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DEGENERATE BERNOULLI NUMBERS AND POLYNOMIALS ASSOCIATED WITH DEGENERATE HERMITE POLYNOMIALS

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Abstract. The article is themed to classify new (fully) degenerate Hermite-Bernoulli polynomials with formulation in terms of p-adic fermionic integrals on $\mathbb{Z}_p$. The entire paper is designed to illustrate new properties in association with Daehee polynomials in a consolidated and generalized form.

1. Introduction

Fix a number p (say prime). We begin by regarding $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$ as the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. The normalized p-adic is given by $\mid p \mid_p = \frac{1}{p}$. Let $\bigcup D(\mathbb{Z}_p)$ be the space of (uniformly) differentiable function on $\mathbb{Z}_p$. Then the p-adic invariant integral on $\mathbb{Z}_p$ (also known as Volkenborn integral on $\mathbb{Z}_p$) for any $f \in \bigcup D(\mathbb{Z}_p)$ is defined as (see [11–14]):

$$I_0(f) = \int_{\mathbb{Z}_p} f(y) d\mu_0(y) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(y). \tag{1}$$

The following equation implies from (1):

$$I_0(f_n) - I_0(f) = \sum_{a=0}^{n-1} \acute{f}(a), \ (n \geq 1), \tag{2}$$

where $f_n(y) = f(y+n)$, (see [9,11]).

Specifically, for $n = 1$ in (2), we have

$$I_0(f_1) - I_0(f) = \acute{f}(0). \tag{3}$$

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We recall the ordinary Bernoulli numbers B_n and the ordinary Bernoulli polynomials $B_n(x)$ obtained by the following Taylor series expansion (see [1–20]):

$$\frac{t}{e^t-1}=\sum_{m=0}^{\infty}B_m\frac{t^m}{m!},\ (\mid t\mid<2\pi) \tag{4}$$

and

$$\frac{t}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!},\ (\mid t\mid<2\pi), \tag{5}$$

respectively.

Here, B_m is the m^{th} Bernoulli number. Also, for every odd $k>1$, it can be observed that $B_m=0$. For each $m\in N$, the explicit formula for the Bernoulli polynomial is

$$B_m(x)=\sum_{l=0}^{m}\binom{m}{l}B_lx^{m-l}. \tag{6}$$

It is noteworthy from (4) and (6) that

$$B_m(x)=d^{m-1}\sum_{a=0}^{d-1}B_m\left(\frac{a+x}{d}\right),\ (d\in\mathbb{N}). \tag{7}$$

It was the efforts of Carlitz who created the idea of degenerate Bernoulli polynomials $\beta_m(\lambda,x)$ (see [2,3]), generating function being formulated as:

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{m=0}^{\infty}\beta_m(\lambda,x)\frac{t^m}{m!},\ (\lambda\neq 0). \tag{8}$$

We generally write $\beta_m(\lambda)$ for $\beta_m(\lambda,0)$, and mention the polynomials $\beta_m(\lambda)$ as degenerate Bernoulli numbers. For instance, $\beta_0(\lambda,x)=1$, $\beta_1(\lambda,x)=x-\frac{1}{2}+\frac{1}{2}\lambda$, $\beta_2(\lambda,x)=x^2-x+\frac{1}{6}-\frac{1}{6}\lambda^2,\ldots$.

Returning to the argument of p-adic integral, from (3), we can get

$$\int_{\mathbb{Z}_p}e^{(x+y)t}d\mu_0(y)=\frac{t}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!}. \tag{9}$$

Here, $B_m(x)$ refers to classical Bernoulli polynomials, (see [17,18]).

Subsequently, Kim and Seo [14] proposed (fully) degenerate Bernoulli polynomials which are reformulated in terms of p-adic invariant integral defined on $\mathbb{Z}_p$:

$$\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+y}{\lambda}}d\mu_0(y)=\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{m=0}^{\infty}B_m(x|\lambda)\frac{t^m}{m!}, \tag{10}$$

where $\lambda\neq 0$. Since $(1+\lambda t)^{\frac{1}{\lambda}}\to e^t$ as λ approaches to 0, it is apparent that (10) descends to (9).

Remember that Kim's degenerate Bernoulli polynomials slightly vary from the Carlitz's degenerate Bernoulli polynomials.

Further, (10) can also be written as

$$\sum_{m=0}^{\infty}\lambda^m\int_{\mathbb{Z}_p}\left(\frac{x+y}{\lambda}\right)_m d\mu_0(y)\frac{t^m}{m!}=\sum_{m=0}^{\infty}B_m(x|\lambda)\frac{t^m}{m!}, \tag{11}$$

where $(z)_m = z(z-1)\cdots(z-m+1)$.

It can be found that

$$\left(\frac{x+y}{\lambda}\right)_m=\lambda^{-m}(x+y|\lambda)_m,\ \text{(see [13])} \tag{12}$$

where $(z|\lambda)_m = z(z-\lambda)(z-2\lambda)\cdots(z-\lambda(m-1)))$.

It is obtainable from (11) and (12) that

$$\int_{\mathbb{Z}_p}(x+y|\lambda)_m d\mu_0(y)=B_m(x|\lambda),\ (m\geq 0). \tag{13}$$

Very recently, Khan [6] intensified the notion of degenerate Bernoulli polynomials $B_n(x|\lambda)$ to degenerate Hermite-Bernoulli polynomials (of second kind) ${}_HB_n(x,y|\lambda)$ computed as:

$$\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}=\sum_{m=0}^{\infty}{}_HB_m(x,y|\lambda)\frac{t^m}{m!}, \tag{14}$$

which is eventually an extended generalization of Carlitz's degenerate Bernoulli polynomials $\beta_n(x,\lambda)$ (see [3]) and 2-variable Kampé de Fériet generalization of Hermite polynomials $H_n(x,y)$ (see [1,4]). Further as $\lambda\to 0$ in (14), ${}_HB_n(x,y|\lambda)$ converts to ${}_HB_n(x,y)$ (Hermite-Bernoulli polynomials) formally given by Dattoli et al. [5, p. 386 (1.6)] as:

$$\left(\frac{t}{e^t-1}\right)e^{xt+yt^2}=\sum_{m=0}^{\infty}{}_HB_m(x,y)\frac{t^m}{m!}. \tag{15}$$

Also, the Daehee polynomials are set forth by Kim et al. [10, 15] as:

$$\frac{\log(1+t)^{\frac{1}{\lambda}}}{t}(1+t)^x=\sum_{m=0}^{\infty}D_m(x)\frac{t^m}{m!}. \tag{16}$$

When $x=0$ in (16), $D_n = D_n(0)$ are the Daehee numbers.

A major theme of the present article is that the study of Bernoulli numbers, its varied generalizations and other consequential sequences can be made feasible with the help of degenerate Bernoulli polynomials. A new class of (fully) degenerate Hermite-Bernoulli polynomials are considered with formulation in terms of p-adic fermionic integrals on $\mathbb{Z}_p$. The entire paper is designed to illustrate new properties in association with Daehee numbers and polynomials.

