KYUNGPOOK Math. J. 58(2018), 37-46
https://doi.org/10.5666/KMJ.2018.58.1.37
pISSN 1225-6951 eISSN 0454-8124

On Generalized Absolute Riesz Summability Factor of Infinite Series

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Abstract. The objective of the present manuscript is to obtain a moderated theorem proceeding with absolute Riesz summability $|\bar{N}, p_n, \gamma; \delta|_k$ by applying almost increasing sequence for infinite series. Also, a set of reduced and well-known factor theorems have been obtained under suitable conditions.

1. Introduction

A sequence is called bounded variation, i.e., $(\lambda_n) \in BV$, if

$$\sum_{n=1}^{\infty} |\Delta\lambda_n| = |\lambda_n - \lambda_{n+1}| < \infty.$$

A positive sequence (g_n) is an almost increasing sequence [1] if $\exists$ a positive increasing sequence (h_n) and two positive constants M and N s.t.

$$Mh_n \leq g_n \leq Nh_n.$$

Definition 1.1. Let $\sum_{n=0}^{\infty} a_n$ be an infinite series with sequence of partial sums (s_n) and is said to be absolute Cesáro summable, if

$$\sum_{n=1}^{\infty} | u_n - u_{n-1}| < \infty, \tag{1.1}$$

where u_n represents the n^{th} sequence to sequence transformation (mean) of (s_n).

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* Corresponding Author.
Received March 6, 2017; revised November 16, 2017; accepted November 30, 2017.
2010 Mathematics Subject Classification: Primary 40F05; Secondary 40D20, 40G05.
Key words: Absolute Summability, Almost Increasing Sequence, Infinite Series, $|\bar{N}, p_n, \gamma; \delta|_k$ Summability.

Definition 1.2.([11]) Let t_n represent the n^{th} $(C, 1)$ means of the sequence (na_n), then series $\sum\limits_{n=0}^{\infty} a_n$ is said to be $|C, 1|_k$ summable for $k \geq 1$, if

$$\sum_{n=1}^{\infty} \frac{1}{n}|t_n|^k < \infty. \tag{1.2}$$

Definition 1.3.([2]) Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty, \quad n \to \infty, \qquad (P_{-n} = p_{-n} = 0, \quad n \geq 1), \tag{1.3}$$

then the sequence-to-sequence transformation σ_n defines the $(\bar{N}, p_n)$ mean of series $\sum a_n$ and given by,

$$\sigma_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k, \ \ P_n \neq 0, \ \ n \in N \tag{1.4}$$

and $lim_{n \to \infty} \sigma_n = s$, then the series $\sum a_n$ is said to be $(\bar{N}, p_n)$ summable generated by the sequence of coefficients (p_n).

Further, if sequence (σ_n) is of bounded variation with index $k \geq 1$, i.e.,

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |\Delta \sigma_{n-1}|^k < \infty, \tag{1.5}$$

then the series $\sum a_n$ is said to be absolutely $(R, p_n)_k$ summable with index k or $|\bar{N}, p_n|_k$ summable.

Definition 1.4.([3]) The series $\sum a_n$ is said to be $|\bar{N}, p_n; \delta|_k$ summable, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |\Delta \sigma_{n-1}|^k < \infty, \tag{1.6}$$

and $|\bar{N}, p_n, \gamma; \delta|_k$ summable, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\gamma(\delta k + k - 1)} |\Delta \sigma_{n-1}|^k < \infty, \tag{1.7}$$

where $k \geq 1, \ \ \delta \geq 0$ and γ is a real number and

$$\Delta \sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \ \ n \geq 1. \tag{1.8}$$

Bor and Seyhan [8] determined the set of sufficient conditions for an infinite series to be absolute Riesz summable $|\bar{N}, p_n; \delta|_k$ by using almost increasing sequence.

In 2002, Bor and Özarslan [7] redesigned the problem of Mazhar [12] under weaker conditions by using a quasi-power increasing sequence and in 2014, Bor [4] generalized the theorem dealing with a general class of power increasing sequences and absolute Riesz summability factors of infinite series.

