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Some Results on Locally Noetherian Modules and Locally Artinian Modules

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Abstract. We prove that if R is a commutative ring, then every maximal ideal of R is idempotent if and only if every locally noetherian (locally artinian) R-module is semisimple.

1. Introduction

Throughout this article, R is a commutative ring with unity. Let M be an R-module. The injective hull of M will be denoted by $E(M)$. We denote the *annihilator* of M (of an element $x \in M$) in R by $Ann(M)$ (by $Ann(x)$), i.e. $Ann(M) = \{r \in R \mid rM = 0\}$ and $Ann(x) = \{r \in R \mid rx = 0\}$. For any ideal $\mathfrak{a}$ of R, let $Ann_M(\mathfrak{a}) = \{x \in M \mid \mathfrak{a}x = 0\}$. We use the notations $N \subseteq M$ and $N \leq M$ to denote that N is a subset of M and N is a submodule of M, respectively. By $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ we denote the field of rational numbers, the ring of integers and the set of natural numbers, respectively. An R-module M is called *locally noetherian* (*locally artinian*) if all its finitely generated submodules are noetherian (artinian). Clearly, any noetherian (artinian) module is locally noetherian (locally artinian). Note also that the class of locally noetherian (locally artinian) modules is closed under submodules, factor modules and arbitrary direct sums. A number of results

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on this two types of modules can be found in [5] and [10].

The investigations in this paper are restricted to the case of modules over commutative rings. We begin by providing a characterization of locally noetherian modules and locally artinian modules. Then we focus on characterizing some kind of rings using (locally) noetherian modules and (locally) artinian modules. The class of rings R for which every noetherian (artinian) R-module is injective, is shown to be exactly that of von Neumann regular rings (Proposition 2.7); while the class of rings R for which every locally noetherian (locally artinian) module is semisimple, is characterized as that of rings in which every maximal ideal is idempotent (Theorem 2.15).

2. Results

We begin by exhibiting some examples of locally noetherian modules and locally artinian modules.

Example 2.1.

(1) Any semisimple module is locally noetherian and locally artinian. Every module over a noetherian (an artinian) ring is locally noetherian (locally arinian). On the other hand, every ring R has a locally noetherian module which is locally artinian, but it is neither noetherian nor artinian (we can take the R-module $M = (R/m)^{(\mathbb{N})}$ where m is a maximal ideal of R).

(2) Assume that R is a Dedekind domain which is not a field. Then every R-module is locally noetherian. Moreover, from [7, Theorem 6.14], it follows that an R-module M is locally artinian if and only if M is a torsion R-module.

Proposition 2.2. *The following conditions are equivalent for an R-module M:*

(1) *M is a locally noetherian (a locally artinian) module;*

(2) *Every nonzero cyclic submodule of M is noetherian (artinian);*

(3) *$R/Ann(x)$ is a noetherian (an artinian) ring for all $0 \neq x \in M$;*

(4) *$R/Ann(N)$ is a noetherian (an artinian) ring for every nonzero finitely generated submodule N of M.*

Proof. (1) $\Rightarrow$ (2) This is clear.

(2) $\Leftrightarrow$ (3) Comes from the isomorphism of R-modules $R/Ann(x) \cong Rx$ for all $0 \neq x \in M$.

(3) $\Rightarrow$ (4) Let $N = \sum_{i=1}^{n} Rx_i$ be a nonzero finitely generated submodule of M such that $x_i \neq 0$ for every $i = 1, \ldots, n$. Then $Ann(N) = \cap_{i=1}^{n} Ann(x_i)$. By hypothesis, $R/Ann(x_i)$ is a noetherian (an artinian) ring for all $i \in \{1, \ldots, n\}$. Note that $R/\cap_{i=1}^{n} Ann(x_i)$ is isomorphic to an R-submodule of the R-module $\oplus_{i=1}^{n} R/Ann(x_i)$. Therefore $R/Ann(N)$ is a noetherian (an artinian) ring.

(4) $\Rightarrow$ (1) This is immediate. $\square$

Corollary 2.3. *The following conditions are equivalent for a finitely generated R-module M:*

(1) *M is a locally noetherian (locally artinian) module;*

(2) *M is a noetherian (an artinian) module;*

(3) *$R/Ann(M)$ is a noetherian (an artinian) ring.*

Proof. By Proposition 2.2. □

The following example illustrates that the condition "M is a finitely generated R-module "in Corollary 2.3 is not superfluous.

