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Results of Graded Local Cohomology Modules with respect to a Pair of Ideals

Fatemeh Dehghani-Zadeh
Department of Mathematics, Yazd Branch, Islamic Azad University, Yazd, Iran
e-mail : dehghanizadeh@iauyazd.ac.ir, fdzadeh@gmail.com

Abstract. Let $R = \bigoplus_{n\in Z} R_n$ be a graded commutative Noetherian ring and let I be a graded ideal of R and J be an arbitrary ideal. It is shown that the i-th generalized local cohomology module of graded module M with respect to the (I, J), is graded. Also, the asymptotic behaviour of the homogeneous components of $H^i_{I,J}(M)$ is investigated for some i's with a specified property.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring with identity and M is an R-module. Also, it is denoted the set of integers (resp. non-negative integers) by $\mathbb{Z}$ (resp. $\mathbb{N}_0$). For a pair of ideals (I, J), the (I, J)-torsion submodule of M is $\{x \in M \mid I^n x \subseteq Jx \quad \text{for} \quad n \gg 1\}$. Let $\xi(R)$ denote the category of R-modules and R-homomorphisms. It is well known that $\Gamma_{I,J}(-) : \xi(R) \to \xi(R)$ is a covariant, R-linear and left exact functor. For a non-negative integer i, the i-th right derived functor of $\Gamma_{I,J}(-)$ is denoted by $H^i_{I,J}(-)$ and is called the i-th local cohomology functor with respect to (I, J). Note that if $J = 0$, then $H^i_{I,J}(-)$ coincides with the ordinary local cohomology functor $H^i_I(-)$, with the support in the closed subset $V(I)$.

In this note, we study the graded structure of the i-th local cohomology module of M defined by a pair of ideals (I, J) and obtain some finiteness results for the homogeneous pieces of the module $H^i_{I,J}(M)$. This paper is divided into three sections. In the second section of the paper, we show that if R is graded ring, M is graded R-module and I is a graded ideal of R, then $H^i_{I,J}(M)$ is graded R-module for each $i \in \mathbb{N}_0$.

Throughout section 3, $R = \bigoplus_{n\geq 0} R_n$ is a graded commutative Noetherian ring, where the base ring R_0 is a commutative Noetherian ring with maximal ideal $\mathfrak{m}_0$. Moreover, $R_+ = \bigoplus_{n>0} R_n$ is the irrelevant ideal of R, also I and J are two graded

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ideal of R and $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is finitely generated graded R-module. Furthermore, the notation $H^i_{I,J}(M)_n$ is applied to denote the n-th graded component of $H^i_{I,J}(M)$ for each $n \in \mathbb{Z}$. The cohomological dimension of M with respect to (I, J) is denoted by $\mathrm{cd}_{I,J}(M)$. Thus, $\mathrm{cd}_{I,J}(M)$ is the largest non-negative integer i such that $H^i_{I,J}(M)$ is not equal to zero. The finiteness dimension of M with respect to (I, J) is denoted by $\mathrm{f}_{I,J}(M)$, where $\mathrm{f}_{I,J}(M)$ is the least non-negative integer i such that $H^i_{I,J}(M)$ is not finitely generated. In the section 3, using the above result, the asymptotic behaviour of R_0-module $H^i_{I,J}(M)_n$ is considered as n tends to $-\infty$ in the following cases:

(i) $I \bigcap R_0 \subseteq J \bigcap R_0$ and $i \leq \mathrm{f}_{I,J}(M)$,

(ii) $I \bigcap R_0 \subseteq J \bigcap R_0$ and $i \geq \mathrm{cd}_{I,J}(M)$.

For any unexplained notations and terminologies we refer the reader to [2] and [7].

This work is motivated by articles of Zamani [10] and Khashyarmanesh [4]. The primary results of the research is presented and published in the Proceeding of the international conference on Algebra in Honour of Patrick Smith and John Clark's 70^{th} Birthdays, August 2013. While editing this paper, we realized that current paper is in the same area of the papers [3] and [5], however it is definitely understood that is with different approach.

2. Graded Structure of $H^i_{I,J}(M)$

Let R denotes a commutative Noetherian ring, and let I and J be two ideals of R. Suppose that $\xi(R)$ is the category of R-modules. As usual, $\Gamma_I(\ ) : \xi(R) \to \xi(R)$ is a functor that assigns to each $M \in \xi(R)$ the module $\Gamma_I(M) = \{x \in M \mid I^t x = 0 \text{ for some } t \in \mathbb{N}_0\}$. The functor $\Gamma_I(\ )$ is covariant, R-linear and left exact. For a non-negative integer i, the i-th right derived functor of $\Gamma_I(\ )$ is denoted by $H^i_I(\ )$ and is called the i-th local cohomology functor with support in $V(I)$.

