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AN EXTENSION OF RANDOM SUMMATIONS OF INDEPENDENT AND IDENTICALLY DISTRIBUTED RANDOM VARIABLES

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Abstract. The main goal of this paper is to study an extension of random summations of independent and identically distributed random variables when the number of summands in random summation is a partial sum of n independent, identically distributed, non-negative integer-valued random variables. Some characterizations of random summations are considered. The central limit theorems and weak law of large numbers for extended random summations are established. Some weak limit theorems related to geometric random sums, binomial random sums and negative-binomial random sums are also investigated as asymptotic behaviors of extended random summations.

1. Introduction

Let $X, X_1, X_2, \dots$ be a sequence of independent, identically distributed (i.i.d.) random variables, having common distribution F_X, mean $E(X) = \mu$, and finite variance $D(X) = \sigma^2 < +\infty$. Suppose that $Y, Y_1, Y_2, \dots, Y_n$ are independent, identically distributed, non-negative, integer-valued random variables, with mean $E(Y) = \alpha$, and finite variance $D(Y) = \tau^2 < +\infty$. Additionally, suppose that the random variables $X, X_1, X_2, \dots$ and $Y, Y_1, Y_2, \dots, Y_n$ are independent. Set, for $n \geq 1, N_n := Y_1 + Y_2 + \cdots + Y_n$ and define $S_{N_n} := \sum_{j=1}^{N_n} X_j$. Then, the random summation S_{N_n} is a remarkable random summation because of the number of summands in random summation is a partial sum of i.i.d. random variables. It is obvious that when $P(N_1 = Y_1) = 1$, the random summation will be returned to the classical random sum $S_{Y_1} = \sum_{j=1}^{Y_1} X_j$.

Since the appearance of the Robbin's results in 1948 (see [21] for more details), the random summations have been investigated in the theory probability, statistics, stochastic processes and various related fields for quite some time by Robbins (1948), Feller (1971), Gnhedenko (1972), Gnedenko and Korolev (1996), Renyi (1970), Korolev and Kruglov (1990), Gut (2005), Hung

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and Thanh (2010), Hung et al. (2008), Durrett (1977), Hu and Cheng (2012), Chen and Shao (2007), Chen and Goldstein (2011), Omey and Vesilob (2015), ...(see [2–9, 15, 17, 19–21], and the references therein).

Recently, a number of results related to the specific cases of random summations like geometric random sums, compound Poisson sums, binomial random sums, negative–binomial random sums, etc. and their applications have been investigated by many authors as Gnedenko (1971), Gnedenko and Korolev (1996), Kruglov and Korolev (1990), Kalashnikov (1997), Vellaisamy and Chaudhuri (1996), Hung et al. (2008, 2010), Sunklodas (2009, 2014, 2015), ... Results of this nature may be found in [7, 10, 11, 13, 17, 22–25] and the references given there.

It makes sense to consider an extension of random summation when N_n, $n \geq 1$, is being a partial sum of independent, identically distributed non-negative integer-valued random variables. This extension comes from the actual requirements, e.g. a negative-binomial random sum is an extension of geometric random sum. Actually, suppose that Y_j, $j = 1, 2, \dots, n$ are n independent, geometric distributed random variables with parameter $p \in (0, 1)$. Then, the sum $S_{Y_1} = X_1 + X_2 + \cdots + X_{Y_1}$ is said to be a geometric random sum. Obviously, the sum $N_n = Y_1 + Y_2 + \cdots + Y_n$ will be a negative-binomial random variable with parameters n and p $(n \geq 1, p \in (0, 1))$, and the random sum S_{N_n} will be an extension of the S_{Y_1}, and we will call it by negative–binomial random summation (see Remark 2.2 in next Section and Theorems 3.4, 3.5 and 3.6 in last Section).

As far as we know, up to the present there is just a little number of the results concerning with convergence rates in limit theorems for an extension of random summations have been discussed like by Chen and Shao (2007) in [3] for case of independent and identically distributed random variables and by Islack (2013) in case of m-dependent and identically distributed random variables (see [12] for more details).

The main purpose of this paper is to investigate the characterizations and asymptotic behaviors of the extended random summations of i.i.d. random variables when the numbers of random summation is a partial sum of n i.i.d. non-negative integer-valued random variables. Some limit theorems for extended random summations like central limit theorem and weak law of large numbers are re-established. Moreover, some limit theorems for binomial random sum (Corollary 3.1), geometric random sums (Corollary 3.2 and Corollary 3.4) and negative-binomial random sums (Theorems 3.3, 3.4 and 3.5) also considered. The received results are extensions of the known results (see [1, 4–10, 16–18, 21–24] and [14]).

The organization of this paper is as follows. Section 2 is devoted to the discussion on some characterizations of random summation as mean, variance, generating function and characteristic function. Section 3 gives some results on asymptotic behaviors of normalization of the extended random summations.