Example 2.4. Let p be a prime number and let $M = \mathbb{Z}(p^\infty)$ be the Prüfer p-group. It is well known that M is an artinian $\mathbb{Z}$-module. On the other hand, $Ann(M) = \cap_{n=1}^{\infty}\mathbb{Z}p^n = 0$. Therefore $\mathbb{Z}/Ann(M)$ is not an artinian ring.

Next, we compare the notions of locally noetherian modules and locally artinian modules. It is well known that every artinian ring is noetherian. From Proposition 2.2, it follows that every locally artinian module is locally noetherian. The converse is not true, in general, as shown below.

Example 2.5. It is clear that $\mathbb{Z}$ and $\mathbb{Q}$ are locally noetherian $\mathbb{Z}$-modules, but neither of them is a locally artinian $\mathbb{Z}$-module.

Recall that a ring R is said to be *zero dimensional* (or of *Krull dimension zero*) and we write $dim(R) = 0$ if every prime ideal of R is maximal. The following result provides a necessary and sufficient condition for a locally noetherian module to be locally artinian.

Proposition 2.6.

(1) *An R-module M is locally artinian if and only if M is a locally noetherian module and the ring $R/Ann(x)$ is zero dimensional for all $0 \neq x \in M$.*

(2) *If M is a noetherian R-module such that the ring $R/Ann(x)$ is zero dimensional for all $0 \neq x \in M$, then M is an artinian module.*

Proof. (1) Let $0 \neq x \in M$. It is well known that a commutative ring is artinian if and only if it is noetherian and zero dimensional. Hence, $R/Ann(x)$ is artinian if and only if it is noetherian and $dim(R/Ann(x)) = 0$. The result follows from Proposition 2.2.

(2) This is a direct consequence of (1). □

Recall that a ring R is called *von Neumann regular* if for every $x \in R$, there exists $y \in R$ such that $x = xyx$. We conclude this section by some characterizations of several classes of rings in terms of (locally) noetherian modules and (locally) artinian modules.

Proposition 2.7. *The following conditions are equivalent for a ring R:*

(1) *Any noetherian R-module is injective;*

(2) *Any artinian R-module is injective;*

(3) *R is a von Neumann regular ring.*

Proof. The implication (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3) come from the fact that any simple R-module is noetherian and artinian.

(3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2), let M be a noetherian (artinian) R-module and let $0 \neq x \in M$. Since M is locally noetherian (locally artinian), $R/Ann(x)$ is a noetherian (artinian) ring (Proposition 2.2). By (3), the ring $R/Ann(x)$ is von Neumann regular, and so it is semisimple. It follows that the cyclic R-module Rx is semisimple. Thus M is a semisimple noetherian (artinian) R-module. Therefore M is a finite direct sum of simple R-modules each of them is injective. So M is injective. □

Proposition 2.8. *The following are equivalent for a commutative ring R:*

(1) *Every locally noetherian R-module is injective;*

(2) *Every locally artinian R-module is injective;*

(3) *Every locally noetherian R-module is projective;*

(4) *Every locally artinian R-module is projective;*

(5) *R is a semisimple ring.*

Proof. Note that every semisimple module is locally noetherian and locally artinian.

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (5) These follow from [3, Corollary p. 236].

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) By [9, Proposition 3.5]. □

The proof of the following proposition is straightforward.

Proposition 2.9. *The following are equivalent for a commutative ring R:*

(1) *Every injective R-module is locally noetherian (locally artinian);*

(2) *Every projective R-module is locally noetherian (locally artinian);*

(3) *Every R-module is locally noetherian (locally artinian);*

(4) *The R-module R is locally noetherian (locally artinian);*

(5) *R is a noetherian (an artinian) ring.*

Recall that an R-module M is called *unfaithful* if $Ann(M) \neq 0$.

Proposition 2.10. *The following conditions on a ring R are equivalent:*

(1) *Every unfaithful R-module is locally noetherian;*

(2) *R is a noetherian ring.*

Proof. (1) $\Rightarrow$ (2) Let $\mathfrak{a}$ be a nonzero ideal of R and let $0 \neq r \in \mathfrak{a}$. Since $Ann(R/Rr) = Rr \neq 0$, R/Rr is a locally noetherian R-module. Hence R/Rr is a noetherian ring by Corollary 2.3. So $\mathfrak{a}/Rr$ is a finitely generated R-module. It is easy to see that $\mathfrak{a}$ is also a finitely generated R-module (since Rr is finitely generated). Therefore R is a noetherian ring.