2. Degenerate Hermite-Bernoulli polynomials and numbers

Consider, $\lambda, t \in \mathbb{C}_p$ and $|\lambda t|_p < p^{-\frac{1}{p-1}}$. With the viewpoint of (10) and (14), we can easily define:

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+x_1}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}d\mu_0(x_1) &= \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} \\ &= \sum_{m=0}^{\infty} {}_HB_m(x,y|\lambda)\frac{t^m}{m!}.\end{aligned} \tag{17}$$

For $y = 0$, (17) reduces to (10) and when $x = y = 0$ in (17), $B_n(\lambda) = {}_HB_n(0,0|\lambda)$ are known as the degenerate Bernoulli numbers.

Mark that $\lim_{\lambda\to 0} {}_HB_n(x,y|\lambda) = {}_HB_n(x,y)$, (see [5,6]).

Theorem 2.1. *We have, for* $m \geq 0$

$$ {}_HB_m(x,y|\lambda) = \sum_{k=0}^{[\frac{m}{2}]}\left(\frac{y}{\lambda}\right)_k \sum_{l=0}^{m-2k} S_1(m-2k,l)\lambda^{m-k-l}B_l(x)\frac{m!}{k!(m-2k)!}. \tag{18}$$

Here, $S_1(m,l)$ *is the first kind stirling number* [6].

Proof. From (17), we observe that

$$\begin{aligned}\sum_{m=0}^{\infty} {}_HB_m(x,y|\lambda)\frac{t^m}{m!} &= (1+\lambda t^2)^{\frac{y}{\lambda}}\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+x_1}{\lambda}}d\mu_0(x_1) \\ &= (1+\lambda t^2)^{\frac{y}{\lambda}}\sum_{m=0}^{\infty}\lambda^m\int_{\mathbb{Z}_p}\left(\frac{x+x_1}{\lambda}\right)_m d\mu_0(x_1)\frac{t^m}{m!},\end{aligned}$$

which on using (12), turns out to be

$$\sum_{m=0}^{\infty} {}_HB_m(x,y|\lambda)\frac{t^m}{m!} = (1+\lambda t^2)^{\frac{y}{\lambda}}\sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}(x+x_1|\lambda)_m d\mu_0(x_1)\frac{t^m}{m!}. \tag{19}$$

Now, by the definition of stirling number, we find

$$\begin{aligned}\int_{\mathbb{Z}_p}(x+x_1|\lambda)_m d\mu_0(x_1) &= \lambda^m\sum_{l=0}^{m}S_1(m,l)\int_{\mathbb{Z}_p}(x+x_1)^l d\mu_0(x_1)\lambda^{-l} \\ &= \sum_{l=0}^{m}S_1(m,l)\lambda^{m-l}B_l(x).\end{aligned} \tag{20}$$

Thus, by (19) and (20), we obtain

$$\begin{aligned}\sum_{m=0}^{\infty} {}_HB_m(x,y|\lambda)\frac{t^m}{m!} &= (1+\lambda t^2)^{\frac{y}{\lambda}}\sum_{m=0}^{\infty}\sum_{l=0}^{m}S_1(m,l)\lambda^{m-l}B_l(x)\frac{t^m}{m!} \\ &= \left(\sum_{k=0}^{\infty}\left(\frac{y}{\lambda}\right)_k\lambda^k\frac{t^{2k}}{k!}\right)\left(\sum_{m=0}^{\infty}\sum_{l=0}^{m}S_1(m,l)\lambda^{m-l}B_l(x)\frac{t^m}{m!}\right),\end{aligned}$$

$$\sum_{m=0}^{\infty} {}_HB_m(x,y|\lambda)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\left(\sum_{k=0}^{[\frac{m}{2}]}\left(\frac{y}{\lambda}\right)_k \sum_{l=0}^{m-2k} S_1(m-2k,l)\lambda^{m-k-l} \times B_l(x)\frac{m!}{k!(m-2k)!}\right)\frac{t^m}{m!}.$$

Coefficients of identical powers of t on comparing, yields the expected result of Theorem 2.1. □

Remark. $y=0$ in Theorem 2.1, gives a familiar looking identity of Kim et al. [14, p. 1272 (Theorem 2.1)]:

Corollary 2.1. *We have, for $m \geq 0$*

$$B_m(x|\lambda) = \sum_{l=0}^{m} S_1(m,l)\lambda^{m-l}B_l(x).$$

Theorem 2.2. *We have, for $m \geq 0$*

(21)

$${}_HB_m(x,y|\lambda) = \sum_{k=0}^{[\frac{m}{2}]}\lambda^k\left(\frac{y}{\lambda}\right)_k \sum_{l=0}^{m-2k}\binom{m-2k}{l}B_l(\lambda)(x|\lambda)_{m-l-2k}\frac{m!}{k!(m-2k)!}.$$

Proof. From (17), we have

$$\begin{aligned}\sum_{m=0}^{\infty} {}_HB_m(x,y|\lambda)\frac{t^m}{m!} &= (1+\lambda t^2)^{\frac{y}{\lambda}}\sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}(x+x_1|\lambda)_m d\mu_0(x_1)\lambda^m\frac{t^m}{m!}\\ &= (1+\lambda t^2)^{\frac{y}{\lambda}}\sum_{m=0}^{\infty}\lambda^m m!\int_{\mathbb{Z}_p}\binom{\frac{x+x_1}{\lambda}}{m}d\mu_0(x_1)\frac{t^m}{m!}\\ &= (1+\lambda t^2)^{\frac{y}{\lambda}}\sum_{m=0}^{\infty}\lambda^m m!\sum_{l=0}^{m}\binom{\frac{x}{\lambda}}{m-l}\int_{\mathbb{Z}_p}\binom{\frac{x_1}{\lambda}}{l}d\mu_0(x_1)\frac{t^m}{m!}\\ &= (1+\lambda t^2)^{\frac{y}{\lambda}}\sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{\frac{x}{\lambda}}{m-l}\frac{\lambda^{m-l}m!}{l!}B_l(\lambda)\frac{t^m}{m!},\end{aligned}$$

$$\sum_{m=0}^{\infty} {}_HB_m(x,y|\lambda)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\left(\sum_{k=0}^{\infty}\lambda^k\left(\frac{y}{\lambda}\right)_k\frac{t^{2k}}{k!}\right)\left(\sum_{l=0}^{m}\binom{m}{l}B_l(\lambda)(x|\lambda)_{m-l}\right)\frac{t^m}{m!}.$$

Coefficients of identical powers of t on comparing, yields the expected result of Theorem 2.2. □

Remark. With $y=0$ in Theorem 2.2, a familiar looking identity of Kim et al. [14, p. 1273 (Theorem 2.2)] follows:

Corollary 2.2. *We have, for $m \geq 0$*

$$B_m(x|\lambda) = \sum_{l=0}^{m}\binom{m}{l}B_l(\lambda)(x|\lambda)_{m-l}.$$

Theorem 2.3. *We have, for $m \geq 0$*

$$ {}_HB_m(x,y|\lambda) = m!\sum_{k=0}^{[\frac{m}{2}]} D_{m-2k}(\lambda,x)\left(\frac{y}{\lambda}\right)_k \frac{\lambda^k}{(m-2k)!k!}, \tag{22} $$

where $\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = D_n(\lambda,x)$ *is the λ-Daehee polynomial* [16].

