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CLASSIFICATION OF A FAMILY OF RIBBON 2-KNOTS WITH TRIVIAL ALEXANDER POLYNOMIAL

Taizo Kanenobu and Toshio Sumi

Abstract. We consider a family of ribbon 2-knots with trivial Alexander polynomial. We give nonabelian $SL(2, \boldsymbol{C})$-representations from the groups of these knots, and then calculate the twisted Alexander polynomials associated to these representations, which allows us to classify this family of knots.

1. Introduction

A ribbon 2-knot is an embedded 2-sphere in S^4 obtained by adding r 1-handles to a trivial 2-link with $r + 1$ components for some r, which is called a ribbon 2-knot of r-fusion (cf. [14, 15]). Yasuda [16–20] studied an enumeration of ribbon 2-knot with ribbon crossing number up to 4, where the Alexander polynomial of each ribbon 2-knot was given but it was not referred about the classification of the knots so much. Takahashi [12] classified ribbon 2-knots of 1-fusion with small ribbon crossing number using the Alexander polynomial, representations of the knot group into $SL(2, \boldsymbol{C})$, and twisted Alexander polynomial. Recently, Kanenobu and Komatsu [2] have enumerated ribbon 2-knots based on the virtual arc presentation of ribbon 2-knots, and Kanenobu and Sumi [3] have attempted the classification of these ribbon 2-knots, where they used the Alexander polynomial, homology of double branched covering space, representations of the knot group into $SL(2, \boldsymbol{F})$, $\boldsymbol{F}$ a finite field, and twisted Alexander polynomial.

In order to classify ribbon 2-knots the Alexander polynomial is a very useful invariant. However, it is difficult to distinguish ribbon 2-knots sharing the same Alexander polynomial. In this paper, we show the effectiveness of the twisted Alexander polynomial in classifying the ribbon 2-knots, which was first achieved by Takahashi [12], and then by the authors [3] as mentioned above. The twisted Alexander polynomial was introduced by Lin [6] for knots in S^3 and by Wada [13] for finitely presentable groups, which is a generalization of the classical

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Alexander polynomial and has many applications. In this paper, we classify a family of ribbon 2-knots of 1-fusion with trivial Alexander polynomial $K_n = R(1, n, -n-1, 1)$, $n \in \boldsymbol{Z}$ (see Sect. 2 for the definition of $R(1, n, -n-1, 1)$). First, we show that the number of irreducible representations $\rho : \pi_1(S^4 - K_n) \to SL(2, \boldsymbol{C})$ up to conjugate is $2n$ (Proposition 3.5), where $n \geq 0$, classifying the knots K_n, $n \geq 0$. Next, we distinguish K_n and K_{-n-1}, which are mirror images one another, by Wada's twisted Alexander polynomials (Proposition 4.1). Our main theorem is the following.

Theorem 1.1. *For the family of ribbon 2-knots K_n, $n \in \boldsymbol{Z}$, of 1-fusion we have the following*:

(i) *K_n has trivial Alexander polynomial.*
(ii) *The mirror image of K_n is isotopic to K_{-n-1}.*
(iii) *K_n is trivial if and only if $n = 0$ or -1.*
(iv) *For m, $n \in \boldsymbol{Z} - \{-1, 0\}$, K_m and K_n are isotopic if and only if $m = n$.*

This paper is organized as follows: In Sect. 2 we define a ribbon 2-knot K_n of 1-fusion and give some properties. In Sect. 3 we decide irreducible representations of the group of the knot K_n into $SL(2, \boldsymbol{C})$ up to conjugate. In Sect. 4 we calculate the twisted Alexander polynomial of K_n associated to the representations given in Sect. 3.

2. Ribbon 2-knot of 1-fusion

We define a ribbon 2-knot $R(p_1, q_1, \dots, p_n, q_n)$ of 1-fusion as follows. Let $L_0 = S^1_0 \cup S^1_1$ be a trivial link with 2 components in $\boldsymbol{R}^3$. We add a band B to L_0 as shown in Fig. 1, where $\tau_{p_1}, \dots, \tau_{p_n}$, $\sigma_{q_1}, \dots, \sigma_{q_n}$ are pairs $(D^3, a \cup \beta)$ of a 3-ball D^3 and a properly embedded arc a and band β as shown in Fig. 2.

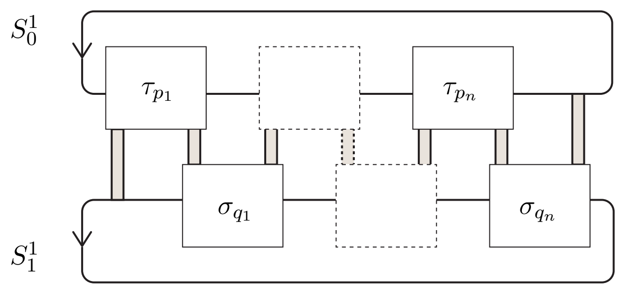

FIGURE 1. Adding a band B to a trivial link $L_0 = S^1_0 \cup S^1_1$.