The following generalization of the local cohomology functor has been presented by Takahashi, Yoshino and Yoshizawa [9]. The i-th generalized local cohomology functor, $H^i_{I,J}(\quad) : \xi(R) \longrightarrow \xi(R)$ is defined by $H^i_{I,J}(M) = H^i(\Gamma_{I,J}(M))$ for all M *in* $\xi(R)$. By using [9, Theorem 3.2], there is a natural isomorphism $H^i_{I,J}(M) \cong \varinjlim_{\mathfrak{a}\in\widetilde{W}(I,J)} H^i_{\mathfrak{a}}(M)$ for any integer i, where $\widetilde{W}(I, J)$ is the set of ideals $\mathfrak{a}$ of R such that $I^n \subseteq \mathfrak{a} + J$ for some integer n.

Now, let $R = \bigoplus_{n\in\mathbb{Z}} R_n$ be a $\mathbb{Z}$-graded commutative ring, I be a graded ideal of R, and J an arbitrary ideal of R and suppose that the R-module M is graded. In the following, the methods of grading $H^i_{I,J}(M)$ are indicated. In this direction, let ${}^*\widetilde{W}(I, J)$ denotes the set of graded ideals $\mathfrak{b}$ of R such that $I^n \subseteq \mathfrak{b} + J$ for some integer n. We define a partial order on ${}^*\widetilde{W}(I, J)$ be letting $\mathfrak{a} \leq \mathfrak{b}$ if $\mathfrak{a} \supseteq \mathfrak{b}$ for all $\mathfrak{a}$ and $\mathfrak{b} \in^* \widetilde{W}(I, J)$. In order relation on ${}^*\widetilde{W}(I, J)$ and the inclusion maps make $\{\Gamma_{\mathfrak{a}}(M)\}_{\mathfrak{a}\in^*\widetilde{W}(I,J)}$ into direct system of graded R-module. We denote the direct limit of this system by ${}^*\Gamma_{I,J}(M) = \varinjlim_{\mathfrak{a}\in^*\widetilde{W}(I,J)} \Gamma_{\mathfrak{a}}(M)$. It follows from [1, Remark

12.2.8] that the module ${}^*\Gamma_{I,J}(M)$ is graded module and whenever $f : M \to M'$ is a homogeneous homomorphism, then ${}^*\Gamma_{I,J}(M) \longrightarrow^* \Gamma_{I,J}(M')$ is homogeneous. Thus ${}^*\Gamma_{I,J}(-)$ can be viewed as a (left exact, addition) functor from ${}^*\xi(R)$ to itself (Here ${}^*\xi(R)$ denotes the category of graded R-modules and homogeneous,) and so we can form its right derived functors ${}^*H^i_{I,J}$ $(i \in \mathbb{N}_0)$ on that category. This will produce, for a graded R-module M, graded local cohomology modules ${}^*H^i_{I,J}(M)$, which are constructed by the following procedure. Since ${}^*\xi(R)$ has enough injectives, we can construct an injective resolution of M in this category, that is, we can construct an exact sequence

$$0 \longrightarrow M \xrightarrow{\alpha} E^0 \xrightarrow{d^0} E^1 \longrightarrow \cdots \longrightarrow E^i \xrightarrow{d^i} E^{i+1} \longrightarrow \cdots$$

in ${}^*\xi(R)$ in which the E^i $(i \in \mathbb{N}_0)$ are injective objects in that category, then we apply the functor ${}^*\Gamma_{I,J}$ to the complex $0 \longrightarrow E^0 \xrightarrow{d^0} E^1 \longrightarrow \cdots \longrightarrow E^i \xrightarrow{d^i} E^{i+1} \longrightarrow \cdots$ the i-th cohomology module of the resulting complex is ${}^*H^i_{I,J}(M) = \mathrm{Ker}^*\Gamma_{I,J}(d^i)/\mathrm{Im}^*\Gamma_{I,J}(d^{i+1})$.