Bor and Özarslan [9,10] have obtained theorems dealing with $|\bar{N}, p_n; \delta|_k$ summability factors of infinite series. In [13–15], Özarslan has used definitions of almost increasing sequence and non-increasing sequence for absolute summability of infinite series. In 2016, Sonker and Munjal [16] determined a theorem on generalized absolute Cesáro summability with the sufficient conditions for infinite series. Further, in 2017, Sonker and Munjal [17] obtained the sufficient conditions for triple matrices to be bounded. Bor [5] applied absolute summability (Cesáro and Nörlund) and established two theorems by using more general conditions for infinite series.

2. Known Result

By using $|\bar{N}, p_n; \delta|_k$ summability, Bor and Seyhan [8] proved the following theorem with the minimal set of sufficient conditions of an infinite series to be absolute Riesz summable.

Theorem 2.1. *Let (p_n) be a sequence of positive numbers such that*

$$P_n = O(np_n) \quad \text{as } n \to \infty. \tag{2.1}$$

Let (X_n) be an almost increasing sequence and suppose that there exist sequences (β_n) and (λ_n) such that

$$|\Delta\lambda_n| \le \beta_n, \tag{2.2}$$

$$\beta_n \to 0 \ \text{ as } n \to \infty, \tag{2.3}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty, \tag{2.4}$$

$$|\lambda_n|X_n = O(1) \ \text{ as } n \to \infty, \tag{2.5}$$

$$\sum_{n=v+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{P_v}\right\}, \tag{2.6}$$

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k-1} |t_n|^k = O(X_m) \text{ as } m \to \infty, \tag{2.7}$$

where

$$t_n = \frac{1}{n+1}\sum_{v=1}^{n} v a_v, \tag{2.8}$$

then the series $\sum a_n\lambda_n$ *is* $|\bar{N}, p_n; \delta|_k$ *summable for* $k \geq 1$ *and* $0 \leq \delta \leq 1/k$.

3. Main Result

Theorem 3.1. *Let* (X_n) *be an almost increasing sequence and the sequences* (β_n) *and* (λ_n) *be such that conditions* (2.2) – (2.5) *of Theorem 2.1 are satisfied. If the following conditions also satisfy,*

$$\sum_{n=v+1}^{\infty} \frac{1}{P_{n-1}}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k} = O\left\{\frac{1}{P_v}\left(\frac{P_v}{p_v}\right)^{1-k+\gamma(\delta k+k-1)}\right\}, \tag{3.1}$$

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k}|t_n|^k = O(X_m), \tag{3.2}$$

$$\sum_{n=1}^{m}\frac{|\lambda_n|}{n} = O(1), \tag{3.3}$$

and

$$\sum_{n=1}^{m}\frac{1}{n}\left(\frac{P_n}{p_n}\right)^{1-k+\gamma(\delta k+k-1)}|t_n|^k = O(X_m) \text{ as } m \to \infty \tag{3.4}$$

then the series $\sum a_n\lambda_n$ *is* $|\bar{N}, p_n, \gamma; \delta|_k$ *summable for* $k \geq 1$, $0 \leq \delta \leq 1/k$ *and* γ *is a real number.*

4. Lemma

Lemma 4.1.([6]) *Under the conditions on* (X_n), (β_n) *and* (λ_n) *as taken in the statement of the Theorem 3.1, the following conditions hold, where (2.4) is satisfied:*

$$n\beta_n X_n = O(1) \text{ as } n \to \infty, \tag{4.1}$$

$$\sum_{n=1}^{\infty}\beta_n X_n < \infty. \tag{4.2}$$

5. Proof of Theorem 3.1

Let (T_n) denotes the $(\bar{N}, p_n)$ mean of the series $\sum a_n\lambda_n$. Then, by definition and changing the order of summation, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{i=0}^{v} a_i\lambda_i = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v\lambda_v. \tag{5.1}$$

Then, for $n \geq 1$, we have

$$\begin{aligned}\bar{\Delta}T_n = T_n - T_{n-1} &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v\lambda_v \\ &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} \frac{P_{v-1}\lambda_v}{v} v a_v.\end{aligned}$$

By Abel's transformation, we have

$$\begin{aligned}\bar{\Delta}T_n &= \frac{n+1}{nP_n}p_nt_n\lambda_n - \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} p_vt_v\lambda_v\frac{v+1}{v} \\ &\quad + \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_vt_v\Delta\lambda_v\frac{v+1}{v} + \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_vt_v\lambda_{v+1}\frac{1}{v} \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}. \end{aligned} \tag{5.2}$$