Regard the band B as the image of an embedding $b : I \times I \to \boldsymbol{R}^3$, $B = b(I \times I)$, so that $S^1_i \cap b(I \times I) = b(I \times \{i\})$, $i = 0$, 1, where I is the unit interval $[0, 1]$. We take disjoint 2-disks $D_0 \cup D_1$ in $\boldsymbol{R}^3$ so that $S^1_i = \partial D_i$, $i = 0$,

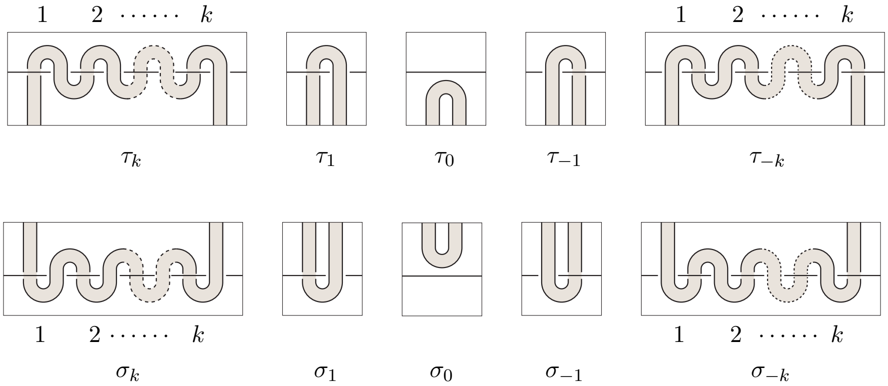

FIGURE 2. τ_p and σ_q.

1. Let $K_0 = (L_0 - b(I \times \partial I)) \cup b(\partial I \times I)$. Then we obtain a ribbon 2-knot $R(p_1, q_1, \ldots, p_n, q_n)$ of 1-fusion in $S^4 = \boldsymbol{R}^4 \cup \{\infty\}$ by the moving pictures:

$$R(p_1, q_1, \ldots, p_n, q_n) \cap (\boldsymbol{R}^3 \times \{t\}) = \begin{cases} K_0 & \text{for } |t| < 1; \\ K_0 \cup B = L_0 \cup B & \text{for } |t| = 1; \\ L_0 & \text{for } 1 < |t| < 2; \\ D_0 \cup D_1 & \text{for } |t| = 2; \\ \emptyset & \text{for } |t| > 2. \end{cases}$$

Any ribbon 2-knot of 1-fusion is represented in this form.

Note that a ribbon 2-knot is *negative-amphicheiral*, that is, a ribbon 2-knot K is ambient isotopic to $-K!$, which is obtained from K by taking the mirror image and then reversing the orientation (see [11, Theorem 2.18], [10, Proposition 4.1]). So, we show that the knot K_n, $n > 0$, is non-positive-amphicheiral and non-invertible. If a ribbon 2-knot has a non-reciprocal Alexander polynomial, that is, $\Delta_K(t) \neq \Delta_K(t^{-1})$ up to $\pm t^k$, then it is not non-positive-amphicheiral and is non-invertible (cf. [11, Proposition 3.26]).

Example 2.1. Figure 3 shows the ribbon 2-knot $K_2 = R(1, 2, -3, 1)$.

Note that $R(p_1, q_1, \ldots, p_n, q_n)$ is isotopic to $R(-q_n, -p_n, \ldots, -q_1, -p_1)$, which is the mirror image of $R(q_n, p_n, \ldots, q_1, p_1)$.

The group of $K = R(p_1, q_1, \ldots, p_n, q_n)$, $G = \pi_1(S^4 - K)$, has a Wirtinger presentation

$$\langle\, x,\, y \mid x^{-1} w^{-1} y w \,\rangle, \quad w = x^{p_1} y^{q_1} \cdots x^{p_n} y^{q_n}, \tag{1}$$

where x and y are meridians of S_0^2 and S_1^2, respectively.

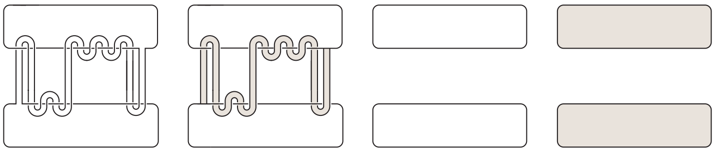

FIGURE 3. The ribbon 2-knot $R(1, 2, -3, 1)$.