(2) $\Rightarrow$ (1) This follows from Proposition 2.9. □

A ring R is called *almost artinian* if $R/\mathfrak{a}$ is an artinian ring for any nonzero ideal $\mathfrak{a}$ of R. Clearly, every artinian ring is almost artinian.

Proposition 2.11. *The following conditions on a ring R are equivalent:*

(1) *Every unfaithful R-module is locally artinian;*

(2) *R is an almost artinian ring;*

(3) *R an artinian ring or R is a one-dimensional noetherian domain.*

Proof. (1) $\Rightarrow$ (2) Let $\mathfrak{a}$ be a nonzero ideal of R. Since $Ann(R/\mathfrak{a}) = \mathfrak{a} \neq 0$, $R/\mathfrak{a}$ is a locally artinian R-module. Hence $R/\mathfrak{a}$ is an artinian ring by Corollary 2.3.

(2) $\Rightarrow$ (1) Let M be an unfaithful R-module. Since $Ann(M) \neq 0$, we have $R/Ann(M)$ is an artinian ring. Therefore M is locally artinian as an $R/Ann(M)$-module and so as an R-module.

(2) $\Leftrightarrow$ (3) See [4]. □

We conclude this paper by characterizing the class of rings over which every locally noetherian (locally artinian) module is semisimple as that of rings whose all their maximal ideals are idempotent. For this purpose, we need a sequence of lemmas.

Lemma 2.12.

(1) *Let M be an R-module and let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R such that $\mathfrak{a} \subseteq \mathfrak{b}$. Then $Hom_R(\mathfrak{b}/\mathfrak{a}, E(M)) \cong Ann_{E(M)}(\mathfrak{a})/Ann_{E(M)}(\mathfrak{b})$.*

(2) *Let $\mathfrak{p}$ be a prime ideal of the ring R. Then $M_{\mathfrak{p}} = 0$ if and only if $Hom_R(M, E(R/\mathfrak{p})) = 0$.*

Proof. (1) Consider the exact sequence $0 \to \mathfrak{b}/\mathfrak{a} \to R/\mathfrak{a} \to R/\mathfrak{b} \to 0$ of R-modules. Since $E(M)$ is injective, $Hom_R(-, E(M))$ is exact by [1, Proposition 16.8]. So we get the following exact sequence:

$$0 \to Hom_R(R/\mathfrak{b}, E(M)) \to Hom_R(R/\mathfrak{a}, E(M)) \to Hom_R(\mathfrak{b}/\mathfrak{a}, E(M)) \to 0.$$

But $Hom_R(R/\mathfrak{b}, E(M)) \cong Ann_{E(M)}(\mathfrak{b})$ and $Hom_R(R/\mathfrak{a}, E(M)) \cong Ann_{E(M)}(\mathfrak{a})$. So $Hom_R(\mathfrak{b}/\mathfrak{a}, E(M)) \cong Ann_{E(M)}(\mathfrak{a})/Ann_{E(M)}(\mathfrak{b})$.

(2) By [2, Lemma 1.10]. □

Lemma 2.13. *Let R be a ring and let $\mathfrak{m}$ be a maximal ideal of R. Then:*

(1) $Ann_{E(R/\mathfrak{m})}(\mathfrak{m}) = R/\mathfrak{m}$.

(2) *The following conditions are equivalent:*

(i) $\mathfrak{m}^2 = \mathfrak{m}$;

(ii) $(\mathfrak{m}R_{\mathfrak{m}})^2 = \mathfrak{m}R_{\mathfrak{m}}$;

(iii) $Ann_{E(R/\mathfrak{m})}(\mathfrak{m}^2) = Ann_{E(R/\mathfrak{m})}(\mathfrak{m})$.

Proof. (1) The inclusion $R/\mathfrak{m} \subseteq Ann_{E(R/\mathfrak{m})}(\mathfrak{m})$ is obvious. Let $0 \neq x \in E(R/\mathfrak{m})$ such that $\mathfrak{m}x = 0$. Hence $\mathfrak{m} = Ann(x)$ and Rx is a simple submodule of $E(R/\mathfrak{m})$. But $R/\mathfrak{m}$ is an essential submodule of $E(R/\mathfrak{m})$, so $Rx = R/\mathfrak{m}$. Thus $Ann_{E(R/\mathfrak{m})}\mathfrak{m} \subseteq R/\mathfrak{m}$, and the result follows.