In order to complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)} |\bar{\Delta}T_n|^k < \infty. \tag{5.3}$$

Using Minkowski's inequality,

$$|T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}|^k \leq 4^k\Big(|T_{n,1}|^k + |T_{n,2}|^k + |T_{n,3}|^k + |T_{n,4}|^k\Big),$$

the equation (5.3) reduces to

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)} |T_{n,r}|^k < \infty \; for \; r = 1, 2, 3, 4. \tag{5.4}$$

Now the L. H. S. of equation (5.4)

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)} |T_{n,1}|^k = \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)} \left|\frac{n+1}{nP_n}p_nt_n\lambda_n\right|^k$$

$$\begin{aligned}
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k}|t_n|^k|\lambda_n| \\
&= O(1)|\lambda_m|\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k}|t_n|^k \\
&\quad +O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}\left(\frac{P_v}{p_v}\right)^{\gamma(\delta k+k-1)-k}|t_v|^k \\
&= O(1)|\lambda_m|X_m+O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n \\
&= O(1)|\lambda_m|X_m+O(1)\sum_{n=1}^{m-1}\beta_n X_n \\
&= O(1)\ as\ \ m\to\infty, \qquad (5.5)
\end{aligned}$$

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)}|T_{n,2}|^k=\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}p_vt_v\lambda_v\frac{v+1}{v}\right|^k$$

$$\begin{aligned}
&= O(1)\sum_{n=2}^{m+1}\frac{1}{P_{n-1}}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k} \\
&\quad \times\sum_{v=1}^{n-1}p_v|\lambda_v||t_v|^k\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}p_v|\lambda_v||t_v|^k\sum_{n=v+1}^{m+1}\frac{1}{P_{n-1}}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k} \\
&= O(1)\sum_{v=1}^{m}p_v|\lambda_v||t_v|^k\frac{1}{P_v}\left(\frac{P_v}{p_v}\right)^{1-k+\gamma(\delta k+k-1)} \\
&= O(1)\sum_{v=1}^{m}|\lambda_v||t_v|^k\left(\frac{P_v}{p_v}\right)^{\gamma(\delta k+k-1)-k} \\
&= O(1)|\lambda_m|\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k}|t_n|^k \\
&\quad +O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}\left(\frac{P_v}{p_v}\right)^{\gamma(\delta k+k-1)-k}|t_v|^k
\end{aligned}$$

$$= O(1)|\lambda_m|X_m + O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n$$

$$= O(1)|\lambda_m|X_m + O(1)\sum_{n=1}^{m-1}\beta_n X_n = O(1) \ as \ m \to \infty, \tag{5.6}$$

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)}|T_{n,3}|^k = \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v t_v \Delta\lambda_v \frac{v+1}{v}\right|^k$$

$$\begin{aligned}
&= O(1)\sum_{n=2}^{m+1}\frac{1}{P_{n-1}}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k} \\
&\quad \times\sum_{v=1}^{n-1}P_v\beta_v|t_v|^k\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_v\beta_v\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}P_v\beta_v|t_v|^k\sum_{n=v+1}^{m+1}\frac{1}{P_{n-1}}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k} \\
&= O(1)\sum_{v=1}^{m}P_v\beta_v|t_v|^k\frac{1}{P_v}\left(\frac{P_v}{p_v}\right)^{1-k+\gamma(\delta k+k-1)} \\
&= O(1)\sum_{v=1}^{m}v\beta_v\frac{1}{v}\left(\frac{P_v}{p_v}\right)^{1-k+\gamma(\delta k+k-1)}|t_v|^k \\
&= O(1)m\beta_m\sum_{v=1}^{m}\frac{1}{v}\left(\frac{P_v}{p_v}\right)^{1-k+\gamma(\delta k+k-1)}|t_v|^k \\
&\quad +O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{i=1}^{v}\frac{1}{i}\left(\frac{P_i}{p_i}\right)^{1-k+\gamma(\delta k+k-1)}|t_i|^k \\
&= O(1)m\beta_m X_m + O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|X_v \\
&= O(1)m\beta_m X_m + O(1)\sum_{v=1}^{m-1}vX_v|\Delta\beta_v| + O(1)\sum_{v=1}^{m-1}\beta_v X_v
\end{aligned}$$

