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The Incomplete Lauricella Functions of Several Variables and Associated Properties and Formulas

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Abstract. Motivated mainly by certain interesting recent extensions of the generalized hypergeometric function [30] and the second Appell function [6], we introduce here the incomplete Lauricella functions $\gamma_A^{(n)}$ and $\Gamma_A^{(n)}$ of n variables. We then systematically investigate several properties of each of these incomplete Lauricella functions including, for example, their various integral representations, finite summation formulas, transformation and derivative formulas, and so on. We provide relevant connections of some of the special cases of the main results presented here with known identities. Several potential areas of application of the incomplete hypergeometric functions in one and more variables are also pointed out.

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1. Introduction, Definitions and Preliminaries

The familiar *incomplete Gamma functions* $\gamma(s,x)$ and $\Gamma(s,x)$ defined by

$$\gamma(s,x) := \int_0^x t^{s-1}\, e^{-t}\, dt \qquad \big(\Re(s) > 0;\ x \geqq 0\big) \tag{1.1}$$

and

$$\Gamma(s,x) := \int_x^\infty t^{s-1}\, e^{-t}\, dt \qquad \big(x \geqq 0;\ \Re(s) > 0 \quad \text{when} \quad x = 0\big), \tag{1.2}$$

respectively, satisfy the following decomposition formula:

$$\gamma(s,x) + \Gamma(s,x) := \Gamma(s) \qquad \big(\Re(s) > 0\big). \tag{1.3}$$

Each of these functions plays an important rôle in the study of the analytic solutions of a variety of problems in diverse areas of science and engineering (see, *e.g.*, [1, 2, 8, 9, 10, 14, 16, 17, 20, 23, 33, 34, 35, 37, 38]).

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}^-$ and $\mathbb{C}$ denote the sets of positive integers, negative integers and complex numbers, respectively,

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \qquad \text{and} \qquad \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}\,.$$

Moreover, the parameter $x \geqq 0$ used above in (1.1) and (1.2) *and* elsewhere in this paper is independent of $\Re(z)$ of the complex number $z \in \mathbb{C}$.

Recently, Srivastava *et al.* [30] introduced and studied in a rather systematic manner the following two families of generalized incomplete hypergeometric functions:

$${}_p\gamma_q\left[\begin{array}{r} (\alpha_1,x), \alpha_2, \cdots, \alpha_p; \\ \beta_1, \cdots, \beta_q; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1;x)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \tag{1.4}$$

and

$${}_p\Gamma_q\left[\begin{array}{r} (\alpha_1,x), \alpha_2, \cdots, \alpha_p; \\ \beta_1, \cdots, \beta_q; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{[\alpha_1;x]_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}, \tag{1.5}$$

where, in terms of the incomplete Gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$ defined by (1.1) and (1.2), respectively, the *incomplete* Pochhammer symbols $(\lambda;x)_\nu$ and $[\lambda;x]_\nu$ $(\lambda;\ \nu \in \mathbb{C};\ x \geqq 0)$ are defined as follows:

$$(\lambda;x)_\nu := \frac{\gamma(\lambda+\nu,x)}{\Gamma(\lambda)} \qquad (\lambda,\nu \in \mathbb{C};\ x \geqq 0) \tag{1.6}$$

and

$$[\lambda;x]_\nu := \frac{\Gamma(\lambda+\nu,x)}{\Gamma(\lambda)} \qquad (\lambda,\nu \in \mathbb{C};\ x \geqq 0), \tag{1.7}$$

so that, obviously, these incomplete Pochhammer symbols $(\lambda;x)_\nu$ and $[\lambda;x]_\nu$ satisfy the following decomposition relation:

$$(\lambda;x)_\nu+[\lambda;x]_\nu:=(\lambda)_\nu \qquad (\lambda;\ \nu\in\mathbb{C};\ x\geqq 0). \tag{1.8}$$

Here, and in what follows, $(\lambda)_\nu$ $(\lambda,\nu\in\mathbb{C})$ denotes the Pochhammer symbol (or the shifted factorial) which is defined (in general) by

$$(\lambda)_\nu:=\frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}=\begin{cases}1 & (\nu=0;\ \lambda\in\mathbb{C}\setminus\{0\})\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}),\end{cases} \tag{1.9}$$

it being understood *conventionally* that $(0)_0:=1$ and assumed *tacitly* that the Γ-quotient exists (see, for details, [35, p. 21 *et seq.*]; see also [25]), $\mathbb{N}$ being (as above) the set of positive integers.

As already observed by Srivastava *et al.* [30], the definitions (1.4) and (1.5) readily yield the following decomposition formula:

$$\begin{aligned}&{}_p\gamma_q\left[\begin{array}{r}(\alpha_1,x),\alpha_2,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{array} z\right]+{}_p\Gamma_q\left[\begin{array}{r}(\alpha_1,x),\alpha_2,\ldots,\alpha_p;\\ \beta_1,\cdots,\beta_q;\end{array} z\right]\\ &\qquad\qquad={}_pF_q\left[\begin{array}{r}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{array} z\right]\end{aligned} \tag{1.10}$$

for the familiar generalized hypergeometric function ${}_pF_q$.