(2) (i) $\Rightarrow$ (ii) Clear since $(\mathfrak{m}R_{\mathfrak{m}})^2 = \mathfrak{m}^2R_{\mathfrak{m}}$.

(ii) $\Rightarrow$ (i) Since $\mathfrak{m}^2 \subseteq \mathfrak{m}$, it is enough to show that $\mathfrak{m}^2R_{\mathfrak{n}} = \mathfrak{m}R_{\mathfrak{n}}$ for any maximal ideal $\mathfrak{n}$ of R. If $\mathfrak{n} = \mathfrak{m}$, then by hypothesis we have the equality. If $\mathfrak{n} \neq \mathfrak{m}$, then $\mathfrak{m} \nsubseteq \mathfrak{n}$ and $\mathfrak{m}^2 \nsubseteq \mathfrak{n}$. So $\mathfrak{m}^2R_{\mathfrak{n}} = \mathfrak{m}R_{\mathfrak{n}} = R_{\mathfrak{n}}$.

(ii) $\Leftrightarrow$ (iii) We have

$$\begin{aligned}(\mathfrak{m}R_{\mathfrak{m}})^2 = \mathfrak{m}R_{\mathfrak{m}} &\Leftrightarrow (\mathfrak{m}/\mathfrak{m}^2)_{\mathfrak{m}} = 0\\ &\Leftrightarrow Hom_R(\mathfrak{m}/\mathfrak{m}^2, E(R/\mathfrak{m})) = 0, \text{ by Lemma 2.12(2)}\\ &\Leftrightarrow Ann_{E(R/\mathfrak{m})}(\mathfrak{m}^2)/Ann_{E(R/\mathfrak{m})}(\mathfrak{m}) = 0, \text{ by Lemma 2.12(1)}\\ &\Leftrightarrow Ann_{E(R/\mathfrak{m})}(\mathfrak{m}^2) = Ann_{E(R/\mathfrak{m})}(\mathfrak{m}).\end{aligned}$$

This completes the proof. □

Lemma 2.14.

(1) *The following conditions are equivalent for an R-module M:*

(i) $Soc(M) = 0$;

(ii) $Ann_M(\mathfrak{m}) = 0$ *for any maximal ideal* $\mathfrak{m}$ *of* R.

(2) *If R is a noetherian ring such that $\mathfrak{m}^2 = \mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R, then R is semisimple.*

Proof. (1) (i) $\Leftrightarrow$ (ii) We have

$$\begin{aligned}Soc(M) \neq 0 \quad &\Leftrightarrow \quad \left\{\begin{array}{l}\text{There is } 0 \neq x \in M \text{ such that}\\ Rx \text{ is a simple module.}\end{array}\right.\\ &\Leftrightarrow \quad \left\{\begin{array}{l}\text{There is } 0 \neq x \in M \text{ such that}\\ Ann(x) = \mathfrak{m} \text{ is a maximal ideal of } R.\end{array}\right.\\ &\Leftrightarrow \quad \left\{\begin{array}{l}\text{There is a maximal ideal } \mathfrak{m} \text{ of } R\\ \text{such that } Ann_M(\mathfrak{m}) \neq 0.\end{array}\right.\end{aligned}$$

(2) Let $\mathfrak{m}$ be a maximal ideal of R. Since R is noetherian, $\mathfrak{m}$ is finitely generated. But $\mathfrak{m}^2 = \mathfrak{m}$. So $\mathfrak{m}$ is a direct summand of R ([6, Lemma 2.43]). Thus, each

maximal ideal of R is a direct summand of R. Therefore R is a semisimple ring by [9, Proposition 3.5]. □

This brings us to the main result of this paper.

Theorem 2.15. *The following conditions are equivalent for a ring R:*

(1) *Any R-module of finite length is semisimple;*

(2) *Any artinian R-module is semisimple;*

(3) *Any locally artinian R-module is semisimple;*

(4) *Any noetherian R-module is semisimple;*

(5) *Any locally noetherian R-module is semisimple;*

(6) $\mathfrak{m}^2 = \mathfrak{m}$ *for any maximal ideal* $\mathfrak{m}$ *of* R.

