1)),((1,1),(1,-1))\}$

Let $T_1 = (a-\frac{1}{n}, b+\frac{1}{n}, c-\frac{1}{n}, d+\frac{1}{n})$ and $T_2 = (a+\frac{1}{n}, b-\frac{1}{n}, c+\frac{1}{n}, d-\frac{1}{n})$ for a sufficiently large $n \in \mathbb{N}$ so that $\|T_i\| = 1$ for $i = 1,2$. Therefore, T is not extreme, which is a contradiction.

Case 2: $Norm(T) = \{((1,1),(1,1)), ((1,-1),(1,-1)), ((1,-1),(1,1))\}$

Let $T_1 = (a+\frac{1}{n}, b-\frac{1}{n}, c-\frac{1}{n}, d+\frac{1}{n})$ and $T_2 = (a-\frac{1}{n}, b+\frac{1}{n}, c+\frac{1}{n}, d-\frac{1}{n})$ for a sufficiently large $n \in \mathbb{N}$ so that $\|T_i\| = 1$ for $i = 1, 2$. Therefore, T is not extreme, which is a contradiction.

Case 3: $Norm(T) = \{((1,1),(1,1)), ((1,1),(1,-1)), ((1,-1),(1,1))\}$

Let $T_1 = (a-\frac{1}{n}, b-\frac{1}{n}, c+\frac{1}{n}, d+\frac{1}{n})$ and $T_2 = (a+\frac{1}{n}, b+\frac{1}{n}, c-\frac{1}{n}, d-\frac{1}{n})$ for a sufficiently large $n \in \mathbb{N}$ so that $\|T_i\| = 1$ for $i = 1, 2$. Therefore, T is not extreme, which is a contradiction.

Case 4: $Norm(T) = \{((1,-1),(1,-1)), ((1,1),(1,-1)), ((1,-1),(1,1))\}$

Let $T_1 = (a+\frac{1}{n}, b+\frac{1}{n}, c+\frac{1}{n}, d+\frac{1}{n})$ and $T_2 = (a-\frac{1}{n}, b-\frac{1}{n}, c-\frac{1}{n}, d-\frac{1}{n})$ for a sufficiently large $n \in \mathbb{N}$ so that $\|T_i\| = 1$ for $i = 1, 2$. Therefore, T is not extreme, which is a contradiction. □

Theorem 2.4.

$$\begin{aligned} extB_{\mathcal{L}(^2l_\infty^2)} &= \{\pm(1,0,0,0), \pm(0,1,0,0), \pm(0,0,1,0), \pm(0,0,0,0,1), \\ &\pm\frac{1}{2}(-1,1,1,1), \pm\frac{1}{2}(1,-1,1,1), \pm\frac{1}{2}(1,1,-1,1), \pm\frac{1}{2}(1,1,1,-1)\}. \end{aligned}$$

Proof. It follows from the proof of Theorem 2.3. □

3. The exposed points of the unit ball of $\mathcal{L}(^2l_\infty^2)$

Theorem 3.1. *Let $f \in \mathcal{L}(^2l_\infty^2)^*$ and $\alpha = f(x_1x_2), \beta = f(y_1y_2), \delta = f(x_1y_2), \gamma = f(x_2y_1)$. Then,*

$$\|f\| = \max\{|\alpha|, |\beta|, |\delta|, |\gamma|, \frac{1}{2}|\alpha-\beta+\delta+\gamma|, \frac{1}{2}|\alpha+\beta-\delta+\gamma|, \frac{1}{2}|\alpha+\beta+\delta-\gamma|\}.$$

Proof. It follows from Theorem 2.3 and the fact that

$$\|f\| = \max_{T \in extB_{\mathcal{L}(^2l_\infty^2)}} |f(T)|.$$

□

Note that if $\|f\| = 1$, then $|\alpha| \le 1, |\beta| \le 1, |\delta| \le 1, |\gamma| \le 1$.

Theorem 3.2 ([18, Theorem 2.3]). *Let E be a real Banach space such that $extB_E$ is finite. Suppose that $x \in extB_E$ satisfies that there exists an $f \in E^*$ with $f(x) = 1 = \|f\|$ and $|f(y)| < 1$ for every $y \in extB_E \backslash \{\pm x\}$. Then, $x \in expB_E$.*

Theorem 3.3. *Let $T((x_1,y_1), (x_2,y_2)) = (a,b,c,d) \in \mathcal{L}(^2l_\infty^2)$. Then, the followings are equivalent:*

(1) *T is exposed;*

(2) *(b,a,d,c) is exposed;*

(3) $(a, -b, -c, d)$ *is exposed;*

(4) (c, d, a, b) *is exposed;*

(5) (c, d, b, a) *is exposed.*

Proof. It follows from Theorem 2.1 and the above remark of Theorem 2.1. $\square$

Now we are in position to describe all the exposed points of the unit ball of $\mathcal{L}(^2 l_\infty^2)$.