In a sequel to the aforementioned work by Srivastava *et al.* [30], Çetinkaya [6] introduced the incomplete second Appell hypergeometric functions γ_2 and Γ_2 in two variables and investigated their various properties including integral representations. Motivated essentially by the demonstrated potential for applications of these incomplete hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$, and the incomplete second Appell hypergeometric functions γ_2 and Γ_2 in many diverse areas of mathematical, physical, engineering and statistical sciences (see, for details, [6, 30] and the references cited therein), here, we aim here at systematically investigating the family of the incomplete Lauricella's functions $\gamma_A^{(n)}$ and $\Gamma_A^{(n)}$ of n variables. For each of these incomplete multivariable hypergeometric functions, we derive various definite and semi-definite integral representations involving the Laguerre polynomials, incomplete gamma functions, and the Bessel and modified Bessel functions. Some transformation and summation formulas of the incomplete Lauricella functions are also presented. We point out relevant connections of some of the special cases of the main results derived here with known identities. Several potential areas of application of the incomplete hypergeometric functions in one and more variables are also indicated. For various other investigations involving generalizations of the hypergeometric function ${}_pF_q$ of p numerator and q denominator parameters, which were motivated essentially by the pioneering work of Srivastava *et al.* [30], the interested reader may be referred to several recent papers on the subject (see, for example, [7, 11, 13, 18, 26, 27, 28, 29, 31, 32] *and* the references cited in each of these papers).

2. The Incomplete Lauricella Functions $\gamma_A^{(n)}$ and $\Gamma_A^{(n)}$ in n Variables

In terms of the incomplete Pochhammer symbol $(\lambda;x)_\nu$ and $[\lambda;x]_\nu$ defined by (1.6) and (1.7), we introduce the families of the incomplete Lauricella hypergeometric functions $\gamma_A^{(n)}$ and $\Gamma_A^{(n)}$ of n variables as follows: For $\alpha,\beta_1,\ldots,\beta_n \in \mathbb{C}$ and $\gamma_1,\ldots,\gamma_n \in \mathbb{C}\setminus\mathbb{Z}_0^-$, we have

$$\begin{aligned}\gamma_A^{(n)}&[(\alpha,x),\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n]\\ &:=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(\alpha;x)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\,\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\ &\qquad(x\geqq 0;\ |x_1|+\cdots+|x_n|<1\quad\text{when}\quad x=0)\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}\Gamma_A^{(n)}&[(\alpha,x),\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n]\\ &=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{[\alpha;x]_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\,\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\ &\qquad(x\geqq 0;\ |x_1|+\cdots+|x_n|<1\quad\text{when}\quad x=0).\end{aligned}\tag{2.2}$$

In view of (1.8), these families of incomplete Lauricella functions satisfy the following decomposition formula:

$$\begin{aligned}\gamma_A^{(n)}&[(\alpha,x),\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n]\\ &+\Gamma_A^{(n)}[(\alpha,x),\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n]\\ &=F_A^{(n)}[\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n],\end{aligned}\tag{2.3}$$

where $F_A^{(n)}$ is the familiar Lauricella function of n variables [35, 36]. It is noted in passing that, in view of the decomposition formula (2.3), it is sufficient to discuss the properties and characteristics of the incomplete Lauricella function $\Gamma_A^{(n)}$.

Theorem 2.1. *The incomplete Lauricella functions $\gamma_A^{(n)}$ and $\Gamma_A^{(n)}$ satisfy the following partial differential equation*:

$$\begin{aligned}x_j(1-x_j)\frac{\partial^2 u}{\partial x_j^2}-x_j\sum_{k=1\ (k\neq j)}^{n}x_k\frac{\partial^2 u}{\partial x_k\partial x_j}+[\gamma_j-(\alpha+\beta_j+1)x_j]\frac{\partial u}{\partial x_j}\\ -\beta_j\sum_{k=1\ (k\neq j)}^{n}x_k\frac{\partial u}{\partial x_k}-\alpha\beta_j u=0\qquad(j=1,\ldots,n),\end{aligned}\tag{2.4}$$

where

$$\begin{aligned}u=u(x_1,\ldots,x_n):=\gamma_A^{(n)}&[(\alpha,x),\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n]\\ &+\Gamma_A^{(n)}[(\alpha,x),\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n].\end{aligned}$$

Proof. In light of the decomposition formula (2.3), it is easy to derive (2.4), since the n-variable Lauricella function $F_A^{(n)}$ satisfies the same system of partial differential equations as in (2.4). □

Remark 2.2. The special cases of (2.1) and (2.2) when $n = 2$ are easily seen to correspond to the following known families of the incomplete second Appell hypergeometric functions in two variables [6]:

$$\gamma_A^{(2)} = \gamma_2[(a,x),b_1,b_2;c_1,c_2;x_1,x_2] \ \text{ and } \ \Gamma_A^{(2)} = \Gamma_2[(a,x),b_1,b_2;c_1,c_2;x_1,x_2], \tag{2.5}$$

respectively. Also, the special cases of (2.1) and (2.2) when $n = 1$ correspond to the following known families of the incomplete Gauss hypergeometric functions in one variable [30]:

$$\gamma_A^{(1)} = {}_2\gamma_1[(a,x),b_1;c_1;x_1] \ \text{ and } \ \Gamma_A^{(1)} = {}_2\Gamma_1[(a,x),b_1;c_1;x_1], \tag{2.6}$$

respectively.