Now, let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of graded R-modules. For each $i \in \mathbb{N}_0$, there is a homogeneous connecting homomorphism ${}^*H^i_{I,J}(M'') \longrightarrow^* H^{i+1}_{I,J}(M')$ and these connecting homomorphisms make the resulting homogeneous long exact sequence

$$\begin{aligned} 0 \longrightarrow^* H^0_{I,J}(M') \longrightarrow^* H^0_{I,J}(M) \longrightarrow^* H^0_{I,J}(M'') \longrightarrow \cdots \\ \longrightarrow^* H^i_{I,J}(M') \longrightarrow^* H^i_{I,J}(M) \longrightarrow^* H^i_{I,J}(M'') \longrightarrow^* H^{i+1}_{I,J}(M') \longrightarrow \cdots . \end{aligned}$$

Obviously, for all $i \in \mathbb{N}_0$ the induced homomorphisms ${}^*H^i_{I,J}(M') \to^* H^i_{I,J}(M)$ and ${}^*H^i_{I,J}(M) \to^* H^i_{I,J}(M'')$ and all the connecting homomorphisms ${}^*H^i_{I,J}(M'') \to^* H^{i+1}_{I,J}(M')$ are homogeneous. Therefore $({}^*H^i_{I,J})_{i\in\mathbb{N}_0}$ is a negative strongly connected sequence of covariant functors from ${}^*\xi(R)$ to itself.

Lemma 2.1. *Assume that J is an ideal of R and that the ideal I is graded. Let M be a graded R-module. Then ${}^*\Gamma_{I,J}(M) = \Gamma_{I,J}(M)$.*

Proof. Suppose $x \in^* \Gamma_{I,J}(M)$. Then there is $\mathfrak{a} \in^* \widetilde{W}(I,J)$ such that $x \in \Gamma_{\mathfrak{a}}(M)$. Since ${}^*\widetilde{W}(I,J) \subseteq \widetilde{W}(I,J)$, $x \in \Gamma_{I,J}(M)$.

Conversely, let $x \in \Gamma_{I,J}(M)$. Therefore $I^n \subseteq \mathrm{Ann}x + J$ for some $n \in \mathbb{Z}$. Assume that $x = x_1 + x_2 + \cdots + x_k$ where $x_i \in M_i$. Setting $\mathfrak{a} = (\mathrm{Ann}x_1)(\mathrm{Ann}x_2)\cdots(\mathrm{Ann}x_k)$, we have $\mathfrak{a} \in^* \widetilde{W}(I,J)$, $x \in \Gamma_{\mathfrak{a}}(M)$, and the proof is completed. □

Theorem 2.2. *Let M be a graded R-module. Let I be a graded ideal, and J an arbitrary ideal of R. Then, there is a natural isomorphism $H^i_{I,J}(M) \cong^* H^i_{I,J}(M)$ for any integer i.*

Proof. Since $H^i_{I,J}$ and ${}^*H^i_{I,J}$ are negative strongly connected sequence of covariant functors from ${}^*\xi(R)$ to ${}^*\xi(R)$. Using [1, Proposition 12.1.3] and [9, Proposition

1.11], $H^i_{I,J}(E) =^* H^i_{I,J}(E) = 0$ for all $i > 0$ and all *injective R-module E. Now the result follows from Lemma 2.1 and [1, Exercise 12.1.7]. $\square$

Theorem 2.3. *Let M be a graded R-module and suppose that I is a graded ideal and J is an ideal of R. Then, there is a natural isomorphism ${}^*H^i_{I,J}(M) \cong \varinjlim_{\mathfrak{a}\in^*\widetilde{W}(I,J)} H^i_{\mathfrak{a}}(M)$.*

Proof. Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of graded R-modules and homogeneous homomorphisms. Then it implies a long exact sequence

$$0 \longrightarrow H^0_{\mathfrak{a}}(L) \longrightarrow H^0_{\mathfrak{a}}(M) \longrightarrow H^0_{\mathfrak{a}}(N) \longrightarrow \cdots \\ \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(N) \longrightarrow H^{i+1}_{\mathfrak{a}}(L) \longrightarrow \cdots$$

for $\mathfrak{a} \in^* \widetilde{W}(I,J)$. Since taking the direct limit is an exact functor, we obtain

$$0 \longrightarrow \varinjlim_{\mathfrak{a}\in^*\widetilde{W}(I,J)} H^0_{\mathfrak{a}}(L) \longrightarrow \varinjlim_{\mathfrak{a}\in^*\widetilde{W}(I,J)} H^0_{\mathfrak{a}}(M) \longrightarrow \varinjlim_{\mathfrak{a}\in^*\widetilde{W}(I,J)} H^0_{\mathfrak{a}}(N) \longrightarrow \cdots \\ \longrightarrow \varinjlim_{\mathfrak{a}\in^*\widetilde{W}(I,J)} H^i_{\mathfrak{a}}(M) \longrightarrow \varinjlim_{\mathfrak{a}\in^*\widetilde{W}(I,J)} H^i_{\mathfrak{a}}(N) \longrightarrow \varinjlim_{\mathfrak{a}\in^*\widetilde{W}(I,J)} H^{i+1}_{\mathfrak{a}}(L) \longrightarrow \cdots.$$