2. Characterizations of random summation

Throughout this paper, we shall denote by $\psi(t)$ and $\varphi(t)$ the generating function and characteristic function of random variables, respectively. We start to consider some characterizations of the random summation S_{N_n} with two following propositions.

Proposition 2.1. *Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with common distribution F_X, mean $E(X) = \mu$, and finite variance $D(X) = \sigma^2 < +\infty$. Suppose that $Y, Y_1, Y_2, \ldots, Y_n$ are i.i.d. non-negative integer-valued random variables with mean $E(Y) = \alpha$, and finite variance $D(Y) = \tau^2 < +\infty$. Additionally, assume that random variables $X, X_1, X_2, \ldots$ and $Y, Y_1, Y_2, \ldots, Y_n$ are independent. Define*

$$S_{N_n} := \sum_{k=1}^{N_n} X_k, \tag{1}$$

where $N_n = Y_1 + Y_2 + \cdots + Y_n$. Then,

(1) *Probability distribution function of random summation S_{N_n} in (1) is defined in form*

$$\begin{aligned} F_{S_{N_n}}(x) = P(S_{N_n} \le x) &= \sum_{k=1}^{\infty} P(N_n = k) \times P(S_k \le x) \\ &= \sum_{k=1}^{\infty} P(N_n = k) \times F_X^{*k}(x), \end{aligned}$$

where $F_X^{*k}(x)$ denotes the k–th convolution power of the distribution function $F_X(x)$.

(2) *The Wald's identity for random summation S_{N_n} in (1) is given by*

$$E(S_{N_n}) = n \times E(Y) \times E(X) = n\alpha\mu.$$

(3) *Variance of random summation in (1) is defined by*

$$D(S_{N_n}) = n\left[E(Y) \times D(X) + D(Y) \times (E(X))^2\right] = n\left[\alpha\sigma^2 + \tau^2\mu^2\right].$$

Proof. (1) It is easy to verify that

$$\begin{aligned} F_{S_{N_n}}(x) = P(S_{N_n} \le x) = E(\mathbb{I}_{(-\infty,x]}(S_{N_n})) &= E(E(\mathbb{I}_{(-\infty,x]}(S_{N_n}) \mid N_n)) \\ &= \sum_{k=1}^{\infty} P(N_n = k) E(\mathbb{I}_{(-\infty,x]}(S_k) \mid N_n = k) \\ &= \sum_{k=1}^{\infty} P(N_n = k) E(\mathbb{I}_{(-\infty,x]}(S_k)) \\ &= \sum_{k=1}^{\infty} P(N_n = k) P(S_k \le x) = \sum_{k=1}^{\infty} P(N_n = k) \times F_X^{*k}(x), \end{aligned}$$

where

$$\mathbb{I}_A(x) = \begin{cases} 1, & \text{if } \ x \in A \\ 0, & \text{if } \ x \notin A. \end{cases}$$

(2) Evidently,

$$\begin{aligned} E(S_{N_n}) &= \sum_{k=1}^{\infty} P(N_n = k) \times E(S_k) = \sum_{k=1}^{\infty} kP(N_n = k) \times E(X) \\ &= nE(Y)E(X) = n\alpha\mu. \end{aligned}$$

(3) It is obvious that

$$\begin{aligned} E(S^2_{N_n}) = E(E(S^2_{N_n} \mid N_n)) &= \sum_{k=1}^{\infty} P(N_n = k) \times E(S^2_k \mid N_n = k) \\ &= \sum_{k=1}^{\infty} P(N_n = k) \times E(S^2_k) \\ &= \sum_{k=1}^{\infty} P(N_n = k) \left(\sum_{j=1}^{k} EX_j^2 + \sum_{\substack{i \neq j \\ i,j=1}}^{k} E(X_i) \times E(X_j) \right) \\ &= \sum_{k=1}^{\infty} P\left(N_n = k\right) \left[kE\left(X^2\right) + \left(k^2 - k\right) \left(E\left(X\right)\right)^2 \right] \\ &= \sum_{k=1}^{\infty} P\left(N_n = k\right) \left[kD\left(X\right) + k^2 (E\left(X\right))^2 \right] \\ &= E\left(N_n\right) \times D\left(X\right) + E\left({N_n}^2\right) \times \left(E\left(X\right)\right)^2. \end{aligned}$$

Therefore,

$$\begin{aligned} D(S_{N_n}) &= E(S^2_{N_n}) - (E(S_{N_n}))^2 \\ &= E(N_n) \times D(X) + E(N_n^2)(E(X))^2 - (E(N_n) \times E(X))^2 \\ &= n \left[E(Y) \times D(X) + D(Y) \times (E(X))^2 \right] = n \left[\alpha\sigma^2 + \tau^2\mu^2 \right]. \qquad \square \end{aligned}$$