3. Integral Representations of the Incomplete Lauricella Function $\Gamma_A^{(n)}$

In this section, we present certain integral representations of the incomplete Lauricella function $\Gamma_A^{(n)}$ by applying (1.2) and (1.7). We also obtain some integral representations involving the Laguerre polynomials, the incomplete gamma functions, and the Bessel and modified Bessel functions.

Theorem 3.1. *The following integral representation for* $\Gamma_A^{(n)}$ in (2.2) *holds true*:

$$\begin{aligned}&\Gamma_A^{(n)}[(\alpha,x),\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n]\\&\quad=\frac{1}{\Gamma(\alpha)}\int_x^\infty e^{-t}\,t^{\alpha-1}\;{}_1F_1\left[\begin{matrix}\beta_1;\\ \\ \gamma_1;\end{matrix}\;x_1t\right]\cdots\;{}_1F_1\left[\begin{matrix}\beta_n;\\ \\ \gamma_n;\end{matrix}\;x_nt\right]dt\end{aligned}\tag{3.1}$$

$$\big(x \geqq 0;\ \Re(x_1+\cdots+x_n)<1;\ \Re(\alpha)>0 \quad \textit{when} \quad x=0\big).$$

Proof. Using the definition of the incomplete Pochhammer symbol $[\alpha;x]_{m_1+\cdots+m_n}$ in (2.2) and considering the integral representation resulting from (1.2) and (1.7), we are led to the desired result (3.1) asserted by Theorem 3.1. □

Theorem 3.2. *The following n-tuple integral representation for* $\Gamma_A^{(n)}$ in (2.2) *holds true*:

$$\begin{aligned}
\Gamma_A^{(n)}[(\alpha,x),\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n] &= \frac{1}{B(\beta_1,\gamma_1-\beta_1)\cdots B(\beta_n,\gamma_n-\beta_n)} \\
&\cdot \int_0^1 \cdots \int_0^1 t_1^{\beta_1-1}\cdots t_n^{\beta_n-1}(1-t_1)^{\gamma_1-\beta_1-1}\cdots(1-t_n)^{\gamma_n-\beta_n-1} \\
&\cdot\ {}_1\Gamma_0\big[(\alpha,x);\text{---};x_1t_1+\cdots+x_nt_n\big]\,dt_1\cdots dt_n \qquad (3.2)\\
&\big(\Re(\gamma_j)>\Re(\beta_j)>0\ \ (j=1,\ldots,n);\ x\geqq 0\big).
\end{aligned}$$

Proof. Upon considering the following elementary identity involving the Beta function $B(\alpha,\beta)$:

$$\frac{(\beta)_\nu}{(\gamma)_\nu}=\frac{B(\beta+\nu,\gamma-\beta)}{B(\beta,\gamma-\beta)}=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta+\nu-1}\,(1-t)^{\gamma-\beta-1}\,dt$$

$$\big(\Re(\gamma)>\Re(\beta)>\max\{0,-\Re(\nu)\}\big)$$

in (2.2) and using the elementary series identity (5.6), if we apply the definition (1.5), we get the desired multiple integral representation (3.2) asserted by Theorem 3.2. □

Theorem 3.3. *The following* $(n+1)$*-tuple integral representation for* $\Gamma_A^{(n)}$ in (2.2) *holds true*:

$$\begin{aligned}
\Gamma_A^{(n)}[(\alpha,x),\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n] &= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)\cdots\Gamma(\beta_n)} \\
&\cdot \int_x^\infty \int_0^\infty \cdots \int_0^\infty e^{-s-t_1-\,\cdots\,-t_n}\, s^{\alpha-1}\, t_1^{\beta_1-1}\cdots t_n^{\beta_n-1} \\
&\cdot\ {}_0F_1(\text{---};\gamma_1;x_1st_1)\ \cdots\ {}_0F_1(\text{---};\gamma_n;x_nst_n)\,ds\,dt_1\ \cdots\ dt_n \qquad (3.3)\\
&\big(x\geqq 0;\ \min\{\Re(\alpha),\Re(\beta_1),\ldots,\Re(\beta_n)\}>0 \quad \textit{when} \quad x=0\big).
\end{aligned}$$

Proof. Using the incomplete Pochhammer symbol $[\alpha;x]_{m_1+\cdots+m_n}$ and the classical Pochhammer symbols $(\beta_1)_{m_1},\ldots,(\beta_n)_{m_n}$ in the definition (2.2) by considering the integral representation resulting from (1.2) and (1.7), we are led to the desired $(n+1)$-tuple integral representation (3.3) asserted by Theorem 3.3. □

Remark 3.4. The Laguerre polynomial $L_n^{(\alpha)}(x)$ of order (index) α and degree n in x, the incomplete gamma function $\gamma(k,x)$, the Bessel function $J_\nu(z)$ and the modified Bessel function $I_\nu(z)$ are expressible in terms of hypergeometric functions as follows (see, *e.g.*, [21]; see also [5, 9, 12, 14, 17, 38, 39]):

$$L_n^{(\alpha)}(x)=\frac{(\alpha+1)_n}{n!}\ {}_1F_1(-n;\alpha+1;x), \qquad (3.4)$$

$$ {}_1F_1(\kappa;\kappa+1;-x) = \kappa x^{-\kappa}\ \gamma(\kappa,x), \tag{3.5} $$

$$ J_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)}\ {}_0F_1\left(\text{---}\ ;\nu+1;-\frac{1}{4}z^2\right) \qquad (\nu\in\mathbb{C}\setminus\mathbb{Z}^-) \tag{3.6} $$

and

$$ I_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)}\ {}_0F_1\left(\text{---}\ ;\nu+1;\frac{1}{4}z^2\right) \qquad (\nu\in\mathbb{C}\setminus\mathbb{Z}^-). \tag{3.7} $$