On the other hand, for any *injective R-module E and any positive integer i, in view of [1, Proposition 12.1.3], we have $H^i_{\mathfrak{a}}(E) = 0$ for each $\mathfrak{a} \in^* \widetilde{W}(I,J)$. Thus we have $\varinjlim_{\mathfrak{a}\in^*\widetilde{W}(I,J)} H^i_{\mathfrak{a}}(E) = 0$. Therefore, the Lemma 2.1 with [1, Exercise 12.1.7] imply the desired isomorphism. $\square$

Remark 2.4. (A) ([9, Theorem 2.4]) Let I be a graded ideal of R and suppose that J is an arbitrary ideal of R, and that M is a graded R-module. Let $\mathfrak{a} = \mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_n \quad (n > 0)$ be homogeneous elements of R which generate I. Then, by [2, Exercises 1.5.19] and [9, Definition 2.2], $C^\bullet_{\mathfrak{a},J} = \bigotimes_{i=1}^n C^\bullet_{\mathfrak{a}_i,J}$ is in ${}^*\xi(R)$, then, for a fixed projective resolution $\mathfrak{p}_0$ of M in ${}^*\xi(R)$, the complex $C^\bullet_{\mathfrak{a},J} \bigotimes \mathfrak{p}_0$ is in ${}^*\xi(R)$. Therefore $H^i(C^\bullet_{\mathfrak{a},J} \bigotimes M)$ has a graded R-module structure; and so $H^i_{I,J}(M)$ inherits a grading in view of [9, Theorem 2.4].
(B) Briefly, if we use the isomorphism of connected sequences $\varinjlim_{\mathfrak{a}\in^*\widetilde{W}(I,J)} H^0_{\mathfrak{a}}(-) \cong H^i_{I,J}(-)$ of [9, Theorem 3.2] to define gradings on the $H^i_{I,J}(M) \quad (i \in \mathbb{N})$ (for graded R-module M), or if we use the isomorphism of connected sequences $H^i_{I,J}(-) \cong H^i(C^\bullet_{\mathfrak{a},J} \bigotimes -)$ of [9, Theorem 2.4], with any choice of homogeneous generators for I, to define gradings on the $H^i_{I,J}(M) \quad (i \in \mathbb{N})$, the resulting gradings are always the same, and are precisely those with respect to which $(H^i_{I,J}\,)_{i\in\mathbb{N}}$ has the *restriction property: see [1, Definition 12.3.5]. Furthermore, as, when these gradings are imposed, there is a unique isomorphism $(H^i_{I,J}\lceil_{^*\xi})_{i\in\mathbb{N}_0} \cong^* (H^i_{I,J})_{i\in\mathbb{N}_0}$ of connected sequences of covariant functors from ${}^*\xi(R)$ to itself. Thus in the rest of this note we give the module $H^i_{I,J}(M)$ with one of the above gradings and we use the notation $H^i_{I,J}(M)_n$ for the n-th homogeneous piece of that module.

It should be mentioned here that some basic properties of local cohomology module cannot exted to generalized local cohomology module. For example, it is known, $H^i_{R_+}(M)_n$ is a finitely generated R_0-module for all $n \in \mathbb{Z}$ and it vanishes for all $n \gg 0$. The following example show that it is not same for $H^i_{I,J}(M)_n$.

Example 2.5. Let $d > 0$ and $(R_0, \mathfrak{m}_0)$ be a d-dimensional local ring. Now put $M = \oplus_{n\in\mathbb{Z}} M_n$, where $M_n = \frac{R}{R_+} = R_0$. Then M has a natural finitely generated graded R-module such that $\Gamma_{R_+}(M) = M$. Consequently, for all $n \in \mathbb{Z}$ we have $H^d_{\mathfrak{m},R_+}(M)_n = H^d_{\mathfrak{m}_0,R_+}(M)_n = H^d_{\mathfrak{m}_0,R_+}(\Gamma_{R_+}(M))_n = H^d_{\mathfrak{m}_0}(M)_n = H^d_{\mathfrak{m}_0}(R_0)$ for all $n \in \mathbb{Z}$. Therefore $H^d_{\mathfrak{m},R_+}(M)_n$ is not finitely generated; and hence non-zero.

3. The Results

We keep the notations and hypotheses presented in the introduction and study the asymptotic behaviour of R_0-module $H^i_{I,J}(M)_n$ as n tends to $-\infty$. Thus the following statements are considered.