Remark 2.1. For the case of $n = 1$, we have the well–known Wald's identity (see for instance [8], Theorem 9.1, page 194)

$$E(S_{Y_1}) = E(Y_1) \times E(X) = E(Y) \times E(X) = \alpha\mu,$$

and

$$D(S_{Y_1}) = E(Y_1) \times D(X) + (E(X))^2 \times D(Y_1) = \alpha\sigma^2 + \mu^2\tau^2.$$

Remark 2.2. On account of the Proposition 2.1, we consider following example, which will be used in next section. Suppose that $Y, Y_1, Y_2, \ldots, Y_n$ are independent, geometric distributed random variables with the success probability $p \in (0, 1)$ (for short, $Y_j \sim \mathcal{G}eo(p), p \in (0, 1)$. We remark that the partial sum $N_n = \sum_{j=1}^{n} Y_j$ will be a negative–binomial distributed random variable with two parameters n and p, $N_n \sim \mathcal{NB}(n,p)$, $n = 1, 2, \dots$, with

$$P(N_n = k) = \binom{k-1}{n-1} p^n (1-p)^{k-n}, \quad k = n, n+1, n+2, \dots.$$

Additionally, assume that random variables $X, X_1, X_2, \dots$ and $Y, Y_1, Y_2, \dots, Y_n$ are independent. Then, the random summation S_{N_n} will become a negative-binomial random summation of random variables $X_1, X_2, \dots$ with following probability distribution function

$$\begin{aligned} F_{S_{N_n}}(x) = P(S_{N_n} \le x) &= \sum_{k=1}^{\infty} \binom{k-1}{n-1} p^n (1-p)^{k-n} \times P(S_k \le x) \\ &= \sum_{k=1}^{\infty} \binom{k-1}{n-1} p^n (1-p)^{k-n} \times F_X^{*k}(x), \quad k = n, n+1, n+2, \dots, \end{aligned}$$

where $F_X^{*k}(x)$ denotes the k–th convolution power of the distribution function $F_X(x)$.

Proposition 2.2. *Assume that the hypotheses of Proportion 2.1 hold. Then*

(1) *Generating function of the random summation S_{N_n} in (1) is given by*

$$\psi_{S_{N_n}}(t) = E(t^{S_{N_n}}) = (\psi_Y \circ \psi_X(t))^n.$$

(2) *Characteristic function of the random summation S_{N_n} in (1) is defined by*

$$\varphi_{S_{N_n}}(t) = E(e^{itS_{N_n}}) = (\psi_Y \circ \varphi_X(t))^n.$$

Proof. Write $p_k = P(N_n = k), k = 0, 1, \dots$. Then,

(1) Direct computation shows that the generating function of the random summation S_{N_n} will be given by

$$\begin{aligned} \psi_{S_{N_n}}(t) &= E\left(t^{S_{N_n}}\right) = E\left(E\left(t^{S_{N_n}} \mid N_n\right)\right) \\ &= \sum_{k=0}^{\infty} p_k E\left(t^{S_{N_n}} \mid N_n = k\right) = \sum_{k=0}^{\infty} p_k E\left(t^{S_k}\right) \\ &= \sum_{k=0}^{\infty} p_k \left[E\left(t^X\right)\right]^k = \sum_{k=0}^{\infty} p_k [\psi_X(t)]^k = E\left([\psi_X(t)]^{N_n}\right) \\ &= E\left(\psi_X(t)^{Y_1} \times \psi_X(t)^{Y_2} \times \cdots \times \psi_X(t)^{Y_n}\right) = \left(E\left(\psi_X(t)^Y\right)\right)^n \\ &= (\psi_Y \circ \psi_X(t))^n. \end{aligned}$$

(2) Upon simple computation, the characteristic function of the random summation S_{N_n} is defined as follows

$$\varphi_{S_{N_n}}(t) = E\left(e^{itS_{N_n}}\right) = E\left(E\left(e^{itS_{N_n}} \mid N_n\right)\right) = \sum_{k=0}^{\infty} p_k E\left(e^{itS_{N_n}} \mid N_n = k\right)$$

$$= \sum_{k=0}^{\infty} p_k E\left(e^{itS_k}\right) = \sum_{k=0}^{\infty} p_k \left[E\left(e^{itX}\right)\right]^k = \sum_{k=0}^{\infty} p_k [\varphi_X(t)]^k$$
$$= E\left([\varphi_X(t)]^{N_n}\right) = E\left(\varphi_X(t)^{Y_1} \times \varphi_X(t)^{Y_2} \times \cdots \times \varphi_X(t)^{Y_n}\right)$$
$$= \left(E\left(\varphi_X(t)^{Y}\right)\right)^n = \left(\psi_Y \circ \varphi_X(t)\right)^n. \qquad \square$$

Remark 2.3. For the case of $n = 1$, when $P(N_1 = Y_1) = 1$, we will return to known results (see for instance [8], Theorem 9.2 and Theorem 9.3, page 194)

(1) $\psi_{S_{Y_1}}(t) = \psi_Y \circ \psi_X(t)$,
(2) $\varphi_{S_{Y_1}}(t) = \psi_Y \circ \varphi_X(t)$.