Now, by applying the relationships (3.4) and (3.5) to (3.1) and (3.6) and (3.7) to (3.3), we can deduce certain interesting integral representations for the incomplete Lauricella hypergeometric function in (2.2), which are asserted by Corollaries 3.5 and 3.6 below. We state here the resulting integral representations without proof.

Corollary 3.5. *Each of the following integral representations holds true*:

$$ \begin{aligned} &\Gamma_A^{(n)}[(\alpha,x),-m_1,\ldots,-m_n;\beta_1+1,\ldots,\beta_n+1;x_1,\ldots,x_n] \\ &= \frac{m_1!\cdots m_n!}{(\beta_1+1)_{m_1}\cdots(\beta_n+1)_{m_n}\Gamma(\alpha)}\int_x^\infty e^{-t}\ t^{\alpha-1}\ L_{m_1}^{(\beta_1)}(x_1t)\cdots L_{m_n}^{(\beta_n)}(x_nt)\,dt \end{aligned} \tag{3.8} $$

and

$$ \begin{aligned} &\Gamma_A^{(n)}[(\alpha,x),\beta_1,\ldots,\beta_n;\beta_1+1,\ldots,\beta_n+1;-x_1,\ldots,-x_n] \\ &= \frac{\beta_1\cdots\beta_n x_1^{-\beta_1}\cdots x_n^{-\beta_n}}{\Gamma(\alpha)}\int_x^\infty e^{-t}\ t^{\alpha-\beta_1-\cdots-\beta_n-1}\ \gamma(\beta_1,x_1t)\cdots\gamma(\beta_n,x_nt)\,dt \end{aligned} \tag{3.9} $$

provided that the integrals involved are convergent.

Corollary 3.6. *Each of the following* $(n+1)$*-tuple integral representations holds true*:

$$ \begin{aligned} &\Gamma_A^{(n)}[(\alpha,x),\beta_1,\ldots,\beta_n;\gamma_1+1,\ldots,\gamma_n+1;-x_1,\ldots,-x_n] \\ &\quad= \frac{\Gamma(\gamma_1+1)\cdots\Gamma(\gamma_n+1)\,x_1^{-\frac{\gamma_1}{2}}\cdots x_n^{-\frac{\gamma_n}{2}}}{\Gamma(\alpha)\Gamma(\beta_1)\cdots\Gamma(\beta_n)} \\ &\quad\cdot\int_x^\infty\int_0^\infty\cdots\int_0^\infty e^{-s-t_1-\cdots-t_n}\ s^{\alpha-\frac{\gamma_1}{2}-\cdots-\frac{\gamma_n}{2}-1}\ t_1^{\beta_1-\frac{\gamma_1}{2}-1}\cdots t_1^{\beta_n-\frac{\gamma_n}{2}-1} \\ &\quad\cdot J_{\gamma_1}\left(2\sqrt{x_1st_1}\,\right)\cdots J_{\gamma_n}\left(2\sqrt{x_nst_n}\,\right)\,ds\,dt_1\cdots dt_n \end{aligned} \tag{3.10} $$

and

$$ \begin{aligned} &\Gamma_A^{(n)}[(\alpha,x),\beta_1,\ldots,\beta_n;\gamma_1+1,\ldots,\gamma_n+1;x_1,\ldots,x_n] \\ &\quad= \frac{\Gamma(\gamma_1+1)\cdots\Gamma(\gamma_n+1)\,x_1^{-\frac{\gamma_1}{2}}\cdots x_n^{-\frac{\gamma_n}{2}}}{\Gamma(\alpha)\Gamma(\beta_1)\cdots\Gamma(\beta_n)} \\ &\quad\cdot\int_x^\infty\int_0^\infty\cdots\int_0^\infty e^{-s-t_1-\cdots-t_n}\ s^{\alpha-\frac{\gamma_1}{2}-\cdots-\frac{\gamma_n}{2}-1}\ t_1^{\beta_1-\frac{\gamma_1}{2}-1}\cdots t_1^{\beta_n-\frac{\gamma_n}{2}-1} \\ &\quad\cdot I_{\gamma_1}\left(2\sqrt{x_1st_1}\,\right)\cdots I_{\gamma_n}\left(2\sqrt{x_nst_n}\,\right)\,ds\,dt_1\cdots dt_n, \end{aligned} \tag{3.11} $$

provided that the integrals involved are convergent.

4. A Derivative Formula

Differentiating both sides of (2.2) with respect to $x_1, \ldots, x_n$ partially $m_1, \ldots, m_n$ times, respectively, we obtain a derivative formula for the incomplete Lauricella hypergeometric function $\Gamma_A^{(n)}$ given in the following theorem.