(i) $\mathrm{Ass}_{R_0} H^i_{I,J}(M)_n$ is asymptotically stable for $n \to -\infty$, if there is some $n_0 \in \mathbb{Z}$ such that $\mathrm{Ass}_{R_0} H^i_{I,J}(M)_n = \mathrm{Ass}_{R_0} H^i_{I,J}(M)_{n_0}$ for all $n \leq n_0$,

(ii) R-module $H^i_{I,J}(M)$ is said to be tame if there is some $n_0 \in \mathbb{Z}$ such that $H^i_{I,J}(M)_n = 0$ for all $n \leq n_0$ or else $H^i_{I,J}(M_n) \neq 0$ for all $n \leq n_0$. Note that, all graded Artinian R-modules are tame.

For ease in access, we firstly quote some known observations about generalized local cohomology modules. At this stage the following remark is needed.

Remark 3.1. Let I and J be two ideals of R, and M be an R-module.

(i) If M is a J-torsion module, then there is a isomorphism $H^i_{I,J}(M) \to H^i_I(M)$ for all $i \geq 0$.

(ii) If J' is an ideal of R such that $J' \subseteq J$, then $H^i_{I+J',J}(M) \cong H^i_{I,J}(M)$ for all integers i.

(iii) If there exists $\mathfrak{a} \in \widetilde{W}(I,J)$ such that $\Gamma_{\mathfrak{a}}(M) = M$, then $\Gamma_{I,J}(M) = M$ and if $\Gamma_{I,J}(M) = 0$, then $\Gamma_{\mathfrak{a}}(M) = 0$ for all $\mathfrak{a} \in \widetilde{W}(I,J)$ and therefore if $\Gamma_{I,J}(M) = 0$, then I contains a non-zero-divisor on M.

Theorem 3.2. *Let $x \in R_+$ and $(0 :_{H^i_{I,J}(M)} x)_n$ is a finitely generated R_0-module for all n and $H^i_{I,J}(M)_n = 0$ for all $n \gg 0$. Then $H^i_{I,J}(M)_n$ is finitely generated as R_0-module for all n.*

Proof. Set $H^i_{I,J}(M) = T$. Since $T_n = 0$ for all $n \gg 0$. Then there exists $n_0 \in \mathbb{N}$ such that $T_n = 0$ for all $n > n_0$. For any given integer p, since $x^{n_0+1-p} T_p \subseteq T_{n_0+1} = 0$, $T_p \subseteq (0 :_T x^{n_0+1-p})_p$. Then, it suffices to prove that for any given t, the R_0-module $(o :_T x^t)_n$ is finitely generated for all n. We prove, by induction on t. Assume that $t > 1$, and this claim is true for the case $t-1$. Assume that $deg(x) = \iota$. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & (0:_T x^{t-1})_n & \longrightarrow & T_n & \longrightarrow & \big(T/(0:_T x^{t-1})\big)_n & \longrightarrow & 0 \\
& x\big\downarrow\alpha & & x\big\downarrow\beta & & x\big\downarrow\gamma & & \\
0 \longrightarrow & (0:_T x^{t-1})_{n+\iota} & \longrightarrow & T_{n+\iota} & \longrightarrow & \big(T/(0:_T x^{t-1})\big)_{n+\iota} & \longrightarrow & 0.
\end{array}$$

Using the snake Lemma to get the following exact sequence $0 \to \mathrm{Ker}\alpha \to \mathrm{Ker}\beta \to \mathrm{Ker}\gamma \to \mathrm{Coker}\alpha \to \mathrm{Coker}\beta \to \mathrm{Coker}\gamma \to 0$. Since both $\mathrm{Ker}\beta$ and $\mathrm{Coker}\alpha$ are finitely generated R_0-modules, it follows $\mathrm{Ker}\gamma = \Big(\big((0:_T x^{t-1}):_T x\big)/(0:_T x^{t-1})\Big)_n$ is finitely generated R_0-module. Therefore, in view of the short exact sequence

$$0 \to (0:_T x^{t-1})_n \to \big((0:_T x^{t-1}):_T x\big)_n \to \Big(\big((0:_T x^{t-1}):_T x\big)/(0:_T x^{t-1})\Big)_n \longrightarrow 0$$

in conjunction with the fact that $\big((0:_T x^{t-1}):_T x\big)_n \cong (0:_T x^t)_n$ the inductive step is completed and the result follows by induction. □