Proof. From (16) and (17), we evaluate that

$$ \begin{aligned} \sum_{m=0}^{\infty} {}_HB_m(x,y|\lambda)\frac{t^m}{m!} &= (1+\lambda t^2)^{\frac{y}{\lambda}} \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+x_1}{\lambda}} d\mu_0(x_1) \\ &= \left(\sum_{k=0}^{\infty}\left(\frac{y}{\lambda}\right)_k \lambda^k \frac{t^{2k}}{k!}\right)\left(\sum_{m=0}^{\infty} D_m(\lambda,x)\frac{t^m}{m!}\right) \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{[\frac{m}{2}]} D_{m-2k}(\lambda,x)\left(\frac{y}{\lambda}\right)_k \frac{\lambda^k t^m}{(m-2k)!k!}, \end{aligned} $$

$$ \sum_{m=0}^{\infty} {}_HB_m(x,y|\lambda)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\left(m!\sum_{k=0}^{[\frac{m}{2}]} D_{m-2k}(\lambda,x)\left(\frac{y}{\lambda}\right)_k \frac{\lambda^k}{(m-2k)!k!}\right)\frac{t^m}{m!}. $$

Coefficients of identical powers of t on comparing, yields the expected result of Theorem 2.3. □

Theorem 2.4. *We have, for $m \geq 0$*

$$ \begin{aligned} &\sum_{l=0}^{m}\binom{m}{l}\frac{(\frac{1}{\lambda})_{l+1}}{l+1} {}_HB_{m-l}(x,y|\lambda) \\ &= m!\sum_{l=0}^{m-2k}\sum_{k=0}^{[\frac{m}{2}]}\binom{m-2k}{l}\left(\frac{x}{\lambda}\right)_{m-2k-l}\left(\frac{y}{\lambda}\right)_k \frac{\lambda^{m-k}D_l(0)}{(m-2k)!k!}. \end{aligned} \tag{23} $$

Proof. From (17), we notice that

$$ \begin{aligned} &\log(1+\lambda t)^{\frac{1}{\lambda}}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} \\ &= \left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)\left(\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+x_1}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} d\mu_0(x_1)\right) \\ &= \left(\sum_{l=0}^{\infty}\frac{(\frac{1}{\lambda})_{l+1}}{(l+1)!}t^{l+1}\right)\left(\sum_{m=0}^{\infty} {}_HB_m(x,y|\lambda)\frac{t^m}{m!}\right) \\ &= t\sum_{m=0}^{\infty}\left(\sum_{l=0}^{m}\binom{m}{l}\frac{(\frac{1}{\lambda})_{l+1}}{l+1} {}_HB_{m-l}(x,y|\lambda)\right)\frac{t^m}{m!}. \end{aligned} \tag{24} $$

Again using (17) and the special case of (16), we find that

$$\begin{aligned}
&\log(1+\lambda t)^{\frac{1}{\lambda}}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}\\
&= t\left(\frac{\log(1+\lambda t)}{\lambda t}\right)(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}\\
&= t\left(\sum_{l=0}^{\infty} D_l(0)\frac{\lambda^l t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\left(\frac{x}{\lambda}\right)_m \lambda^m\frac{t^m}{m!}\right)\left(\sum_{k=0}^{\infty}\left(\frac{y}{\lambda}\right)_k \lambda^k\frac{t^{2k}}{k!}\right) \qquad (25)\\
&= \sum_{m=0}^{\infty}\left(m!\sum_{l=0}^{m-2k}\sum_{k=0}^{[\frac{m}{2}]}\binom{m-2k}{l}\frac{D_l(0)\lambda^{m-k}}{(m-2k)!k!}\left(\frac{x}{\lambda}\right)_{m-2k-l}\left(\frac{y}{\lambda}\right)_k\right)\frac{t^m}{m!}.
\end{aligned}$$

Therefore, from (24) and (25), the expected result of Theorem 2.4 is achieved. □

Remark. $y = 0$ in Theorem 2.4, gives a familiar looking identity of Kim et al. [13, p. 907 (Theorem 2.1)].

Corollary 2.3. *We have, for* $m \geq 0$

$$\sum_{l=0}^{m}\binom{m}{l}\frac{(\frac{1}{\lambda})_{l+1}}{(l+1)}\beta_{m-l,\lambda}(x) = \sum_{l=0}^{m}\binom{m}{l}\lambda^l (x|\lambda)_{m-l}D_l.$$

Theorem 2.5. *We have, for* $m \geq 0$

$$\begin{aligned}
&{}_HB_m(x+1,y|\lambda) - {}_HB_m(x,y|\lambda)\\
&= m\sum_{k=0}^{m-1}\sum_{l=0}^{k}\binom{m-1}{k}\binom{k}{l}H_{m-k-1}(x,y;\lambda)\frac{\left(\frac{1}{\lambda}\right)_{l+1}}{l+1}D_{k-l}, \qquad (26)
\end{aligned}$$

where $H_m(x,y;\lambda)$ *is the degenerate Hermite polynomial* (*see* [6]).

Proof. From the p-adic integral (17), we evaluate

$$\begin{aligned}
&\sum_{m=0}^{\infty} {}_HB_m(x+1,y|\lambda)\frac{t^m}{m!} - \sum_{m=0}^{\infty} {}_HB_m(x,y|\lambda)\frac{t^m}{m!} \qquad (27)\\
&= \int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+1+x_1}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}d\mu_0(x_1)\\
&\quad - \int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+x_1}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}d\mu_0(x_1)\\
&= [(1+\lambda t)^{\frac{1}{\lambda}} - 1]\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+x_1}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}d\mu_0(x_1)\\
&= \left(\sum_{l=0}^{\infty}\frac{\left(\frac{1}{\lambda}\right)_{l+1}}{(l+1)!}t^{l+1}\right)(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x_1}{\lambda}}d\mu_0(x_1)\\
&= t\left(\sum_{l=0}^{\infty}\frac{\left(\frac{1}{\lambda}\right)_{l+1}}{l+1}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}H_m(x,y;\lambda)\frac{t^m}{m!}\right)\left(\sum_{k=0}^{\infty}D_k\frac{t^k}{k!}\right). \qquad (28)
\end{aligned}$$

Finally, the expected result of Theorem 2.5 is achieved on comparing the coefficients of $\frac{t^m}{m!}$ in (27) and (28). □