3. Asymptotic behaviors of the extended random summations

Throughout the forthcoming, unless otherwise specified, we shall denote by $\xrightarrow{d}$ and $\xrightarrow{P}$ the convergence in distribution and in probability, respectively. The following limit theorems are extensions of classical results for random summations of i.i.d. random variables.

Theorem 3.1 (Weak law of large numbers)**.** *Let $X, X_1, X_2, \cdots$ be a sequence of i.i.d. random variables with finite mean $\mu = E(X)$. Suppose that $Y, Y_1, Y_2, \ldots, Y_n$ are i.i.d., non-negative integer-valued random variables with finite mean $\alpha = E(Y)$. Moreover, assume that random variables $X, X_1, X_2, \ldots$ and $Y, Y_1, Y_2, \ldots, Y_n$ are independent. Write $S_{N_n} := \sum_{i=1}^{N_n} X_i$, where $N_n = Y_1 + Y_2 + \cdots + Y_n$. Then, the weak law of large numbers for random summation states in form*

$$\frac{S_{N_n}}{n} \xrightarrow{P} \alpha\mu \quad as \quad n \to \infty. \tag{2}$$

Proof. In view of the continuity theorem for characteristic function (see [8] for more details), it suffices to prove that

$$\varphi_{\frac{S_{N_n}}{n}}(t) \to \varphi_{\alpha\mu}(t) \quad \text{as} \quad n \to \infty, \quad \text{for} \quad -\infty < t < +\infty.$$

According to Proportion 2.2, the characteristic function of random summation $\frac{S_{N_n}}{n}$ is given by

$$\varphi_{\frac{S_{N_n}}{n}}(t) = \varphi_{S_{N_n}}\left(\frac{t}{n}\right) = \left[\psi_Y \circ \varphi_X\left(\frac{t}{n}\right)\right]^n.$$

Setting $h(t) = \ln\left[\psi_Y \circ \varphi_X(t)\right]$. Direct computation shows that $h(0) = 0$, and the derivative of the function $h(t)$ is calculated by

$$h'(t) = \frac{dh(t)}{dt} = \frac{\frac{d}{dt}[\psi_Y \circ \varphi_X(t)]}{\psi_Y \circ \varphi_X(t)} = \frac{E\left[Y(\varphi_X(t))^{Y-1}\varphi_X'(t)\right]}{E\left[(\varphi_X(t))^Y\right]}.$$

It is easily seen that $h^{'}(0) = E\left[Yi\mu\right] = iE(Y)\mu = i\alpha\mu$. Then, by letting $n \to \infty$, we have

$$\lim_{n\to\infty} \varphi_{\frac{S_{N_n}}{n}}(t) = \lim_{n\to\infty} \exp\left[nh(t/n)\right] = \lim_{t/n\to 0} \exp\left[\frac{h(t/n) - h(0)}{t/n - 0} \times t\right]$$
$$= \exp\left(h^{'}(0)\right) = \exp(\alpha\mu i t) = \varphi_{\alpha\mu}(t) \quad \text{for all } t.$$

Consequently,

$$\frac{S_{N_n}}{n} \xrightarrow{P} \alpha\mu \quad \text{as} \quad n \to \infty.$$

This finishes the proof. $\square$

Theorem 3.2 (Central limit theorem)**.** *Let $X, X_1, X_2, \dots$ be a sequence of i.i.d. random variables with mean $\mu = E(X)$ and positive, finite variance $D(X) = \sigma^2 < +\infty$. Suppose that $Y, Y_1, Y_2, \dots, Y_n$ is a sequence of i.i.d., positive integer-value random variables with mean $\alpha = E(Y) \in (0, +\infty)$ and positive, finite variance $D(Y) = \tau^2 < +\infty$. Additionally, assume that random variables $X, X_1, X_2, \dots$ and $Y, Y_1, Y_2, \dots, Y_n$ are independent. Write $S_{N_n} := \sum_{j=1}^{N_n} X_j$, where $N_n := Y_1 + Y_2 + \cdots + Y_n$. Then,*

$$\frac{S_{N_n} - n\alpha\mu}{\sqrt{n[\alpha\sigma^2 + \mu^2\tau^2]}} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as} \quad n \to \infty.$$

Proof. According to the assumptions on sequence $Y, Y_1, Y_2, \dots$, it is easily seen that the central limit theorem for a sequence of $Y_1, Y_2, \dots$ holds, i.e.,

$$\frac{N_n - E(N_n)}{\sqrt{D(N_n)}} = \frac{N_n - n\alpha}{\tau\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as} \quad n \to \infty. \tag{3}$$