Theorem 3.3. *Suppose that $(R_0, \mathfrak{m}_0)$ is local. Let I and J be two graded ideals of R such that $I \cap R_0 \subseteq J \cap R_0$. Suppose that $H^i_{I,J}(M)_n = 0$ for all $n \gg 0$ and $\sqrt{(I+J)\cap R_+} = \sqrt{R_+}$. Then $cd_{I,J}(M) = dim(M/(\mathfrak{m}_0 + J)M)$.*

Proof. We prove this by induction on $d = \dim\big(M/(\mathfrak{m}_0 + J)M\big)$. If $d = 0$, then $\Gamma_{I,J}(M) = M \neq 0$ and $H^i_{I,J}(M) = 0$ for all $i > 0$. Now assume that the statement is proved for any finitely generated graded R-module N such that $\dim(N/(\mathfrak{m}_0 + J)N) = d-1$. By our hypothesis and using Remark 3.1(ii), it is supposed that I is generated by homogeneous elements of positive degree. In view of Remark 3.1(iii), it is replaced M with $M/\Gamma_{I,J}(M)$, so that it is assumed that $\Gamma_{I,J}(M) = 0$. It follows that $\Gamma_I(M) = 0$. Then there exists a homogeneous element $x \in I$ which avoids all members of $\mathrm{Ass}(M)$ and $\mathrm{Ass}\big(M/(J + \mathfrak{m}_0)M\big)$. It is clear $\dim\big(M/xM/(\mathfrak{m}_0 + J)M/xM\big) = d-1$. Therefore, the exact sequence $0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$ yields a long exact sequence

$$H^{i-1}_{I,J}(M) \xrightarrow{x} H^{i-1}_{I,J}(M) \longrightarrow H^{i-1}_{I,J}(M/xM) \longrightarrow H^i_{I,J}(M) \xrightarrow{x} H^i_{I,J}(M),$$

which shows that $H^i_{I,J}(M) \cong xH^i_{I,J}(M)$ for all $i > d$. As $x \in R_+$, it follows that $H^i_{I,J}(M) = 0$ for all $i > d$. It thus remains to show that $H^d_{I,J}(M) \neq 0$. By using the exact sequence $0 \longrightarrow (\mathfrak{m}_0 + J)M \longrightarrow M \longrightarrow M/(\mathfrak{m}_0 + J)M \longrightarrow 0$, the exact sequence

$$H^i_{I,J}\big((\mathfrak{m}_0 + J)M\big) \longrightarrow H^i_{I,J}(M) \longrightarrow H^i_{I,J}\big(M/(\mathfrak{m}_0 + J)M\big) \text{ is obtained.}$$

Also, from $\dim\big((\mathfrak{m}_0 + J)M/(\mathfrak{m}_0 + J)(\mathfrak{m}_0 + J)M\big) \leq d$ it is resulted $H^i_{I,J}\big((\mathfrak{m}_0 + J)M\big) = 0$ and $H^i_{I,J}(M) \cong H^i_{I,J}\big(M/(\mathfrak{m}_0 + J)M\big)$ for all $i > d$. Since $M/(\mathfrak{m}_0 + J)M$ is J-torsion. It follows that $H^i_{I,J}\big(M/(\mathfrak{m}_0 + J)M\big) \cong H^i_I\big(M/(\mathfrak{m}_0 + J)M\big)$ for all i. By assumption on $I + J$, we also have $H^i_I\big(M/(\mathfrak{m}_0 + J)M\big) \cong H^i_{\mathfrak{m}}\big(M/(\mathfrak{m}_0 + J)M\big)$ for all i. It follows $H^d_{I,J}(M) \neq 0$, see [1, Theorem 6.1.4]. □

Theorem 3.4. *Suppose that $(R_0, \mathfrak{m}_0)$ is local. Let I and J be graded ideals of R such that $I \cap R_0 \subseteq J \cap R_0$ and $H^i_{I,J}(M)_n = 0$ for all $n \gg 0$. Then $H^i_{I,J}(M)_n$ is finitely generated for all n.*

Proof. Set $D = \dim(M)$. By [9, Theorem 3.2] and using Grothendieck's vanishing theorem the result is clear for all $i > D$. It remains to show that $H^i_{I,J}(M)_n$ is finitely generated for all n and $i \leq D$. It is proved this by induction on i. The result is clear when $i = 0$. So let $i > 0$ and assume that the result has been proved for smaller values of i. Since $i > 0$, $H^i_{I,J}(M) \cong H^i_{I,J}(M/\Gamma_{I,J}(M))$. Hence, by replacing M with $M/\Gamma_{I,J}(M)$, it is assumed that there exists a homogeneous element $x \in I$ of degree t which is a non-zero-divisor on M. Now, the application of local cohomology with respect to a pair of ideals to the exact sequence $0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$ leads to an exact sequence of R_0-modules

$$H^{i-1}_{I,J}(M/xM)_{n+t} \longrightarrow H^i_{I,J}(M)_n \xrightarrow{x} H^i_{I,J}(M)_{n+t} \longrightarrow H^i_{I,J}(M/xM)_{n+t} \longrightarrow H^{i+1}_{I,J}(M)_n.$$