Theorem 2.6. *We have, for* $m \geq 0$

$$ {}_HB_{m+1}(x,y|\lambda) = \sum_{k=0}^{m+1} \binom{m+1}{k} \lambda^k D_k \, {}_H\beta_{m-k+1}(x,y|\lambda), \tag{29} $$

where ${}_H\beta_m(x,y|\lambda)$ *is Carlitz's degenerate Hermite-Bernoulli polynomials.*

Proof. From (17), it can be conveniently shown that

$$ \begin{aligned} &\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+x_1}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}} d\mu_0(x_1) \\ &= \frac{\log(1+\lambda t)}{\lambda t} \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}} \\ &= \left(\sum_{k=0}^{\infty} D_k \frac{(\lambda t)^k}{k!} \right) \left(\sum_{m=0}^{\infty} {}_H\beta_m(x,y|\lambda) \frac{t^m}{m!} \right) \\ &= \sum_{m=1}^{\infty} \left(\sum_{k=0}^{m} \binom{m}{k} \lambda^k D_k \, {}_H\beta_{m-k}(x,y|\lambda) \right) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m+1} \binom{m+1}{k} \lambda^k D_k \, {}_H\beta_{m-k+1}(x,y|\lambda) \right) \frac{t^{m+1}}{(m+1)!} \\ &= t \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m+1} \binom{m+1}{k} \lambda^k D_k \frac{{}_H\beta_{m-k+1}(x,y|\lambda)}{m+1} \right) \frac{t^m}{m!}. \end{aligned} \tag{30} $$

Again from (17), we can show

$$ \begin{aligned} \frac{1}{t} \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}} &= \frac{1}{t} \sum_{m=0}^{\infty} {}_HB_m(x,y|\lambda) \frac{t^m}{m!} \\ &= \frac{1}{t} \sum_{m=1}^{\infty} {}_HB_m(x,y|\lambda) \frac{t^m}{m!} \\ &= \frac{1}{t} \sum_{m=0}^{\infty} {}_HB_{m+1}(x,y|\lambda) \frac{t^{m+1}}{(m+1)!} \\ &= \sum_{m=0}^{\infty} \frac{{}_HB_{m+1}(x,y|\lambda)}{m+1} \frac{t^m}{m!}. \end{aligned} \tag{31} $$

Ultimately, the expected result of Theorem 2.6 is achieved on comparing the coefficients of $\frac{t^m}{m!}$ in (30) and (31). □

Remark. On setting $y = 0$ in Theorem 2.6, the corresponding corollary follows:

Corollary 2.4. *We have, for $m \geq 0$*

$$B_{m+1}(x|\lambda) = \sum_{k=0}^{m+1} \binom{m+1}{k} \lambda^k D_k \beta_{m-k+1}(x|\lambda). \tag{32}$$

Theorem 2.7. *For $d \in \mathbb{N}$, we state*

$$_H B_m(x, y|\lambda) = d^{m-1} \sum_{a=0}^{d-1} {}_H B_m\left(\frac{a+x}{d}, y|\frac{\lambda}{d}\right)$$

and

$$\sum_{k=0}^{[\frac{m}{2}]} \frac{(y|\lambda)_k}{k!(m-2k)!} B_{m-2k}(x|\lambda) = d^{m-1} \sum_{a=0}^{d-1} {}_H B_m\left(\frac{a+x}{d}, y|\frac{\lambda}{d}\right). \tag{33}$$

Proof. From (3) and (7), one can get

$$\begin{aligned}
&((1+\lambda t)^{\frac{d}{\lambda}} - 1) \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+x_1}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}} d\mu_0(x_1) \\
&= \log(1+\lambda t)^{\frac{1}{\lambda}} \sum_{a=0}^{d-1} (1+\lambda t)^{\frac{x+a}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}},
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+x_1}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}} d\mu_0(x_1) &= \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{d}{\lambda}} - 1} \sum_{a=0}^{d-1} (1+\lambda t)^{\frac{x+a}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}} \\
&= \frac{1}{d} \sum_{a=0}^{d-1} \sum_{m=0}^{\infty} d^m {}_H B_m\left(\frac{a+x}{d}, y|\frac{\lambda}{d}\right) \frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty} \left(d^{n-1} \sum_{a=0}^{d-1} {}_H B_n\left(\frac{a+x}{d}, y|\frac{\lambda}{d}\right)\right) \frac{t^n}{n!}.
\end{aligned}$$

Therefore, from (17) we get

$$_H B_m(x, y|\lambda) = d^{m-1} \sum_{a=0}^{d-1} {}_H B_m\left(\frac{a+x}{d}, y|\frac{\lambda}{d}\right). \tag{34}$$

On the other hand,

$$\begin{aligned}
&\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+x_1}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}} d\mu_0(x_1) \\
&= \left(\sum_{k=0}^{\infty} \left(\frac{y}{\lambda}\right)_k \lambda^k \frac{t^{2k}}{k!}\right) \int_{\mathbb{Z}_p} \left(\frac{x+x_1}{\lambda}\right)_m d\mu_0(x_1) \lambda^m \\
&= \left(\sum_{k=0}^{\infty} (y|\lambda)_k \frac{t^{2k}}{k!}\right) \left(\sum_{m=0}^{\infty} B_m(x|\lambda) \frac{t^m}{m!}\right).
\end{aligned}$$

Now, from (17), (34) and the preceding equation, the expected result of Theorem 2.7 is achieved. □

Remark. $y = 0$ in Theorem 2.7, gives a familiar looking identity of Kim et al. [14, p. 1275 (Theorem 2.3)].

Corollary 2.5. *For $d \in \mathbb{N}$, we state*

$$B_m(x|\lambda) = d^{m-1} \sum_{a=0}^{d-1} B_m\left(\frac{a+x}{d} | \frac{\lambda}{d}\right).$$

Theorem 2.8. *We have, for $m \geq 0$*

$$ {}_H\beta_m(x,y|\lambda) = \sum_{k=0}^{m} \binom{m}{k} b_k \lambda^k {}_H B_{m-k}(x,y|\lambda), \tag{35}$$

where b_k is the Binomial number of second kind.

Proof. From (17), we can show that

$$\begin{aligned} &\frac{\lambda t}{\log(1+\lambda t)} \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+x_1}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}} d\mu_0(x_1) \\ &= \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}} = \sum_{m=0}^{\infty} {}_H\beta_m(x,y|\lambda) \frac{t^m}{m!}. \end{aligned} \tag{36}$$

We also attain

$$\begin{aligned} &\frac{\lambda t}{\log(1+\lambda t)} \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+x_1}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}} d\mu_0(x_1) \\ &= \frac{\lambda t}{\log(1+\lambda t)} \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}} \\ &= \left(\sum_{k=0}^{\infty} b_k \lambda^k \frac{t^k}{k!} \right) \left(\sum_{m=0}^{\infty} {}_H B_m(x,y|\lambda) \frac{t^m}{m!} \right) \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^{n} \binom{m}{k} b_k \lambda^k {}_H B_{m-k}(x,y|\lambda) \frac{t^m}{m!}. \end{aligned} \tag{37}$$

Now, comparing the coefficients of $\frac{t^m}{m!}$ in (36) and (37) gives the required theorem. □

Remark. For $y = 0$ in Theorem 2.8, the corresponding corollary follows.