The Alexander polynomial of a ribbon 2-knot K, $\Delta_K(t) \in \boldsymbol{Z}[t^{\pm 1}]$, is defined up to $\pm t^n$, which we normalize so that $\Delta_K(1) = 1$ and $(d/dt)\Delta_K(1) = 0$ (cf. [1, 4, 7]). For a ribbon 2-knot of 1-fusion we have the following.

Proposition 2.2. *The normalized Alexander polynomial of the ribbon 2-knot $R(p_1, q_1, \dots, p_n, q_n)$ of 1-fusion is*

$$\begin{aligned} &t^{-q_1-q_2-\cdots-q_n}\left(1 - t^{p_1} + t^{p_1+q_1} - t^{p_1+q_1+p_2} + \cdots \right. \\ &\qquad\qquad \left. - t^{p_1+q_1+\cdots+p_n} + t^{p_1+q_1+\cdots+p_n+q_n}\right) \\ = {}& t^{p_n+p_{n-1}+\cdots+p_1}\left(1 - t^{-q_n} + t^{-q_n-p_n} - t^{-q_n-p_n-q_{n-1}} + \cdots \right. \\ &\qquad\qquad \left. - t^{-q_n-p_n-q_{n-1}-\cdots-q_1} + t^{-q_n-p_n-q_{n-1}-\cdots-q_1-p_1}\right). \end{aligned}$$

3. Representation to $\boldsymbol{SL(2, C)}$

Let G be a finitely presented group. Two representations, namely homomorphisms, ρ, $\rho' : G \to SL(2, \boldsymbol{C})$ are called *conjugate* if $\rho(g) = C\rho'(g)C^{-1}$ for some $C \in SL(2, \boldsymbol{C})$ and for any $g \in G$. A representation $\rho : G \to SL(2, \boldsymbol{C})$ is said to be *abelian* if $\rho(G)$ is an abelian subgroup of $SL(2, \boldsymbol{C})$. A representation ρ is called *reducible* if there exists a proper invariant subspace of $\boldsymbol{C}^2$ under the action of $\rho(G)$. This is equivalent to saying that ρ can be conjugate to a representation whose image consists of upper triangular matrices. It is easy to see that every abelian representation is reducible, but the converse does not hold. When ρ is not reducible, it is called *irreducible*.

The following is due to Riley [8, 9].

Proposition 3.1. *If two matrices X, Y are conjugate in $SL(2, \boldsymbol{C})$ and $XY \neq YX$, then there exists a matrix $C \in SL(2, \boldsymbol{C})$ such that*

$$CXC^{-1} = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix}, \quad CYC^{-1} = \begin{pmatrix} s & 0 \\ u & s^{-1} \end{pmatrix},$$

where s, $u \in \boldsymbol{C}$ with $s \neq 0$ and $(s, u) \neq (\pm 1, 0)$.

Furthermore, if there exists a matrix $D \in SL(2, \boldsymbol{C})$ such that

$$DXD^{-1} = \begin{pmatrix} s' & 1 \\ 0 & s'^{-1} \end{pmatrix}, \quad DYD^{-1} = \begin{pmatrix} s' & 0 \\ u' & s'^{-1} \end{pmatrix},$$

where s', $u' \in \boldsymbol{C}$ with $s' \neq 0$ and $(s', u') \neq (\pm 1, 0)$, then $(s', u') = (s, u)$ or (s^{-1}, u).

Let us consider the presentation Eq. (1) of the group G of the ribbon 2-knot $R(p_1, q_1, \dots, p_n, q_n)$ of 1-fusion. Then since x and y are conjugate, by Proposition 3.1 any nonabelian representation $G \to SL(2, \boldsymbol{C})$ is conjugate to a representation $\rho : G \to SL(2, \boldsymbol{C})$ given by

$$\rho(x) = X = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix}, \quad \rho(y) = Y = \begin{pmatrix} s & 0 \\ u & s^{-1} \end{pmatrix} \tag{2}$$

for some s, $u \in \boldsymbol{C}$ with $s \neq 0$ and $(s, u) \neq (\pm 1, 0)$; such a representation ρ is parametrized by the trace $s + s^{-1}$ and u. Furthermore, it is easy to prove the following.