As x avoids all members of $\mathrm{Ass}(M)$, $\dim(M/xM) = D - 1$. Thus, we can deduced from the induction hypothesis that the R_0-module $H^{i-1}_{I,J}(M/xM)_{t+n}$ is finitely generated for all n. The above long exact sequence implies that $(0 :_{H^i_{I,J}(M)} x)_n$ is finitely generated as R_0-module for all n.

The result now follows by Theorem 3.2. □

Definition 3.5. An R-module T is called (I, J)-*cofinite* if $\mathrm{Ext}^i_R(R/I, T)$ is a finite R-module, for every $i \geq 0$ and $\mathrm{Supp}M \subseteq W(I, J)$, where $W(I, J)$ is the set of prime ideals $\mathfrak{p}$ of R such that $I^n \subseteq \mathfrak{p} + J$ for some positive integer n.

Theorem 3.6. *Make the assumptions of the statement of Theorem 3.3 and impose the further condition $\mathfrak{q}_0$ is an $\mathfrak{m}_0$-primary ideal. Then $H^i_{I,J}(M)/(\mathfrak{q}_0 + J)H^i_{I,J}(M)$ is Artinian and (I, J)-cofinite and so tame for all $i \geq d$.*

Proof. By Theorem 3.3, the result is clear for all $i > d$. It remains to show that $H^d_{I,J}(M)/(\mathfrak{q}_0 + J)H^d_{I,J}(M)$ is Artinian, and (I, J)-cofinite. As $\mathfrak{q}_0$ is an $\mathfrak{m}_0$-primary ideal, there is some $t \in \mathbb{N}$ such that $\mathfrak{m}_0^t \subseteq \mathfrak{q}_0$. It suffices to show that $H^d_{I,J}(M)/(\mathfrak{m}_0 + J)^t H^d_{I,J}(M)$ is Artinian. First it is proved that the module $H^d_{I,J}(M)/(\mathfrak{m}_0 + J)H^d_{I,J}(M)$ is Artinian. To do this, induction on $d = \dim(M/(\mathfrak{m}_0 + J)M)$ is used. If $d = 0$, then $\Gamma_{I,J}(M) = M$ and hence $M/(\mathfrak{m}_0 + J)M = \Gamma_{I,J}(M)/(\mathfrak{m}_0 + J)\Gamma_{I,J}(M)$. Therefore, since the radical of the annihilator of $\Gamma_{I,J}(M)/(\mathfrak{m}_0 + J)\Gamma_{I,J}(M)$ is equal to $\mathfrak{m}$, the R-module $\Gamma_{I,J}(M)/(\mathfrak{m}_0 + J)\Gamma_{I,J}(M)$ is Artinian and (I, J)-cofinite. Now suppose, inductively that $d > 0$, and the result has been proved for $d - 1$. In view of $H^d_{I,J}(M) \cong H^d_{I,J}(M/\Gamma_{I,J}(M))$, it suffices to consider the case where $\Gamma_{I,J}(M) = 0$. Hence, there is an element $x \in I$, such that x is a non-zero-divisor on M. Now, consider the exact sequence $0 \to M \xrightarrow{x} M \longrightarrow M/xM \to 0$ to get the following exact sequence

$$H^{d-1}_{I,J}(M) \longrightarrow H^{d-1}_{I,J}(M/xM) \longrightarrow H^d_{I,J}(M) \xrightarrow{x} H^d_{I,J}(M) \longrightarrow 0,$$

which, in turn, yields the exact sequence

$$H^{d-1}_{I,J}(M/xM)/(\mathfrak{m}_0+J)H^{d-1}_{I+J}(M/xM) \longrightarrow H^d_{I,J}(M)/(\mathfrak{m}_0+J)H^d_{I,J}(M) \xrightarrow{x} H^d_{I,J}(M)/(\mathfrak{m}_0+J)H^d_{I,J}(M) \longrightarrow 0.$$

By using the above exact sequence in conjunction with the inductive hypothesis to see that the R-module $\left(0 :_{H^d_{I,J}(M)/(\mathfrak{m}_0+J)H^d_{I,J}(M)} x\right)$ is Artinian and (I,J)-cofinite. Therefore, in view of $\Gamma_{(x)}\big(H^d_{I,J}(M)/(\mathfrak{m}_0+J)H^d_{I,J}(M)\big) = H^d_{I,J}(M)/(\mathfrak{m}_0+J)H^d_{I,J}(M)$, and using [6, Proposition 4.1], the module $H^d_{I,J}(M)/(\mathfrak{m}_0+J)H^d_{I,J}(M)$ is Artinian and (I,J)-cofinite.