Corollary 2.6. *We have, for $m \geq 0$*

$$\beta_m(x|\lambda) = \sum_{k=0}^{m} \binom{m}{k} b_k \lambda^k B_{m-k}(x|\lambda).$$

3. Generalized degenerate Hermite-Bernoulli polynomials and numbers

Consider a Dirichlet character χ and the conductor d ($d \in \mathbb{N}$) associated with it ($d \equiv 1 \pmod 2$). Let X be a subset of $\mathbb{Q}_p$, alike $\mathbb{Z}_p$, then

$$X = \lim_{\leftarrow N}(\mathbb{Z}/dp^N\mathbb{Z});$$

$$a + dp^N\mathbb{Z}_p = \{y \in X | y \equiv a \pmod{dp^N}\};$$

$$X^* = \bigcup_{0<a<dp}(a + dp\mathbb{Z}_p).$$

If the sets are of the form $a + dp^N\mathbb{Z}_p$, then we usually take $0 \leq a < dp^N$. One is allowed to write

$$\int_X f(y)d\mu_0(y) = \int_{\mathbb{Z}_p} f(y)d\mu_0(y), \ (f \in \cup D(\mathbb{Z}_p)).$$

Thus, we presently define the generalized version of degenerate Hermite-Bernoulli polynomials attached with χ in the form:

$$\begin{aligned} &\int_X \chi(x_1)(1+\lambda t)^{\frac{x+x_1}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} d\mu_0(x_1) \\ &= \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{d}{\lambda}}-1}\sum_{a=0}^{d-1}(-1)^a\chi(a)(1+\lambda t)^{\frac{x+a}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} \\ &= \sum_{n=0}^{\infty} {}_HB_{n,\chi}(x,y|\lambda)\frac{t^n}{n!}. \end{aligned} \tag{38}$$

For $y = 0$, ${}_HB_{n,\chi}(x,y|\lambda)$ reduces to $B_{n,\chi}(x|\lambda)$ defined by Kim [14] as:

$$\begin{aligned} \int_X \chi(x_1)(1+\lambda t)^{\frac{x+x_1}{\lambda}} d\mu_0(x_1) &= \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{d}{\lambda}}-1}\sum_{a=0}^{d-1}(1+\lambda t)^{\frac{x+a}{\lambda}}\chi(a) \\ &= \sum_{m=0}^{\infty} B_{m,\chi}(x|\lambda)\frac{t^m}{m!}. \end{aligned} \tag{39}$$

On adjusting $x = y = 0$ in (38), $B_{n,\chi}(\lambda) = {}_HB_{n,\chi}(0,0|\lambda)$ are marked as the generalized degenerate Bernoulli numbers attached to χ.

Moreover, as $\lambda \to 0$, then

$$\begin{aligned} \int_X \chi(y)e^{(x+y)t} d\mu_0(y) &= \frac{t}{e^{dt}-1}\sum_{a=0}^{d-1}\chi(a)e^{(a+x)t} \\ &= \sum_{m=0}^{\infty} B_{m,\chi}(x)\frac{t^m}{m!}, \ \text{(see [12,19])}. \end{aligned} \tag{40}$$

Theorem 3.1. *We have, for $m \geq 0$*

$$ {}_HB_{m,\chi}(x,y|\lambda) = \sum_{k=0}^{[\frac{m}{2}]} \left(\frac{y}{\lambda}\right)_k B_{m-2k,\chi}(x|\lambda) \frac{\lambda^{m-k} m!}{(m-2k)!k!}. \tag{41} $$

Proof. From (38), we evaluate

$$ \begin{aligned} &\sum_{m=0}^{\infty} {}_HB_{m,\chi}(x,y|\lambda) \frac{t^m}{m!} \\ &= (1+\lambda t^2)^{\frac{y}{\lambda}} \int_X \chi(x_1)(1+\lambda t)^{\frac{x+x_1}{\lambda}} d\mu_0(x_1) \\ &= \left(\sum_{k=0}^{\infty} \left(\frac{y}{\lambda}\right)_k \lambda^k \frac{t^{2k}}{k!}\right) \left(\sum_{m=0}^{\infty} \lambda^m \int_X \chi(x_1) \left(\frac{x+x_1}{\lambda}\right)_m d\mu_0(x_1) \frac{t^m}{m!}\right). \end{aligned} $$

Therefore, from (13) and (39), we deduce

$$ \sum_{m=0}^{\infty} {}_HB_{m,\chi}(x,y|\lambda) \frac{t^m}{m!} = \left(\sum_{k=0}^{\infty} \lambda^k \left(\frac{y}{\lambda}\right)_k \frac{t^{2k}}{k!}\right) \left(\sum_{m=0}^{\infty} \lambda^m B_{m,\chi}(x|\lambda) \frac{t^m}{m!}\right). $$

Coefficients of $\frac{t^m}{m!}$ on comparison, provides the expected identity of Theorem 3.1. □

Theorem 3.2. *Let $d \in \mathbb{N}$. We have, for $m \geq 0$*

$$ {}_HB_{m,\chi}(x,y|\lambda) = d^{m-1} \sum_{a=0}^{d-1} (-1)^a \chi(a) {}_HB_m \left(\frac{a+x}{d}, y|\frac{\lambda}{d}\right). \tag{42} $$

Proof. From (38), we have

$$ \begin{aligned} &\sum_{m=0}^{\infty} {}_HB_{m,\chi}(x,y|\lambda) \frac{t^m}{m!} \\ &= \frac{1}{d} \frac{\frac{d}{\lambda}\log(1+\lambda t)}{(1+\lambda t)^{\frac{d}{\lambda}} - 1} \sum_{a=0}^{d-1} (-1)^a \chi(a) (1+\lambda t)^{\frac{d}{\lambda} \cdot \frac{x+a}{d}} (1+\lambda t^2)^{\frac{y}{\lambda}} \\ &= \sum_{m=0}^{\infty} \left(d^{m-1} \sum_{a=0}^{d-1} (-1)^a \chi(a) {}_HB_m \left(\frac{a+x}{d}, y|\frac{\lambda}{d}\right)\right) \frac{t^m}{m!}. \end{aligned} $$

From (3), we have

$$ \begin{aligned} &\int_X \chi(x_1)(1+\lambda t)^{\frac{x+x_1}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}} d\mu_0(x_1) \\ &= \frac{\frac{1}{\lambda}\log(1+\lambda t)}{(1+\lambda t)^{\frac{d}{\lambda}} - 1} \sum_{a=0}^{d-1} (-1)^a (1+\lambda t)^{\frac{x+a}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}} \\ &= \sum_{m=0}^{\infty} {}_HB_{m,\chi}(x,y|\lambda) \frac{t^m}{m!}. \end{aligned} $$