Theorem 3.4. $expB_{\mathcal{L}(^2 l_\infty^2)} = extB_{\mathcal{L}(^2 l_\infty^2)}$.

Proof. It is enough to show that $extB_{\mathcal{L}(^2 l_\infty^2)} \subset expB_{\mathcal{L}(^2 l_\infty^2)}$.

Claim: $T = (1, 0, 0, 0)$ is exposed.

Let $f \in \mathcal{L}(^2 l_\infty^2)^*$ be such that $\alpha = 1, 0 = \beta = \delta = \gamma$. Then $f(T) = 1, |f(S)| < 1$ for every $S \in extB_{\mathcal{L}(^2 l_\infty^2)} \backslash \{\pm T\}$. By Theorem 3.2, T is exposed. By Theorem 3.3, $(0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$ are exposed.

Claim: $T = \frac{1}{2}(-1, 1, 1, 1)$ is exposed.

Let $f \in \mathcal{L}(^2 l_\infty^2)^*$ be such that $\alpha = 1, 0 = \beta, \delta = \gamma = \frac{1}{2}$. Then $f(T) = 1, |f(S)| < 1$ for every $S \in extB_{\mathcal{L}(^2 l_\infty^2)} \backslash \{\pm T\}$. By Theorem 3.2, T is exposed. By Theorem 3.3, $\frac{1}{2}(1, -1, 1, 1), \frac{1}{2}(1, 1, -1, 1), \frac{1}{2}(1, 1, 1, -1)$ are exposed. $\square$

4. The smooth points of the unit ball of $\mathcal{L}(^2 l_\infty^2)$

Theorem 4.1. *Let* $T((x_1, y_1),\ (x_2, y_2)) = (a, b, c, d) \in \mathcal{L}(^2 l_\infty^2)$. *Then, the followings are equivalent:*

(1) T *is smooth;*

(2) (b, a, d, c) *is smooth;*

(3) $(a, -b, -c, d)$ *is smooth;*

(4) (c, d, a, b) *is smooth;*

(5) (c, d, b, a) *is smooth.*

Proof. It follows from Theorem 2.1 and the above remark of Theorem 2.1. $\square$

Theorem 4.2. *Let* $T((x_1, y_1),\ (x_2, y_2)) = (a, b, c, d) \in \mathcal{L}(^2 l_\infty^2)$. *Then,* $T \in smB_{\mathcal{L}(^2 l_\infty^2)}$ *if and only if* $(0 < |a + b| < 1, |a + b| + |c + d| = 1, |a - b| + |c - d| < 1)$ *or* $(0 < |a - b| < 1, |a - b| + |c - d| = 1, |a + b| + |c + d| < 1)$.

Proof. By Theorem 4.1, we may assume that $a \geq |b|$ and $c \geq |d|$.

($\Rightarrow$): Suppose that T is smooth. For $(u_1, v_1), (u_2, v_2) \in S_{l_\infty^2}$, let $\delta_{(u_1,v_1),(u_2,v_2)} \in \mathcal{L}(^2 l_\infty^2)^*$ such that

$\delta_{(u_1,v_1),(u_2,v_2)}(L) = L((u_1, v_1), (u_2, v_2))$ for $L \in \mathcal{L}(^2 l_\infty^2)$. Then $\|\delta_{(u_1,v_1),(u_2,v_2)}\| \leq 1$. Note that, by Theorem 3.1, $1 = \|\delta_{(1,1),(1,\pm 1)}\| = \|\delta_{(1,-1),(-1,1)}\| = \|\delta_{(-1,1),(1,1)}\|$.