$$= O(1) \ as \ m \to \infty, \tag{5.7}$$

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)}|T_{n,4}|^k = \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v t_v \lambda_{v+1}\frac{1}{v}\right|^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{1}{P_{n-1}}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k}$$
$$\times\sum_{v=1}^{n-1}P_v\frac{|\lambda_{v+1}|}{v}|t_v|^k\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_v\frac{|\lambda_{v+1}|}{v}\right)^{k-1}$$
$$= O(1)\sum_{v=1}^{m}P_v\frac{|\lambda_{v+1}|}{v}|t_v|^k\sum_{n=v+1}^{m+1}\frac{1}{P_{n-1}}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)-k}$$
$$= O(1)\sum_{v=1}^{m}P_v\frac{|\lambda_{v+1}|}{v}|t_v|^k\frac{1}{P_v}\left(\frac{P_v}{p_v}\right)^{1-k+\gamma(\delta k+k-1)}$$
$$= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|\frac{1}{v}\left(\frac{P_v}{p_v}\right)^{1-k+\gamma(\delta k+k-1)}|t_v|^k$$
$$= O(1)|\lambda_{m+1}|\sum_{v=1}^{m}\frac{1}{v}\left(\frac{P_v}{p_v}\right)^{1-k+\gamma(\delta k+k-1)}|t_v|^k$$
$$+O(1)\sum_{v=1}^{m-1}\Delta|\lambda_{v+1}|\sum_{i=1}^{v}\frac{1}{i}\left(\frac{P_i}{p_i}\right)^{1-k+\gamma(\delta k+k-1)}|t_i|^k$$
$$= O(1)|\lambda_{m+1}|X_m+O(1)\sum_{v=1}^{m-1}\Delta|\lambda_{v+1}|X_v$$
$$= O(1)|\lambda_{m+1}|X_m+\sum_{v=1}^{m-1}\beta_{v+1}X_{v+1}$$
$$= O(1)\ as\ m\to\infty, \tag{5.8}$$

Collecting (5.1) - (5.8), we have

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)}|T_{n,r}|^k<\infty\ for\ r=1,2,3,4. \tag{5.9}$$

Hence proof of the theorem is completed. □

6. Corollaries

Corollary 6.1. ([6]) *Let (X_n) be an almost increasing sequence and the sequences (β_n) and (λ_n) be such that conditions (2.2) – (2.5) and (3.3) are satisfied. If the following conditions also satisfy,*

$$\sum_{n=1}^{m}\frac{p_n}{P_n}|t_n|^k=O(X_m)\ as\ m\to\infty, \tag{6.1}$$

(6.2) $$\sum_{n=1}^{m} \frac{1}{n}|t_n|^k = O(X_m) \text{ as } m \to \infty,$$

then the series $\sum a_n\lambda_n$ *is* $|\bar{N}, p_n|_k$ *summable for* $k \geq 1$.

Proof. On putting $\gamma = 1$ and $\delta = 0$ in Theorem 3.1, we will get (6.1) and (6.2). We omit the details as the proof is similar to that of Theorem 3.1 and we use (6.1) and (6.2) instead of (3.2) and (3.4).

Corollary 6.2. *Let* (X_n) *be an almost increasing sequence and the sequences* (β_n) *and* (λ_n) *be such that conditions* (2.2) − (2.5) *and* (3.3) *are satisfied. If the following conditions also satisfy,*

(6.3) $$\sum_{n=1}^{m} \frac{p_n}{P_n}|t_n| = O(X_m) \text{ as } m \to \infty,$$

(6.4) $$\sum_{n=1}^{m} \frac{1}{n}|t_n| = O(X_m) \text{ as } m \to \infty,$$

then the series $\sum a_n\lambda_n$ *is* $|\bar{N}, p_n|$ *summable .*

Proof. On putting $\gamma = 1$, $\delta = 0$ and $k = 1$ in Theorem 3.1, we will get (6.3) and (6.4). We omit the details as the proof is similar to that of Theorem 3.1 and we use (6.3) and (6.4) instead of (3.2) and (3.4). □

Acknowledgments. The authors are very thankful to the Department of Science and Technology (India) for providing the financial support to the second author under INSPIRE Scheme (Innovation in Science Pursuit for Inspired Research Scheme).

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