From the above two equations, we get

$$ {}_HB_{m,\chi}(x,y|\lambda) = d^{m-1}\sum_{a=0}^{d-1}(-1)^a\chi(a)\,{}_HB_m\left(\frac{a+x}{d}, y|\frac{\lambda}{d}\right). $$

□

Remark. For $y = 0$ in Theorem 3.2, a familiar looking statement of Kim et al. [14, p. 1276 (Theorem 2.4)] follows:

Corollary 3.1. *Let $d \in \mathbb{N}$. We have, for $m \geq 0$*

$$ B_{m,\chi}(x|\lambda) = d^{m-1}\sum_{a=0}^{d-1}\chi(a)B_m\left(\frac{a+x}{d}|\frac{\lambda}{d}\right). $$

Theorem 3.3. *We have, for $m \geq 0$*

(43)

$$ {}_HB_{m,\chi}(x,y|\lambda) = \sum_{l=0}^{m-2k}\sum_{k=0}^{[\frac{m}{2}]}\binom{m-2k}{l}\left(\frac{x}{\lambda}\right)_{m-l-2k}\left(\frac{y}{\lambda}\right)_k B_{l,\chi}(\lambda)\frac{\lambda^{m-2k-l}m!}{(m-2k)!k!}. $$

Proof. From (38), we evaluate

$$ \begin{aligned} &\sum_{m=0}^{\infty}{}_HB_{m,\chi}(x,y|\lambda)\frac{t^m}{m!} \\ &= \left(\int_X \chi(a)(1+\lambda t)^{\frac{x_1}{\lambda}}d\mu_0(x_1)\right)(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} \\ &= \frac{\frac{1}{\lambda}\log(1+\lambda t)}{(1+\lambda t)^{\frac{d}{\lambda}}-1}\sum_{a=0}^{d-1}(1+\lambda t)^{\frac{a}{\lambda}}\chi(a)\left(\sum_{m=0}^{\infty}\left(\frac{x}{\lambda}\right)_m\frac{(\lambda t)^m}{m!}\right)\left(\sum_{k=0}^{\infty}\left(\frac{y}{\lambda}\right)_k\frac{(\lambda t^2)^k}{k!}\right) \\ &= \left(\sum_{l=0}^{\infty}B_{l,\chi}(\lambda)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\lambda^m\left(\frac{x}{\lambda}\right)_m\frac{t^m}{m!}\right)\left(\sum_{k=0}^{\infty}\lambda^k\left(\frac{y}{\lambda}\right)_k\frac{t^{2k}}{k!}\right). \end{aligned} $$

Coefficients of $\frac{t^m}{m!}$ on comparison, provides the expected result of Theorem 3.3. □

Theorem 3.4. *We have, for $m \geq 0$*

$$ {}_HB_{m,\chi}(x,y|\lambda) = \sum_{k=0}^{[\frac{m}{2}]}\sum_{l=0}^{m-2k}\left(\frac{y}{\lambda}\right)_k S_1(m-2k,l)B_{l,\chi}(x)\frac{\lambda^{m-k-l}m!}{(m-2k)!k!}, \tag{44} $$

where $B_{m,\chi}(x)$ is the generalized Bernoulli polynomial defined in (40).

Proof. We write, from (38)

$$ \begin{aligned} &\sum_{m=0}^{\infty}{}_HB_{m,\chi}(x,y|\lambda)\frac{t^m}{m!} \\ &= (1+\lambda t^2)^{\frac{y}{\lambda}}\int_X \chi(x_1)(1+\lambda t)^{\frac{x+x_1}{\lambda}}d\mu_0(x_1) \end{aligned} $$

$$= \left(\sum_{k=0}^{\infty}\left(\frac{y}{\lambda}\right)_k \frac{(\lambda t^2)^k}{k!}\right)\sum_{m=0}^{\infty}\lambda^m \int_X \chi(x_1)\left(\frac{x+x_1}{\lambda}\right)_m d\mu_0(x_1)\frac{t^m}{m!}$$
$$= \left(\sum_{k=0}^{\infty}\left(\frac{y}{\lambda}\right)_k \frac{(\lambda t^2)^k}{k!}\right)\sum_{m=0}^{\infty}\lambda^m \sum_{l=0}^{m} S_1(m,l)\int_X \chi(x_1)\left(\frac{x+x_1}{\lambda}\right)^l d\mu_0(x_1)\frac{t^m}{m!}$$
$$= \left(\sum_{k=0}^{\infty}\left(\frac{y}{\lambda}\right)_k \frac{(\lambda t^2)^k}{k!}\right)\sum_{m=0}^{\infty}\sum_{l=0}^{m} S_1(m,l)B_{l,\chi}(x)\lambda^{m-l}\frac{t^m}{m!}.$$

Finally, the expected result of Theorem 3.4 is achieved on equating the coefficients of $\frac{t^m}{m!}$. □

Remark. For $y = 0$ in Theorem 3.4, a familiar looking statement of Kim et al. [14, p. 1277 (Corollary 2.5)] follows:

Corollary 3.2. *We have, for* $n \geq 0$

$$B_{m,\chi}(x|\lambda) = \sum_{l=0}^{m} S_1(m,l)\lambda^{m-l}B_{l,\chi}(x).$$

4. Identities of symmetry for the degenerate Hermite-Bernoulli polynomials

Let $\lambda, t \in \mathbb{C}_p$ so that $|\lambda t|_p < p^{-\frac{1}{p-1}}$. For $\mu_1, \mu_2 \in \mathbb{N}$, one can define with ease

$$\begin{aligned}
&\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{\mu_1 x_1}{\lambda}}(1+\lambda t)^{\frac{\mu_1\mu_2}{\lambda}x}(1+\lambda t^2)^{\frac{\mu_1{}^2\mu_2{}^2}{\lambda}y}d\mu_0(x_1)\\
&= \frac{\mu_1 \log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{\mu_1}{\lambda}}-1}(1+\lambda t)^{\frac{\mu_1\mu_2}{\lambda}x}(1+\lambda t^2)^{\frac{\mu_1{}^2\mu_2{}^2}{\lambda}y} \qquad (45)\\
&= \sum_{m=0}^{\infty} {}_HB_{m,\frac{\lambda}{\mu_1}}(\mu_2 x, \mu_2{}^2 y)\frac{(\mu_1 t)^m}{m!}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mu_1\mu_2\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x}{\lambda}}d\mu_0(x)}{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{\mu_1\mu_2}{\lambda}x}d\mu_0(x)} &= \frac{(1+\lambda t)^{\frac{\mu_1\mu_2}{\lambda}-1}}{(1+\lambda t)^{\frac{\mu_2}{\lambda}-1}} \qquad (46)\\
&= \sum_{k=0}^{\infty} S_k\left(\mu_1 - 1|\frac{\lambda}{\mu_2}\right)\frac{(\mu_2 t)^k}{k!}, \text{ (see [8]).}
\end{aligned}$$

Note that $\lim_{\lambda\to 0} S_k(n|\lambda) = S_k(n)$.