Obviously,

$$\begin{aligned} |\delta_{(1,1),(1,1)}(T)| &= |(a+b)+(c+d)|,\ |\delta_{(1,-1),(-1,1)}(T)| = |(a+b)-(c+d)|, \\ |\delta_{(1,1),(1,-1)}(T)| &= |(a-b)-(c-d)|,\ |\delta_{(-1,1),(1,1)}(T)| = |(a-b)+(c-d)|. \end{aligned}$$

Hence, if $T \in smB_{\mathcal{L}(^2l^2_\infty)}$, then $\|T\| = 1$, so, by Theorem 2.1, $(|a+b|+|c+d| = 1, |a-b|+|c-d| < 1)$ or $(|a-b|+|c-d| = 1, |a+b|+|c+d| < 1)$. Suppose that $|a+b|+|c+d| = 1, |a-b|+|c-d| < 1$. We will show that $0 < |a+b| < 1$. Otherwise. Then $a+b = 1$ or $a+b = 0$. Suppose that $a+b = 1$. Let $f_1 \in \mathcal{L}(^2l^2_\infty)^*$ such that $f_1(x_1x_2) = 1 = f_1(y_1y_2), f_1(x_1y_2) = 0 = f_1(x_2y_1)$, and let $f_2 \in \mathcal{L}(^2l^2_\infty)^*$ such that $1 = f_2(x_1x_2) = f_2(y_1y_2) = f_2(x_1y_2) = f_2(x_2y_1)$. Since $f_1 \neq f_2, 1 = \|f_j\| = f_j(T)$ for $j = 1, 2$, T is not smooth, which is a contradiction. Suppose that $a+b = 0$. Then $c+d = 1$. Let $g_1 \in \mathcal{L}(^2l^2_\infty)^*$ such that $g_1(x_1x_2) = 0 = g_1(y_1y_2), g_1(x_1y_2) = 1 = g_1(x_2y_1)$. Since $g_1 \neq f_2, 1 = \|g_1\| = g_1(T)$, T is not smooth, which is a contradiction. Therefore, if $|a+b|+|c+d| = 1, |a-b|+|c-d| < 1$, then $0 < |a+b| < 1$. Similarly as the case that $|a+b|+|c+d| = 1, |a-b|+|c-d| < 1$, if $|a-b|+|c-d| = 1, |a+b|+|c+d| < 1$, then we should have $0 < |a-b| < 1$.

($\Leftarrow$): Let $f \in \mathcal{L}(^2l^2_\infty)^*$ such that $1 = \|f\| = f(T)$ with $\alpha = f(x_1x_2), \beta = f(y_1y_2), \delta = f(x_1y_2), \gamma = f(x_2y_1)$.

Case 1: $0 < |a+b| < 1, |a+b|+|c+d| = 1, |a-b|+|c-d| < 1$

By Theorem 2.1, $\|T\| = 1$. Note that

$$(*)\quad a+b+c+d = 1 = a\alpha + b\beta + c\delta + d\gamma.$$

By Theorem 2.1, it follows that, for a sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned} (**)\quad 1 &\geq \|ax_1x_2 + (b \pm \frac{1}{n})y_1y_2 + (c \mp \frac{1}{n})x_1y_2 + dx_2y_1\| \\ 1 &\geq \|ax_1x_2 + (b \pm \frac{1}{n})y_1y_2 + cx_1y_2 + (d \mp \frac{1}{n})x_2y_1\| \\ 1 &\geq \|(a \pm \frac{1}{n})x_1x_2 + by_1y_2 + (c \mp \frac{1}{n})x_1y_2 + dx_2y_1\| \end{aligned}$$

From $(**)$, $1 \geq f(ax_1x_2 + (b \pm \frac{1}{n})y_1y_2 + (c \mp \frac{1}{n})x_1y_2 + dx_2y_1) = 1 + \frac{1}{n}|\beta - \delta|$, hence $\beta = \delta$ and $1 \geq f(ax_1x_2 + (b \pm \frac{1}{n})y_1y_2 + cx_1y_2 + (d \mp \frac{1}{n})x_2y_1) = 1 + \frac{1}{n}|\beta - \gamma|$, hence $\beta = \gamma$. $1 \geq f((a \pm \frac{1}{n})x_1x_2 + by_1y_2 + (c \mp \frac{1}{n})x_1y_2 + dx_2y_1) = 1 + \frac{1}{n}|\alpha - \delta|$, hence $\alpha = \delta$. Therefore, by $(*)$, $1 = \alpha = \beta = \delta = \gamma$, hence f is uniquely determined. Therefore, T is smooth.

Case 2: $0 < |a-b| < 1, |a-b|+|c-d| = 1, |a+b|+|c+d| < 1$

By Case 1 and Theorem 4.1, T is smooth. □

Remark. Nine months after the acceptance of this paper in Kyungpook Math. J., the author found that W. Cavalcante and D. Pellegrino had proved in [2], independently, Theorems 2.1, 2.4 and 3.4.

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