Theorem 4.1. *The following derivative formula for* $\Gamma_A^{(n)}$ *holds true*:

$$\begin{aligned}
&\frac{\partial^{m_1+\cdots+m_n}}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}}\left\{\Gamma_A^{(n)}[(\alpha,x),\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n]\right\}\\
&\quad=\frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}}\cdot\Gamma_A^{(n)}[(\alpha+m_1+\cdots+m_n,x),\\
&\qquad\beta_1+m_1,\ldots,\beta_n+m_n;\gamma_1+m_1,\ldots,\gamma_n+m_n;x_1,\ldots,x_n],
\end{aligned} \tag{4.1}$$

provided that each member of the assertion (4.1) *exists.*

5. A Set of Transformation Formulas

Here we give two transformation formulas for the incomplete Lauricella hypergeometric function $\Gamma_A^{(n)}$ of n variables.

Theorem 5.1. *Each of the following transformation formulas holds true*:

$$\begin{aligned}
&\Gamma_A^{(n)}[(\alpha,x),\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n]=(1-x_1)^{-\alpha}\\
&\quad\cdot\,\Gamma_A^{(n)}\left[\big(\alpha,x(1-x_1)\big),\gamma_1-\beta_1,\beta_2,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;\right.\\
&\qquad\left.\frac{x_1}{x_1-1},\frac{x_2}{1-x_1},\cdots,\frac{x_n}{1-x_1}\right]
\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}
&\Gamma_A^{(n)}[(\alpha,x),\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n]=(1-x_1-\cdots-x_n)^{-\alpha}\\
&\quad\cdot\,\Gamma_A^{(n)}\left[\big(\alpha,x(1-x_1-\cdots-x_n)\big),\gamma_1-\beta_1,\ldots,\gamma_n-\beta_n;\gamma_1,\ldots,\gamma_n;\right.\\
&\qquad\left.\frac{x_1}{x_1+\cdots+x_n-1},\cdots,\frac{x_n}{x_1+\cdots+x_n-1}\right].
\end{aligned} \tag{5.2}$$

Proof. If we first apply Kummer's transformation formula (see, *e.g.*, [21, p. 125, Eq. (2)]):

$${}_1F_1(\alpha;\beta;z)=e^z\,{}_1F_1(\beta-\alpha;\beta;-z) \tag{5.3}$$

to (3.1) and then set

$$\tau = (1 - x_1)t \qquad \text{and} \qquad d\tau = (1 - x_1)dt$$

in the resulting integral, we get the first transformation formula (5.1). A similar argument will establish the second transformation formula (5.2). □

Corollary 5.2. *The following expansion formula holds true*:

$$ {}_1\Gamma_0^{(n)}\big[(\alpha, x); \text{——}\,; z\big] = (1 - z)^{-\alpha}\ \big[\alpha; x(1 - z)\big]_0 \tag{5.4}$$

for the incomplete hypergeometric function ${}_1\Gamma_0$ *defined by* (1.5) *for* $p - 1 = q = 0$, $[\lambda; x]_0$ *being the incomplete Pochhammer symbol given by* (1.6) *for* $\nu = 0$.

Proof. Upon setting $\gamma_j = \beta_j \ \ (j = 1, \ldots, n)$ in (5.2), we find that

$$\begin{aligned}\sum_{m_1,\ldots,m_n=0}^{\infty} &[\alpha; x]_{m_1+\cdots+m_n} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \\ &= (1 - x_1 - \cdots - x_n)^{-\alpha}\ \big[\alpha; x\,(1 - x_1 - \ldots - x_n)\,\big]_0,\end{aligned} \tag{5.5}$$

which, in view of the elementary series identity [36, p. 52, Eq. 1.6(3)]:

$$\sum_{m_1,\ldots,m_n=0}^{\infty} \Omega\,(m_1 + \cdots + m_n)\ \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} = \sum_{m=0}^{\infty} \Omega(m)\ \frac{(x_1 + \cdots + x_m)^m}{m!} \tag{5.6}$$

can easily be simplified to yield the assertion (5.4) of Corollary 5.2 when we put $x_1 + \cdots + x_n = z$. □

Remark 5.3. Since

$$(\lambda; x)_0\Big|_{x=0} = [\lambda; x]_0\Big|_{x=0} = 1,$$

in its special case when $x = 0$, Corollary 5.2 would reduce immediately to the binomial expansion given by

$$ {}_1F_0(\alpha; \text{——}\,; z) = \sum_{n=0}^{\infty} (\alpha)_n\ \frac{z^n}{n!} = (1 - z)^{-\alpha} \qquad (|z| < 1;\ \alpha \in \mathbb{C}). \tag{5.7}$$

Remark 5.4. In light of the expansion formula (5.4) asserted by Corollary 5.2, n-tuple integral representation for $\Gamma_A^{(n)}$ in Theorem 3.2 can be rewritten as follows:

holds true:

$$\Gamma_A^{(n)}[(\alpha,x),\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n]=\frac{1}{B(\beta_1,\gamma_1-\beta_1)\cdots B(\beta_n,\gamma_n-\beta_n)}$$
$$\cdot\int_0^1\cdots\int_0^1 t_1^{\beta_1-1}\cdots t_n^{\beta_n-1}(1-t_1)^{\gamma_1-\beta_1-1}\cdots(1-t_n)^{\gamma_n-\beta_n-1}$$
$$\cdot\ (1-x_1t_1-\cdots-x_nt_n)^{-\alpha}\ \left[\alpha;x\,(1-x_1t_1-\cdots-x_nt_n)\right]_0\,dt_1\cdots dt_n \tag{5.8}$$
$$\big(\Re(\gamma_j)>\Re(\beta_j)>0\ \ (j=1,\ldots,n);\ x\geqq 0\big).$$