By an argument analogous, on account of above assumptions for sequence $X, X_1, X_2, \dots$, the central limit theorem confirms that

$$\frac{S_n - E(S_n)}{\sqrt{D(S_n)}} = \frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as} \quad n \to \infty. \tag{4}$$

According to Corollary 4 from [21], combining the (3) with (4), we conclude that

$$\frac{S_{N_n} - E(S_{N_n})}{\sqrt{D(S_{N_n})}} = \frac{S_{N_n} - n\alpha\mu}{\sqrt{n[\alpha\sigma^2 + \mu^2\tau^2]}} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as} \quad n \to \infty.$$

(See also [5], Chapter VIII, Theorem 4, page 265 or [8], Theorem 3.2, page 346 or [20], Chapter VIII, Section 7, Theorem 2, page 473). The proof is completed. $\square$

Let $X, X_1, X_2, \dots$ be a sequence of i.i.d. random variables with mean $E(X) = \mu$, and finite variance $D(X) = \sigma^2 < +\infty$. Suppose that $Y, Y_1, Y_2, \dots, Y_n$ are independent, Bernoulli distributed random variables, $Y_j \sim \mathcal{B}ernoulli(p)$, $p \in (0,1)$. Moreover, assume that they are independent of all X_j, $j = 1, 2, \dots$. For $n \geq 1$, setting a partial sum $N_n = \sum_{j=1}^{n} Y_j$. It is easily seen that the N_n is a binomial random variable, $N_n \sim B_n(p)$, $n \geq 1$, $p \in (0,1)$. Thus, the random sum $S_{N_n} := X_1 + X_2 + \cdots + X_{N_n}$ has become a binomial random summation. The following corollary is a direct result of Theorem 3.2.

Corollary 3.1. *Assume that the hypotheses of Theorem 3.2 hold. Then*

$$\frac{S_{N_n} - np\mu}{\sqrt{n[p\sigma^2 + \mu^2 p(1-p)]}} \xrightarrow{d} \mathcal{N}(0,1) \quad as \quad n \to \infty.$$

Proof. Under the above assumption that $Y \sim \mathcal{B}er(p)$, $p \in (0,1)$, it follows that

$$E(Y) = p \quad \text{and} \quad D(Y) = p(1-p) \leq \frac{1}{4}$$

and

$$E(S_{N_n}) = n \times E(Y) \times E(X) = np\mu,$$

with

$$D(S_{N_n}) = n[E(Y) \times D(X) + D(Y) \times (E(X))^2] = np\sigma^2 + \mu^2 np(1-p).$$

Then, on account of the above Theorem 3.2, we finish the proof. □

Assume that $Y, Y_1, Y_2, \ldots, Y_n$ are independent, geometric distributed random variables with common success probabilities p, (in short, $Y_j \sim \mathcal{G}eo(p)$, $p \in (0,1)$). Setting $N_n = \sum_{j=1}^{n} Y_j$, $n \geq 1$ and $S_{N_n} = \sum_{j=1}^{N_n} X_j$. It is to be noticed that N_n is a negative-binomial random variable with two parameters n and p, in short, $N_n \sim \mathcal{NB}(n,p)$, $n \geq 1$, $p \in (0,1)$. Thus, the random sum S_{N_n} has become a negative-binomial random summation. The forthcoming theorems will be related to these negative-binomial random summations (see Remark 2.2).

Theorem 3.3. *Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d., positive random variables with a finite expectation $E(X) = \mu \in (0, +\infty)$. Suppose that $Y, Y_1, Y_2, \ldots,$ Y_n are independent, geometric random variables with common success probabilities p, $Y_j \sim \mathcal{G}eo(p)$, $p \in (0,1)$. Moreover, assume that random variables $X, X_1, X_2, \ldots$ and $Y, Y_1, Y_2, \ldots, Y_n$ are independent. Setting $N_n = \sum_{j=1}^{n} Y_j$, $n \geq 1$ and $S_{N_n} = \sum_{j=1}^{N_n} X_j$. Then,*

$$\frac{S_{N_n}}{E(N_n)} \xrightarrow{d} \mathcal{G} \quad as \quad p \to 0^+,$$

where $\mathcal{G}$ is a Gamma random variable with two parameters n and $\frac{n}{\mu}$ (in short, $\mathcal{G} \sim Gamma(n, \frac{n}{\mu})$).