Now, the result follows by an easy induction on t. □

The following corollary, which is an immediate consequence of Theorem 3.6, extends [8, Corollary 2.5].

Corollary 3.7. *In addition to Theorem 3.6 assume that J is ideal of R_0. Then $H^i_{I,J}(M)$ is tame for all $i \geq d$.*

Proof. It follows immediately by using Theorem 3.6 and Nakayama's Lemma. □

Remark 3.8. Let $R = \bigoplus_{n\geq 0} R_n$. Then the following hold:

(i) If $T = \bigoplus_{n\in\mathbb{Z}} T_n$ is a finitely generated graded R-module, then $T_n = 0$ for all $n \ll 0$.

(ii) If $T = \bigoplus_{n\in\mathbb{Z}} T_n$ is Noetherian (resp. Artinian) graded R-module, then T_n is Noetherian (resp. Artinian) R_0-module for all $n \in \mathbb{Z}$.

Theorem 3.9. *Suppose that $(R_0, \mathfrak{m}_0)$ is local. Let I and J be graded ideals of R such that $I \cap R_0 \subseteq J \cap R_0$ and let $i \leq f_{I,J}(M,N) = f$ and $H^i_{I,J}(M)_n = 0$ for all $n \gg 0$. Then $Ass_{R_0} H^i_{I,J}(M)$ is asymptotically stable as $n \to -\infty$.*

Proof. We prove our claim by induction of i. The case $i = 0$ is clear as $H^0_{I,J}(M)_n = \Gamma_{I,J}(M)_n = 0$ for all $n \ll 0$. It follows that $\mathrm{Ass}_{R_0}\Gamma_{I,J}(M)_n$ is empty for all $n \ll 0$. So, let $i > 0$ and assume that the result has been proved for smaller values of i. Since $H^i_{I,J}(M) \cong H^i_{I,J}(M/\Gamma_{I,J}(M))$ for all $i > 0$. Hence, by replacing M with $M/\Gamma_{I,J}(M)$, it is assumed that there exists a homogeneous element $x \in I$, say of degree t, which is a non-zero-divisor on M. Now, the application of local cohomology with respect to (I,J) to the exact sequence $0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$ leads to an exact sequence

$$H^j_{I,J}(M) \xrightarrow{x} H^j_{I,J}(M) \longrightarrow H^j_{I,J}(M/xM) \longrightarrow H^{j+1}_{I,J}(M) \xrightarrow{x} H^{j+1}_{I,J}(M).$$

This sequence shows that, for all $j < i-1$, $H^j_{I,J}(M/xM)_n = 0$ for all $n \ll 0$ and that $i-1 \leq f_{I,J}(M/xM)$. Hence, by inductive hypothesis, there exists a finite subset A of $\mathrm{Spec}(R_0)$ such that $\mathrm{Ass}_{R_0}(H^{i-1}_{I,J}(M/xM)_n) = A$ for all $n \ll 0$. Next, since $i \leq f_{I,J}(M)$, we may use Remark 3.8 to see that $H^{i-1}_{I,J}(M)_n = 0$ for all $n \ll 0$. Therefore, we have the exact sequence

$$0 \longrightarrow H^{i-1}_{I,J}(M/xM)_{n+t} \longrightarrow H^{i}_{I,J}(M)_n \xrightarrow{x} H^{i}_{I,J}(M)_{n+t},$$

which, in turn, yields $A \subseteq \mathrm{Ass}_{R_0}(H^i_{I,J}(M)_n) \subseteq A \bigcup \mathrm{Ass}_{R_0}(H^i_{I,J}(M)_{n+t})$ for all $n \ll 0$. This shows that $\mathrm{Ass}_{R_0}(H^i_{I,J}(M)_n) \subseteq \mathrm{Ass}_{R_0} H^i_{I,J}(M)_{n+t}$ for all $n \ll 0$. Since, by Theorem 3.4, the set $\mathrm{Ass}_{R_0}(H^i_{I,J}(M)_{n+t})$ is finite for all $n \ll 0$. This completes the inductive step and the proof. □

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