Finally, by applying the following known relationship of the complementary error function $\operatorname{erfc}(z)$ with the incomplete gamma function $\Gamma(s,x)$ (see, for example, [36, p. 40, Eq. 1.3(28)]):

$$\operatorname{erfc}(z)=\frac{1}{\sqrt{\pi}}\,\Gamma\left(\frac{1}{2},z^2\right),$$

a special case of Corollary 5.2 yields the following result:

$${}_1\Gamma_0\left[\left(\frac{1}{2},x\right);\text{---};z\right]=\frac{1}{\sqrt{1-z}}\operatorname{erfc}\big(\sqrt{x(1-z)}\,\big). \tag{5.9}$$

6. Finite Sums Involving $\Gamma_A^{(n)}$

Here we consider some finite sum formulas associated with the incomplete Lauricella function $\Gamma_A^{(n)}$.

Theorem 6.1. *The following finite sum formula for* $\Gamma_A^{(n)}$ *holds true*:

$$\sum_{k=0}^{m}\Gamma_A^{(n)}[(\alpha,x),-k,-m+k,\beta_3,\ldots,\beta_n;1,1,\gamma_3,\ldots,\gamma_n;x_1,\ldots,x_n]$$
$$=(m+1)\Gamma_A^{(n-1)}[(\alpha,x),-m,\beta_3,\ldots,\beta_n;2,\gamma_3,\ldots,\gamma_n;x_1+x_2,x_3,\ldots,x_n] \tag{6.1}$$
$$\big(|x_1|+\cdots+|x_n|<1;\ \Re(\alpha)>0\quad \textit{when}\quad x=0\big).$$

Proof. We make use the integral representation (3.1) and the following well-known identity for the Laguerre polynomials (see, *e.g.*, [21, p. 209, Eq. (3)]):

$$\sum_{k=0}^{m}L_k^{(\lambda)}(x)L_{m-k}^{(\mu)}(y)=L_m^{(\lambda+\mu+1)}(x+y) \tag{6.2}$$

for $\lambda=\mu=0$. Thus, in view of the ${}_1F_1$ representation (3.4) for the Laguerre polynomials, we get the desired finite sum formula (6.1) asserted by Theorem 6.1. □

Theorem 6.2. *The following multiple finite sum formula for $\Gamma_A^{(n)}$ holds true*:

$$\sum_{k_1=0}^{m_1}\cdots\sum_{k_s=0}^{m_s}\Gamma_A^{(2s)}[(\alpha,x),-k_1,-m_1+k_1,\ldots,-k_s,-m_s+k_s;1,\ldots,1;x_1,\ldots,x_{2s}]$$

$$= (m_1+1)\cdots(m_s+1)\Gamma_A^{(s)}[(\alpha,x),-m_1,\ldots,-m_s;2,\ldots,2;x_1+x_2,\ldots,x_{2s-1}+x_{2s}] \tag{6.3}$$

$$\big(|x_1|+\cdots+|x_n|<1;\ \Re(\alpha)>0 \quad \textit{when} \quad x=0\big).$$

Proof. By iterating the method used in proving the finite summation formula (6.1), which is based upon the identity (6.2) and the integral representation (3.1), the ${}_1F_1$ representation (3.4) for the Laguerre polynomials yields the desired multiple summation formula (6.3) asserted by Theorem 6.2. □

Theorem 6.3. *The following finite sum formula for $\Gamma_A^{(n)}$ holds true*:

$$\begin{aligned}\sum_{k=0}^{m}\binom{\lambda+k}{k}&\Gamma_A^{(n)}[(\alpha,x),-k,-k,\beta_3,\ldots,\beta_n;\lambda+1,\lambda+1,\gamma_3,\ldots,\gamma_n;x_1,\ldots,x_n]\\ &=\frac{(\lambda+1)_{m+1}}{m!(\alpha-1)}\,(x_1-x_2)^{-1}\\ &\quad\cdot\Gamma_A^{(n)}[(\alpha-1,x),-m,-m-1,\beta_3,\ldots,\beta_n;\lambda+1,\lambda+1,\gamma_3,\ldots,\gamma_n;x_1,\ldots,x_n]\\ &\quad+x_1\rightleftharpoons x_2 \qquad \big(x\geqq 0;\ \alpha\neq 1;\ \Re(\alpha)>0 \quad \textit{when} \quad x=0\big),\end{aligned} \tag{6.4}$$

where $x_1 \rightleftharpoons x_2$ indicates the presence of a second term that originates from the first term by interchanging x_1 and x_2.