Proof. Let us denote by $\varphi(t)$ and $\psi(t)$ the characteristic function and generating functions, respectively. It can be verified that the characteristic function of a Gamma distributed random variable is $\varphi_{\mathcal{G}}(t) = \left(\frac{1}{1 - i\mu\frac{t}{n}}\right)^n$, and the generating function of $Y \sim \mathcal{G}eo(p)$ is given by

$$\psi_Y(t) = \frac{pt}{1-(1-p)t}.$$

On account of Proposition 2.2, the characteristic function of $\frac{S_{N_n}}{E(N_n)}$ will be given as follows

$$(5)\quad \varphi_{\frac{S_{N_n}}{E(N_n)}}(t)=\varphi_{S_{N_n}}\left(\frac{p}{n}t\right)=\left[\psi_Y\circ\varphi_X\left(\frac{p}{n}t\right)\right]^n=\left[\frac{p\varphi_X\left(\frac{p}{n}t\right)}{1-(1-p)\,\varphi_X\left(\frac{p}{n}t\right)}\right]^n.$$

According to Taylor's expansion, there exists a real number c between 0 and $p\frac{t}{n}$, such that

$$\varphi_X\left(\frac{p}{n}t\right)=\varphi_X(0)+\frac{p}{n}t\varphi_X'(c)=1+\frac{p}{n}t\varphi_X'(c).$$

Therefore

$$\begin{aligned}\varphi_{\frac{S_{N_n}}{E(N_n)}}(t)&=\left[\frac{p\varphi_X\left(\frac{p}{n}t\right)}{1-(1-p)\,\varphi_X\left(\frac{p}{n}t\right)}\right]^n=\left[\frac{p\left(1+\frac{p}{n}t\varphi_X'(c)\right)}{1-(1-p)\left(1+\frac{p}{n}t\varphi_X'(c)\right)}\right]^n\\&=\left[\frac{1+\frac{p}{n}t\varphi_X'(c)}{1-\frac{t}{n}\varphi_X'(c)+\frac{p}{n}t\varphi_X'(c)}\right]^n.\end{aligned}$$

Letting $p\to 0^+$, so $\frac{p}{n}t\to 0^+$, it follows that $c\to 0$. Then,

$$(6)\quad\begin{aligned}\lim_{p\to 0^+}\varphi_{\frac{S_{N_n}}{E(N_n)}}(t)&=\left[\frac{1}{1-\frac{t}{n}\varphi_X'(0)}\right]^n\\&=\left[\frac{1}{1-\frac{t}{n}iE(X)}\right]^n=\left(\frac{1}{1-i\mu\frac{t}{n}}\right)^n=\varphi_{\mathcal{G}}(t).\end{aligned}$$

This finished the proof. $\square$

It is worth pointing out that when $P(N_1=Y_1)=1$, we have a desired geometric sum S_{N_1} and the following corollary will be hold and it is analogous to Renyi's results in 1957 for geometric sum of i.i.d. positive-valued random variables (see [13, 17, 20], for more details).

Corollary 3.2. *Let $X, X_1, X_2, \dots$ be a sequence of i.i.d., positive-valued random variables with finite expectation $E(X)=\mu\in(0,+\infty)$. Let Y_1 be a geometric random variable with success probability $p\in(0,1)$. Moreover, assume that random variables $X, X_1, X_2, \dots$ and Y_1 are independent. Write $S_{Y_1}=\sum_{j=1}^{Y_1}X_j$. Then,*

$$\frac{S_{Y_1}}{E(Y_1)}\xrightarrow{d}\mathcal{W}^{(\mu)}\quad as\quad p\to 0^+,$$

where $\mathcal{W}^{(\mu)}$ is an exponential random variable with expectation $E(\mathcal{W}^{(\mu)})=\mu$, (in short, $\mathcal{W}^{(\mu)}\sim Exp(\mu^{-1})$).

Proof. Applying (5) and (6) when $n = 1$, with $N_1 = Y_1$ and $S_{N_1} = \sum_{i=1}^{Y_1} X_i$, we conclude that

$$\lim_{p\to 0^+} \varphi_{\frac{S_{Y_1}}{E(Y_1)}} = \left(\frac{1}{1-i\mu t}\right) = \varphi_{\mathcal{W}^{\mu}}(t).$$

It follows that

$$\frac{S_{Y_1}}{E(Y_1)} \xrightarrow{d} \mathcal{W}^{(\mu)} \quad \text{as} \quad p \to 0^+.$$

The proof is complete finished □

Suppose that $Y, Y_1, Y_2, \ldots, Y_n$ are independent, geometric distributed random variables with common success probability p, $Y_j \sim \mathcal{G}eo(p)$, $p \in (0,1)$. For every $n \geq 1$, we will denote by $N_n = \sum_{j=1}^{n} Y_j$ the partial sum. The random sum N_n is a negative-binomial random variable with parameters n and $p, N_n \sim \mathcal{NB}(n,p)$, $n \geq 1$, $p \in (0,1)$. Thus, the random summation $S_{N_n}^2 = \sum_{j=1}^{N_n} X_j^2$ is a negative-binomial random summation of squares of standard normal distributed random variables.