Theorem 4.1. *For* $\mu_1, \mu_2 \in \mathbb{N}$, *the below symmetry identity holds well*:

$$\sum_{j=0}^{m}\binom{m}{j}\mu_1{}^{m-j-1}\mu_2{}^j \sum_{k=0}^{j}\binom{j}{k} {}_HB_{m-j,\frac{\lambda}{\mu_1}}(\mu_2 x, \mu_2{}^2 z)$$

$$\times S_k\left(\mu_1-1|\frac{\lambda}{2}\right)B_{j-k,\frac{\lambda}{\mu_2}}(\mu_1 y)$$

$$(47)\qquad =\sum_{j=0}^{m}\binom{m}{j}{\mu_2}^{m-j-1}{\mu_1}^{j}\sum_{k=0}^{j}\binom{j}{k}{}_HB_{m-j,\frac{\lambda}{\mu_2}}(\mu_1 x,{\mu_1}^2 z)$$

$$\times S_k\left(\mu_2-1|\frac{\lambda}{\mu_1}\right)B_{j-k,\frac{\lambda}{\mu_1}}(\mu_2 y).$$

Proof. We consider

$$I(\mu_1,\mu_2|\lambda)=\left(\frac{\mu_1\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{\mu_1}{\lambda}}-1}\right)\frac{(1+\lambda t)^{\frac{\mu_1\mu_2}{\lambda}(x+y)}(1+\lambda t^2)^{\frac{{\mu_1}^2{\mu_2}^2}{\lambda}z}}{\mu_1\mu_2\log(1+\lambda t)^{\frac{1}{\lambda}}}$$

$$\times\left(\frac{\mu_2\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{\mu_2}{\lambda}}-1}\right)\left((1+\lambda t)^{\frac{\mu_1\mu_2}{\lambda}}-1\right)$$

or

$$I(\mu_1,\mu_2)=\frac{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{\mu_1}{\lambda}(x_1+\mu_2 x)}d\mu_0(x_1)}{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{\mu_1\mu_2}{\lambda}x}d\mu_0(x)}\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{\mu_2}{\lambda}(x_1+\mu_1 y)}d\mu_0(x_1),$$

where $I(\mu_1,\mu_2|\lambda)$ is symmetric in μ_1 and μ_2.

Now, using (45) and (46) together with the symmetry of $I(\mu_1,\mu_2|\lambda)$, we get

$$I(\mu_1,\mu_2|\lambda)=\left(\sum_{m=0}^{\infty}{}_HB_{m,\frac{\lambda}{\mu_1}}(\mu_2 x,{\mu_2}^2 z)\frac{{\mu_1}^m t^m}{m!}\right)\left(\sum_{k=0}^{\infty}S_k\left(\mu_1-1|\frac{\lambda}{\mu_2}\right)\frac{{\mu_2}^k t^k}{k!}\right)$$

$$\times\left(\sum_{j=0}^{\infty}B_{j,\frac{\lambda}{\mu_2}}(\mu_1 y)\frac{{\mu_2}^j t^j}{j!}\right).\frac{1}{\mu_1}$$

$$I(\mu_1,\mu_2|\lambda)=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}{\mu_1}^{m-j-1}{\mu_2}^{j}\sum_{k=0}^{j}\binom{j}{k}{}_HB_{m-j,\frac{\lambda}{\mu_1}}(\mu_2 x,{\mu_2}^2 z)\right.$$

$$\left.\times S_k\left(\mu_1-1|\frac{\lambda}{\mu_2}\right)B_{j-k,\frac{\lambda}{\mu_2}}(\mu_1 y)\right)\frac{t^m}{m!}.$$

A similar calculation yields

$$I(\mu_1,\mu_2|\lambda)=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}{\mu_2}^{m-j-1}{\mu_1}^{j}\sum_{k=0}^{j}\binom{j}{k}{}_HB_{m-j,\frac{\lambda}{\mu_2}}(\mu_1 x,{\mu_1}^2 z)\right.$$

$$\left.\times S_k\left(\mu_2-1|\frac{\lambda}{\mu_1}\right)B_{j-k,\frac{\lambda}{\mu_1}}(\mu_2 y)\right)\frac{t^m}{m!}.$$

Comparing the coefficient of $\frac{t^m}{m!}$ in last two equations, the required symmetry identity is constructed. □

Remark. For $z=0$ in Theorem 4.1, the below corollary naturally follows:

Corollary 4.1. *We have, for $m \geq 0$*

$$\sum_{j=0}^{m}\binom{m}{j}{\mu_1}^{m-j-1}{\mu_2}^{j}\sum_{k=0}^{j}\binom{j}{k}B_{m-j,\frac{\lambda}{\mu_1}}(\mu_2 x)S_k\left(\mu_1-1|\frac{\lambda}{\mu_2}\right)B_{j-k,\frac{\lambda}{\mu_2}}(\mu_1 y)$$

$$=\sum_{j=0}^{m}\binom{m}{j}{\mu_2}^{m-j-1}{\mu_1}^{j}\sum_{k=0}^{j}\binom{j}{k}B_{m-j,\frac{\lambda}{\mu_2}}(\mu_1 x)S_k\left(\mu_2-1|\frac{\lambda}{\mu_1}\right)B_{j-k,\frac{\lambda}{\mu_1}}(\mu_2 y).$$

Remark. For $y=0$ in Corollary 4.1, another corollary naturally follows:

Corollary 4.2. *We have, for $m \geq 0$*

$$\sum_{j=0}^{m}\binom{m}{j}{\mu_1}^{m-j-1}{\mu_2}^{j}\sum_{k=0}^{j}\binom{j}{k}B_{m-j,\frac{\lambda}{\mu_1}}(\mu_2 x)S_k\left(\mu_1-1|\frac{\lambda}{\mu_2}\right)$$

$$=\sum_{j=0}^{m}\binom{m}{j}{\mu_2}^{m-j-1}{\mu_1}^{j}\sum_{k=0}^{j}\binom{j}{k}B_{m-j,\frac{\lambda}{\mu_2}}(\mu_1 x)S_k\left(\mu_2-1|\frac{\lambda}{\mu_1}\right).$$

Remark. Further, if we take $\mu_2=1$ in Corollary 4.2, then we get the following equality:

Corollary 4.3. *We have, for $\mu_1 \in \mathbb{N}$*

$$B_{m,\lambda}(\mu_1 x)=\sum_{j=0}^{m}\binom{m}{j}B_{m-j,\frac{\lambda}{\mu_1}}(x)S_j(\mu_1-1|\lambda).$$