Proof. Applying the relationship (3.4) and the following known result (see, *e.g.*, [21, p. 206, Eq. (10)]):

$$\begin{aligned}\sum_{k=0}^{m}\frac{k!}{(\lambda+1)_k}\,&L_k^{(\lambda)}(x)L_k^{(\lambda)}(y)\\ &=\frac{(m+1)!}{(\lambda+1)_m}\,(x-y)^{-1}\left[L_m^{(\lambda)}(x)L_{m+1}^{(\lambda)}(y)-L_{m+1}^{(\lambda)}(x)L_m^{(\lambda)}(y)\right]\end{aligned} \tag{6.5}$$

to the integral representation (3.1), we get the desired finite sum formula (6.5) asserted by Theorem 6.3. □

Remark 6.4. By suitably iterating the above process, we obtain

$$\sum_{k_1=0}^{m_1}\sum_{k_2=0}^{m_2}\sum_{k_3=0}^{m_3}\binom{a_1+k_1}{k_1}\binom{a_2+k_2}{k_2}\binom{a_3+k_3}{k_3}$$
$$\cdot\,\Gamma_A^{(n)}\big[(\alpha,x),-k_1,-k_1,-k_2,-k_2,-k_3,-k_3,\beta_7,\ldots,\beta_n;$$
$$a_1+1,a_1+1,a_2+1,a_2+1,a_3+1,a_3+1,\gamma_7,\ldots,\gamma_n;x_1,\ldots,x_n\big]$$
$$=\frac{(a_1+1)_{m_1+1}(a_2+1)_{m_2+1}(a_3+1)_{m_3+1}}{m_1!\;m_2!\;m_3!(\alpha-1)(\alpha-2)(\alpha-3)}(x_1-x_2)^{-1}\Big[(x_3-x_4)^{-1}\Big\{(x_5-x_6)^{-1}$$
$$\cdot\,\Gamma_A^{(n)}\big[(\alpha-3,x),-m_1,-m_1-1,-m_2,-m_2-1,-m_3,-m_3-1,\beta_7,\ldots,\beta_n;$$
$$a_1+1,a_1+1,a_2+1,a_2+1,a_3+1,a_3+1,\gamma_7,\ldots,\gamma_n;x_1,\ldots,x_n\big]$$
$$+\,x_5\rightleftharpoons x_6\Big\}+x_3\rightleftharpoons x_4\Big]+x_1\rightleftharpoons x_2, \tag{6.6}$$

where the right-hand side obviously has 2^3 terms. Similarly, we can derive a more general multiple finite summation formula for $\Gamma_A^{(n)}$ in the following form:

$$\sum_{k_1=0}^{m_1}\cdots\sum_{k_s=0}^{m_s}\binom{a_1+k_1}{k_1}\cdots\binom{a_s+k_s}{k_s}\cdot\,\Gamma_A^{(2s)}[(\alpha,x),-k_1,-k_1,\ldots,-k_s,$$
$$-\,k_s;a_1+1,a_1+1,\ldots,a_s+1,a_s+1;x_1,\cdots,x_{2s}], \tag{6.7}$$

whose detailed expression is being left as an exercise for the interested reader.

The special cases of the identities in this section when $x=0$ are seen to reduce to the corresponding known results due to Padmanabham and Srivastava [19]. Moreover, the special cases of the results in this section when $x=0$ and $n=2$ can be seen to yield the known identities due to Srivastava [24].

7. The Incomplete Lauricella Function $\Gamma_A^{(2s)}$ as an s-Fold Sum

By interpreting the first two ${}_1F_1$ functions occurring on the right-hand side of (3.1) as a Cauchy product, it is easily seen that the incomplete Lauricella function $\Gamma_A^{(n)}$ can be expressed as a series whose terms are composed of ${}_3F_2$ and $\Gamma_A^{(n-2)}$ as follows:

$$\Gamma_A^{(n)}[(\alpha,x),\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n]$$
$$=\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta_1)_m}{(\gamma_1)_m}\frac{x_1^m}{m!}\,{}_3F_2\left[\begin{array}{c}-m,1-\gamma_1-m,\beta_2;\\ \\ 1-\beta_1-m,\gamma_2;\end{array}\;-\frac{x_2}{x_1}\right]$$
$$\cdot\,\Gamma_A^{(n-2)}[(\alpha+m,x),\beta_3,\ldots,\beta_n;\gamma_3,\ldots,\gamma_n;x_3,\ldots,x_n] \tag{7.1}$$
$$(x\geqq 0;\;|x_1|+\cdots+|x_n|<1\quad\text{when}\quad x=0).$$

More generally, by iterating the above process s times, this last sum formula (7.1) would finally express $\Gamma_A^{(2s)}$ as a multiple series whose terms are s-tuple products of the hypergeometric ${}_3F_2$ functions:

$$\Gamma_A^{(2s)}[(\alpha,x),\beta_1,\ldots,\beta_{2s};\gamma_1,\ldots,\gamma_{2s};x_1,\ldots,x_{2s}] = \sum_{m_1,\ldots,m_s=0}^{\infty} [\alpha;x]_{m_1+\cdots+m_s}$$

$$\cdot \prod_{j=1}^{s} \left\{ \frac{(\beta_{2j-1})_{m_j}}{(\gamma_{2j-1})_{m_j}} \frac{x_{2j-1}^{m_j}}{m_j!} \, {}_3F_2 \left[\begin{array}{c} -m_j, 1-\gamma_{2j-1}-m_j, \beta_{2j}; \\ 1-\beta_{2j-1}-m_j, \gamma_{2j}; \end{array} -\frac{x_{2j}}{x_{2j-1}} \right] \right\} \tag{7.2}$$

$$(x \geqq 0;\ |x_1|+\cdots+|x_{2s}|<1 \quad \text{when} \quad x=0).$$

Remark 7.1. The special case of (7.1) when $n=2$ can easily be rewritten in terms of the incomplete Appell function Γ_2. Also the special case of (7.1) when $x=0$ yields a known result (see, *e.g.*, [36, p. 181, Problem 38(ii)]). Furthermore, by setting $x=0$ in the results presented in this section, we are led to the corresponding known identities due to Padmanabham and Srivastava [19].