Theorem 3.4. *Let $X, X_1, X_2, \ldots$ be a sequence of independent, standard normal distributed random variables, $X_j \sim \mathcal{N}(0,1)$, $j = 1, 2, \ldots$. Suppose that $Y, Y_1, Y_2, \cdots, Y_n$ are independent, geometric distributed random variables with common success probability p, $Y_j \sim \mathcal{G}eo(p)$, $p \in (0,1)$. Moreover, assume that random variables $X, X_1, X_2, \ldots$ and $Y, Y_1, Y_2, \ldots, Y_n$ are independent. For every $n \geq 1$, write $N_n = \sum_{j=1}^{n} Y_j$ and define $S_{N_n}^2 = \sum_{j=1}^{N_n} X_j^2$. Then,*

$$\frac{S_{N_n}^2}{E(N_n)} \xrightarrow{d} \mathcal{G} \quad as \quad p \to \ 0^+,$$

where $\mathcal{G}$ is a gamma distributed random variable, $\mathcal{G} \sim Gamma(n,n)$.

Proof. Let us denote by $\varphi_{X^2}(t)$ the characteristic function of random variable X^2 and by $\varphi_{\mathcal{G}}(t)$ the characteristic function of random variables $\mathcal{G}$. It is easily seen that the characteristic function of a gamma random variable $\varphi_{\mathcal{G}}(t)$ is given by

$$\varphi_{\mathcal{G}}(t) = \left(\frac{n}{n-it}\right)^n.$$

Since $X_j \sim \mathcal{N}(0,1)$, we inference that X_j^2 will be a chi-squared random variable with degree of freedom 1. (in short, $X_j^2 \sim \chi_1^2(1)$.) Therefore, the characteristic function of X_j^2 is defined as follows

$$\varphi_{X^2}(t) = \frac{1}{\sqrt{1-2it}}.$$

Since $Y \sim \mathcal{G}eo(p)$, the generating function of Y is given by

$$\psi_Y(t) = \frac{pt}{1-(1-p)\,t}.$$

Then, according to Proposition 2.2, the characteristic function of $S^2_{N_n}$ is defined by

$$\varphi_{S^2_{N_n}}(t) = [\psi_Y \circ \varphi_{X^2}(t)]^n = \left[\frac{p\varphi_{X^2}(t)}{1-(1-p)\varphi_{X^2}(t)}\right]^n.$$

Thus, the characteristic function of $\frac{S^2_{N_n}}{E(N_n)}$ is given as follows

$$\begin{aligned}\varphi_{\frac{S^2_{N_n}}{E(N_n)}}(t) = \varphi_{S^2_{N_n}}\left(\frac{p}{n}t\right) &= \left[\frac{p\varphi_{X^2}\left(\frac{p}{n}t\right)}{1-(1-p)\varphi_{X^2}\left(\frac{p}{n}t\right)}\right]^n \\ &= \left(\frac{\sqrt{1-2i\frac{p}{n}t}+1-p}{2-2i\frac{t}{n}-p}\right)^n.\end{aligned}$$

Letting $p \to 0^+$, we conclude that

$$\lim_{p\to 0^+}\varphi_{\frac{S^2_{N_n}}{E(N_n)}}(t) = \left(\frac{n}{n-it}\right)^n = \varphi_{\mathcal{G}}(t). \tag{7}$$

We have the complete proof. □

It is worth noticing that in case of $P(N_1 = Y_1) = 1$, we have a result related to geometric sum of square standard normal distributed random variables, $S_{Y_1} = \sum_{j=1}^{Y_1} X_j^2$. This geometric sum should be considered as a $\chi^2_{Y_1}$-squared random variable with geometric degrees of freedom Y_1.

Corollary 3.3. *Let $X, X_1, X_2, \ldots$ be a sequence of independent, standard normal distributed random variables, $X \sim \mathcal{N}(0,1)$. Let Y_1 be a geometric random variable with parameter p, $Y_1 \sim \mathcal{G}eo(p)$, $p \in (0,1)$. Moreover, assume that random variables $X, X_1, X_2, \ldots$ and Y_1 are independent. Set $S^2_{Y_1} = \sum_{j=1}^{Y_1} X_j^2$. Then,*

$$\frac{S^2_{Y_1}}{E(Y_1)} \xrightarrow{d} \mathcal{W}^{(1)} \quad as \quad p \to 0^+,$$

where $\mathcal{W}^{(1)} \sim Exp(1)$ is an exponential random variable with mean 1.

Proof. The proof is immediate from (7) with $n = 1$, i.e.,

$$\lim_{p\to 0^+}\varphi_{\frac{S^2_{Y_1}}{E(Y_1)}}(t) = \left(\frac{1}{1-it}\right) = \varphi_{\mathcal{W}^1}(t).$$

□

The following theorem is concerning with the negative-binomial random summation $S_{N_n} = \sum_{j=1}^{N_n} X_j$, where $N_n = Y_1 + Y_2 + \cdots + Y_n$, with $Y_j \sim \mathcal{G}eo(p)$, $p \in (0,1)$.