Proof. (1) $\Rightarrow$ (3) Let M be a locally artinian R-module and let $0 \neq x \in M$. By Proposition 2.6, M is locally noetherian and so the R-module Rx is artinian and noetherian. Thus Rx is an R-module of finite length. By hypothesis, Rx is semisimple. Therefore M is a semisimple module.

(3) $\Rightarrow$ (2) $\Rightarrow$ (1) These are obvious.

(1) $\Rightarrow$ (6) Let S be a simple R-module. Suppose $Soc(E(S)/S) \neq 0$. Let L/S be a simple submodule of $E(S)/S$. Then $0 \subseteq S \subseteq L$ is a composition series of the R-module L. By hypothesis, L is semismple. So S is a direct summand of L. Hence $L = S \oplus T$ for some submodule T of L. Hence $E(S) = E(L) = E(S) \oplus E(T)$. Thus, $E(T) = 0$ and so $T = 0$. Therefore $S = L$, a contradiction. It follows that $Soc(E(S)/S) = 0$ for every simple R-module S. Now let $\mathfrak{m}$ be a maximal ideal of R. Then $Ann_{E(R/\mathfrak{m})/(R/\mathfrak{m})}(\mathfrak{m}) = 0$ by Lemma 2.14(1). Let $x \in Ann_{E(R/\mathfrak{m})}(\mathfrak{m}^2)$. Then $\mathfrak{m}^2 x = 0$ and so $\mathfrak{m}\mathfrak{m}x = 0$. Hence $\mathfrak{m}x \subseteq Ann_{E(R/\mathfrak{m})}(\mathfrak{m}) = R/\mathfrak{m}$ (Lemma 2.13(1)). This implies that $\mathfrak{m}\bar{x} = 0$ in $E(R/\mathfrak{m})/(R/\mathfrak{m})$ and so $\bar{x} \in Ann_{E(R/\mathfrak{m})/(R/\mathfrak{m})}(\mathfrak{m}) = 0$. Thus $x \in R/\mathfrak{m} = Ann_{E(R/\mathfrak{m})}(\mathfrak{m})$. Consequently, $Ann_{E(R/\mathfrak{m})}(\mathfrak{m}^2) = Ann_{E(R/\mathfrak{m})}(\mathfrak{m})$. Therefore $\mathfrak{m}^2 = \mathfrak{m}$ by Lemma 2.13(2).

(6) $\Rightarrow$ (5) Let M be a locally noetherian R-module and let $0 \neq x \in M$. By Proposition 2.2, $R/Ann(x)$ is a noetherian ring. Let $\mathfrak{m}$ be a maximal ideal of R containing $Ann(x)$. Then $\mathfrak{m}/Ann(x)$ is a maximal ideal of $R/Ann(x)$ and $(\mathfrak{m}/Ann(x))^2 = (\mathfrak{m}^2 + Ann(x))/Ann(x) = (\mathfrak{m} + Ann(x))/Ann(x) = \mathfrak{m}/Ann(x)$. By Lemma 2.14(2), $R/Ann(x)$ is a semisimple ring and so $Rx \cong R/Ann(x)$ is a semisimple R-module. Therefore M is semisimple.

(5) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (5) Let M be a locally noetherian R-module and let $0 \neq x \in M$. Then Rx is a noetherian R-module and so it is semisimple. Thus, M is a semisimple R-module.

(5) $\Rightarrow$ (3) This follows from the fact that any locally artinian R-module is locally noetherian (Corollary 2.6). □

Remark 2.16. Let K be a field and let $A = K[X_1, X_2, \ldots]$ be the polynomial ring with countably many commuting indeterminates X_i, $i \geq 1$. Let $\mathfrak{a}$ be the ideal of A generated by the elements X_1^2 and $X_n^2 - X_{n-1}$ for every $n \geq 2$. Let $R = A/\mathfrak{a}$ and let $\mathfrak{m}$ be the ideal of R generated by all $\overline{X_i} = X_i + \mathfrak{a}$, $i \geq 1$. By [8, p. 635], R is a local ring and $\mathfrak{m}$ is its maximal ideal. Moreover, we have $\mathfrak{m}^2 = \mathfrak{m}$. It is clear that R is not a regular ring. By Theorem 2.15, every locally noetherian R-module and every locally artinian R-module is semisimple. In particular, every noetherian R-module and every artinian R-module is semisimple. On the other hand, there exists a noetherian (an artinian) R-module which is not injective by Proposition 2.7.

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