8. Potential Areas of Application of Incomplete Hypergeometric Functions

The familiar decomposition of the gamma function $\Gamma(z)$ into the incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$ is well-recognized to be a non-trivial step as the closed-form solution of a considerably large number of problems in (for example) applied mathematics, astrophysics, nuclear and molecular physics, statistics and engineering, transport theory and fluid flow, diffraction and plasma wave problems, number theory and random walks, Lorentz-Doppler line broadening, design of particle acceleration, and so on, can be expressed in terms of the incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$ defined by (1.1) and (1.2), respectively.

In their pioneering work, Srivastava *et al.* [30] showed that the generalized incomplete hypergeometric functions

$${}_p\gamma_q \quad (p,q\in\mathbb{N}_0) \qquad \text{and} \qquad {}_p\Gamma_q \quad (p,q\in\mathbb{N}_0)$$

are useful in engineering and applied sciences. In particular, they applied these generalized incomplete hypergeometric functions in such diverse areas as (for example) communication theory, probability theory and groundwater pumping modelling. The generalized Marcum q- and Q-functions given, in terms of the *modified* Bessel function $I_\nu(z)$ in (3.7), by

$$q_M(\alpha,\beta) := \frac{1}{\alpha^{M-1}} \int_0^\beta t^M \, e^{-\frac{1}{2}(t^2+\alpha^2)} \, I_{M-1}(\alpha t)dt \tag{8.1}$$

and

$$Q_M(\alpha,\beta) := \frac{1}{\alpha^{M-1}} \int_\beta^\infty t^M \, e^{-\frac{1}{2}(t^2+\alpha^2)} \, I_{M-1}(\alpha t)dt, \tag{8.2}$$

respectively, arise in performance analysis of several types of communications (see, for details and also for citations of related earlier works, [30]). Their special case when $M = 1$ were introduced initially by Marcum [15] during the study of the statistical theory of target detection by pulsed radar. Annamalai and Tellambura [3] studied the Cauchy-Schwarz bounds on these functions and discussed their applications in wireless communications. Simon and Alouini (see [22]) applied these functions in the unified study of *digital communication over fading channels.* As a matter of fact, the generalized Marcum functions in (8.1) and (8.2) are very special cases of the generalized incomplete hypergeometric functions

$${}_p\gamma_q \quad (p, q \in \mathbb{N}_0) \qquad \text{and} \qquad {}_p\Gamma_q \quad (p, q \in \mathbb{N}_0)$$

as specified below:

$$q_M(\sqrt{2\omega}, \sqrt{2x}) = e^{-\omega} \; {}_1\gamma_1 \left[\begin{array}{cc} (M,x); & \\ & \omega \\ M; & \end{array} \right] \tag{8.3}$$

and

$$Q_M(\sqrt{2\omega}, \sqrt{2x}) = e^{-\omega} \; {}_1\Gamma_1 \left[\begin{array}{cc} (M,x); & \\ & \omega \\ M; & \end{array} \right], \tag{8.4}$$

which, in light of (1.10), yield the following decomposition formula:

$$q_M(\alpha,\beta) + Q_M(\alpha,\beta) = 1 \tag{8.5}$$

satisfied by the generalized Marcum functions defined above by (8.1) and (8.2).

In view of the above-mentioned developments, therefore, it is quite natural to expect that the incomplete hypergeometric functions in two and more variables, too, will provide closed-form solutions to a variety of problems in at least some of the many diverse areas of science and engineering. For example, the various integral formulas and integral representations (which are given in this paper) would substantially aid in the evaluation of single, double and multiple definite integrals involving simpler complete and incomplete hypergeometric functions in one and more variables. In particular, the Eulerian type integral representations may be interpreted as the familiar Riemann-Liouville fractional integrals. These integrals, together with the Laplace transform formulas, are potentially useful in solving some families of fractional differential equations (see, for details, [10]). Multivariable hypergeometric functions and their incomplete counterparts are, of course, useful also in solving systems of partial differential equations.

9. Concluding Remarks and Observations

In our present investigation, with the help of the incomplete Pochhammer symbols $(\lambda; x)_\nu$ and $[\lambda; x]_\nu$, we have introduced the incomplete Lauricellla functions $\gamma_A^{(n)}$ and $\Gamma_A^{(n)}$ of n variables, whose special cases when $n = 1$ and $n = 2$ reduce to the incomplete Gauss hypergeometric functions and the incomplete second Appell functions of two variables (see [30] and [6]), respectively. We have investigated their such diverse properties as integral representations and finite summation formulas. The special cases of the results obtained in this paper when $x = 0$ would reduce to the corresponding known results for the Appell and Lauricellla functions (see, for details, [4, 19, 24, 35, 36]). We have provided relevant connections of some of the special cases of the main results derived here with known identities. Several potential areas of application of these incomplete hypergeometric functions in one, two and more variables have also been indicated.

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