Theorem 3.5. *Let $X, X_1, X_2, \dots$ be a sequence of i.i.d., non-negative random variables with mean zero $E(X) = 0$ and finite variance $D(X) = \sigma^2 < +\infty$. Suppose that $Y, Y_1, Y_2, \dots, Y_n$ are independent, geometric distributed random variables with common success probability p, $Y_j \sim \mathcal{G}eo(p)$, $p \in (0,1)$. Moreover, assume that random variables $X, X_1, X_2, \dots$ and $Y, Y_1, Y_2, \dots, Y_n$ are independent. Write $N_n = \sum_{j=1}^{n} Y_j$ and define $S_{N_n} = \sum_{j=1}^{N_n} X_j$. Then,*

$$\frac{S_{N_n}}{\sqrt{E(N_n)}} \xrightarrow{d} \sum_{j=1}^{n} \mathcal{L}_j \quad as \quad p \to 0^+, \tag{8}$$

where $\mathcal{L}_1, \mathcal{L}_1, \dots, \mathcal{L}_n$ are n independent, Laplace distributed random variables with parameters zero and $\frac{\sqrt{2n}}{\sigma}$, $\mathcal{L}_j \sim Laplace\left(0, \frac{\sqrt{2n}}{\sigma}\right)$, $j = 1, 2, \dots, n$.

Proof. An easily computation that, for a geometric random variable $Y \sim \mathcal{G}eo(p)$, the generating function is given by

$$\psi_Y(t) = \frac{pt}{1 - (1-p)\,t}.$$

According to Proposition 2.2, the characteristic function of $\frac{S_{N_n}}{\sqrt{E(N_n)}}$ will be calculated as follows

$$\begin{aligned}\varphi_{\frac{S_{N_n}}{\sqrt{E(N_n)}}}(t) = \varphi_{S_{N_n}}\left(\sqrt{\frac{p}{n}}t\right) &= \left[\psi_Y \circ \varphi_X\left(\sqrt{\frac{p}{n}}t\right)\right]^n \\ &= \left[\frac{p\varphi_X\left(\sqrt{\frac{p}{n}}t\right)}{1 - (1-p)\,\varphi_X\left(\sqrt{\frac{p}{n}}t\right)}\right]^n.\end{aligned}$$

On account of Taylor expansion, there exists a real number c between 0 and $\sqrt{\frac{p}{n}}t$, we obtain

$$\begin{aligned}\varphi_X\left(\sqrt{\frac{p}{n}}t\right) &= \varphi_X(0) + \sqrt{\frac{p}{n}}t\varphi'_X(0) + \frac{pt^2}{2n}\varphi''_X(c) \\ &= 1 + \sqrt{\frac{p}{n}}tiE(X) + \frac{pt^2}{2n}\varphi''_X(c) \\ &= 1 + \frac{pt^2}{2n}\varphi''_X(c).\end{aligned}$$

Therefore

$$\begin{aligned}\varphi_{\frac{S_{N_n}}{\sqrt{E(N_n)}}}(t) &= \left[\frac{p\varphi_X\left(\sqrt{\frac{p}{n}}t\right)}{1 - (1-p)\,\varphi_X\left(\sqrt{\frac{p}{n}}t\right)}\right]^n = \left[\frac{p\left[1 + \frac{pt^2}{2n}\varphi''(c)\right]}{1 - (1-p)\left[1 + \frac{pt^2}{2n}\varphi''(c)\right]}\right]^n \\ &= \left[\frac{1 + \frac{pt^2}{2n}\varphi''(c)}{1 - \frac{t^2}{2n}\varphi''(c) + \frac{pt^2}{2n}\varphi''(c)}\right]^n.\end{aligned}$$

Letting $p \to 0^+$, it follows $\sqrt{\frac{p}{n}}t \to 0^+$. Thus $c \to 0$. Then,

$$\lim_{p\to 0^+} \varphi_{\frac{S_{N_n}}{\sqrt{E(N_n)}}}(t) = \left[\frac{1}{1 - \frac{t^2}{2n}\varphi''(0)}\right]^n = \left[\frac{1}{1 + \frac{\sigma^2 t^2}{2n}}\right]^n = \varphi_{\sum\limits_{j=1}^{n} \mathcal{L}_j}(t).$$

This finishes the proof. □

Corollary 3.4. *Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d., non-negative random variables with mean $E(X) = 0$ and finite variance $D(X) = \sigma^2 < +\infty$. Assume that $Y_1 \sim \mathcal{G}eo(p)$ and independent of all X_j, $j \geq 1$. Then,*

$$\frac{S_{Y_1}}{\sqrt{E(Y_1)}} \xrightarrow{d} \mathcal{L} \quad as \quad n \to \infty,$$

where $S_{Y_1} = \sum_{j=1}^{Y_1} X_j$ and $\mathcal{L} \sim Laplace(0, \frac{\sqrt{2}}{\sigma})$.

Proof. On account of (8) for $n = 1$, the corollary is proven. □

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