Theorem 4.2. *We have, for $\mu_1, \mu_2 \in \mathbb{N}$*

$$\sum_{l=0}^{m}\binom{m}{l}{\mu_1}^{l-1}{\mu_2}^{m-l}B_{m-l,\frac{\lambda}{\mu_2}}(\mu_1 y)\sum_{i=0}^{\mu_1-1}{}_HB_{l,\frac{\lambda}{\mu_1}}\left(\mu_2 x+\frac{\mu_2}{\mu_1}i,{\mu_2}^2 z\right) \tag{48}$$

$$=\sum_{l=0}^{m}\binom{m}{l}{\mu_2}^{l-1}{\mu_1}^{m-l}B_{m-l,\frac{\lambda}{\mu_1}}(\mu_2 y)\sum_{i=0}^{\mu_2-1}{}_HB_{l,\frac{\lambda}{\mu_2}}\left(\mu_1 x+\frac{\mu_1}{\mu_2}i,{\mu_1}^2 z\right).$$

Proof. We consider

$$I(\mu_1,\mu_2|\lambda)=\left(\frac{\mu_1\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{\mu_1}{\lambda}}-1}\right)(1+\lambda t)^{\frac{\mu_1\mu_2}{\lambda}(x+y)}(1+\lambda t^2)^{\frac{{\mu_1}^2{\mu_2}^2}{\lambda}z}$$

$$\times\frac{(1+\lambda t)^{\frac{\mu_1\mu_2}{\lambda}}-1}{(1+\lambda t)^{\frac{\mu_2}{\lambda}z}-1}\left(\frac{\mu_2\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{\mu_2}{\lambda}}-1}\right)\frac{1}{\mu_1}$$

$$=\frac{1}{\mu_1}\left(\frac{\mu_1\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{\mu_1}{\lambda}}-1}\right)(1+\lambda t)^{\frac{\mu_1\mu_2}{\lambda}x}(1+\lambda t^2)^{\frac{{\mu_1}^2{\mu_2}^2}{\lambda}z}$$

$$\times\left(\sum_{i=0}^{\mu_1-1}(1+\lambda t)^{\frac{\mu_2}{\lambda}i}\right)\left(\frac{\mu_2\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{\mu_2}{\lambda}}-1}\right)(1+\lambda t)^{\frac{\mu_1\mu_2}{\lambda}y}$$

$$= \frac{1}{\mu_1}\left(\frac{\mu_1\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{\mu_1}{\lambda}}-1}\right)\sum_{i=0}^{\mu_1-1}(1+\lambda t)^{\frac{\mu_1\mu_2}{\lambda}x+\frac{\mu_2}{\lambda}i}(1+\lambda t^2)^{\frac{{\mu_1}^2{\mu_2}^2}{\lambda}z}$$
$$\times\left(\frac{\mu_2\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{\mu_2}{\lambda}}-1}\right)(1+\lambda t)^{\frac{\mu_1\mu_2}{\lambda}y}$$
$$= \frac{1}{\mu_1}\left(\sum_{i=0}^{\mu_1-1}\sum_{l=0}^{\infty}{}_HB_{l,\frac{\lambda}{\mu_1}}\left(\mu_2x+\frac{\mu_2}{\mu_1}i,{\mu_2}^2z\right){\mu_1}^l\frac{t^l}{l!}\right)$$
$$\times\left(\sum_{m=0}^{\infty}B_{m,\frac{\lambda}{\mu_2}}(\mu_1y){\mu_2}^n\frac{t^m}{m!}\right),$$

$$I(\mu_1,\mu_2|\lambda) = \sum_{m=0}^{\infty}\left(\sum_{l=0}^{m}\binom{m}{l}{\mu_1}^{l-1}{\mu_2}^{n-l}B_{m-l,\frac{\lambda}{\mu_2}}(\mu_1y)\right.$$
$$\left.\times\sum_{i=0}^{\mu_1-1}{}_HB_{l,\frac{\lambda}{\mu_1}}\left(\mu_2x+\frac{\mu_2}{\mu_1}i,{\mu_2}^2z\right)\right)\frac{t^m}{m!}.$$

A similar calculation yields

$$I(\mu_1,\mu_2|\lambda) = \sum_{m=0}^{\infty}\left(\sum_{l=0}^{m}\binom{m}{l}{\mu_2}^{l-1}{\mu_1}^{m-l}B_{n-l,\frac{\lambda}{\mu_1}}(\mu_2y)\right.$$
$$\left.\times\sum_{i=0}^{\mu_2-1}{}_HB_{l,\frac{\lambda}{\mu_2}}\left(\mu_1x+\frac{\mu_1}{\mu_2}i,{\mu_1}^2z\right)\right)\frac{t^m}{m!}.$$

Comparing the coefficient of $\frac{t^m}{m!}$ in last two equations, the required symmetry identity is constructed. □

Remark. For $z = 0$ in Theorem 4.2, the below corollary naturally follows:

Corollary 4.4. *We have, for $m \geq 0$*

$$\sum_{l=0}^{m}\binom{m}{l}{\mu_1}^{l-1}{\mu_2}^{m-l}B_{m-l,\frac{\lambda}{\mu_2}}(\mu_1y)\sum_{i=0}^{\mu_1-1}B_{l,\frac{\lambda}{\mu_1}}\left(\mu_2x+\frac{\mu_2}{\mu_1}i\right)$$
$$= \sum_{l=0}^{m}\binom{m}{l}{\mu_2}^{l-1}{\mu_1}^{m-l}B_{m-l,\frac{\lambda}{\mu_1}}(\mu_2y)\sum_{i=0}^{\mu_2-1}B_{l,\frac{\lambda}{\mu_2}}\left(\mu_1x+\frac{\mu_1}{\mu_2}i\right).$$

Remark. Further for $y = 0$ in Corollary 4.4, another corollary naturally follows:

Corollary 4.5. *We have, for $\mu_1, \mu_2 \in \mathbb{N}$*

$${\mu_1}^{l-1}\sum_{i=0}^{\mu_1-1}B_{l,\frac{\lambda}{\mu_1}}\left(\mu_2x+\frac{\mu_2}{\mu_1}i\right) = {\mu_2}^{l-1}\sum_{i=0}^{\mu_2-1}B_{l,\frac{\lambda}{\mu_2}}\left(\mu_1x+\frac{\mu_1}{\mu_2}i\right).$$

Remark. Let $\mu_2 = 1$ in above corollary, then it reduces to the equality.

Corollary 4.6. *We have, for $m \geq 0$*

$$B_{l,\lambda}(\mu_1 x) = {\mu_1}^{l-1} \sum_{i=0}^{\mu_1 - 1} B_{l,\frac{\lambda}{\mu_1}} \left(x + \frac{1}{\mu_1} i \right).$$

Concluding remarks. In this paper, we have approached the degenerate Hermite-Bernoulli polynomials in the context of p-adic invariant integral on $\mathbb{Z}_p$. The preliminary steps towards accomplishment of results characterized in this paper are of common nature and are capable of extending to newly defined families of special polynomials.

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