Lemma 3.2. *A nonabelian representation ρ in Eq. (2) is reducible if and only if either $u = -(s - s^{-1})^2$ or $u = 0$.*

From now on we focus on the family of ribbon 2-knots $K_n = R(1, n, -n-1, 1)$, $n \in \boldsymbol{Z}$ of 1-fusion. Let $G_n = \pi_1(S^4 - K_n)$. Then

$$G_n = \langle\, x, y \mid w_n x = y w_n \,\rangle, \quad w_n = x y^n x^{-n-1} y.$$

We define a nonabelian representation

$$\rho : G_n \to SL(2, \boldsymbol{C})$$

by the correspondence Eq. (2), where s, $u \in \boldsymbol{C}$ with $s \neq 0$ and $(s, u) \neq (\pm 1, 0)$. Then, we have the following.

Proposition 3.3. *Suppose $n > 0$. The parameters s and u satisfy:*

$$s = \xi_n^k \quad (k = 1, 2, \dots, 2n, 2n+2, 2n+3, \dots, 4n+1); \tag{3}$$

$$u^2 + \left(p^2 - 4\right) u + \epsilon p + 2 = 0, \tag{4}$$

where $\xi_n = \exp \frac{\pi\sqrt{-1}}{2n+1}$, $p = s + s^{-1}$, and $\epsilon = (\xi_n^k)^{2n+1} = (-1)^k$.

We use the following lemma in the proof of Proposition 3.3.

Lemma 3.4. *For $i \in \boldsymbol{Z}$, we have:*

$$X^i = \begin{pmatrix} s^i & f_i \\ 0 & s^{-i} \end{pmatrix}, \quad Y^i = \begin{pmatrix} s^i & 0 \\ u f_i & s^{-i} \end{pmatrix},$$

where

$$f_i = \begin{cases} \dfrac{s^i - s^{-i}}{s - s^{-1}} & \text{if } s \neq \pm 1; \\ i s^{i-1} & \text{if } s = \pm 1. \end{cases}$$

Proof. Induction on i. $\square$

Proof of Proposition 3.3. Let

$$W_n = XY^nX^{-n-1}Y = \begin{pmatrix} (W_n)_{11} & (W_n)_{12} \\ (W_n)_{21} & (W_n)_{22} \end{pmatrix}.$$

Then using Lemma 3.4, we have:

$$\begin{aligned}
(W_n)_{11} &= s + u\left(s + s^{-n}f_n + s^{n+1}f_{-n-1}\right) + u^2 f_n f_{-n-1} \\
&= s + u(1-s^2)f_n f_{n+1} - u^2 f_n f_{n+1};
\end{aligned} \tag{5}$$

$$\begin{aligned}
(W_n)_{12} &= 1 + s^n f_{-n-1} + us^{-1}f_{-n-1}f_n \\
&= -s^{n+1}f_n - us^{-1}f_n f_{n+1};
\end{aligned} \tag{6}$$

$$\begin{aligned}
(W_n)_{21} &= u + us^{-n-1}f_n + u^2 s^{-1} f_{-n-1} f_n \\
&= us^{-n}f_{n+1} - u^2 s^{-1} f_n f_{n+1};
\end{aligned} \tag{7}$$

$$\begin{aligned}
(W_n)_{22} &= s^{-1} + us^{-2}f_n f_{-n-1} \\
&= s^{-1} - us^{-2}f_n f_{n+1},
\end{aligned}$$

where we use $f_{-k} = -f_k$ and $s^k f_{k+1} - s^{k+1} f_k = 1$ for $k \in \boldsymbol{Z}$.

Let

$$R_n = W_nX - YW_n = \begin{pmatrix} (R_n)_{11} & (R_n)_{12} \\ (R_n)_{21} & (R_n)_{22} \end{pmatrix}.$$

Then

$$(R_n)_{11} = 0;$$

$$(R_n)_{12} = (W_n)_{11} - (s - s^{-1})(W_n)_{12}; \tag{8}$$

$$(R_n)_{21} = (s - s^{-1})(W_n)_{21} - u(W_n)_{11}; \tag{9}$$

$$(R_n)_{22} = (W_n)_{21} - u(W_n)_{12}. \tag{10}$$

From the relation $w_n x = y w_n$, it should hold that $R_n = W_nX - YW_n = O$. Using Eqs. (6) and (7), we have $(W_n)_{21} - u(W_n)_{12} = uf_{2n+1}$. Then from $(R_n)_{22} = 0$, Eq. (10) yields either $u = 0$ or $f_{2n+1} = 0$. If $u = 0$, then by Eqs. (5) and (6) $(W_n)_{11} = s$ and $(W_n)_{12} = -s^{n+1}f_n$. Substituting them into Eq. (8) we have $(R_n)_{12} = s - (s - s^{-1})(-s^{n+1}f_n) = s^{2n+1} \neq 0$, and so $u \neq 0$. From $f_{2n+1} = 0$ we obtain Eq. (3).

Next, using Eqs. (5) and (6), we have

$$(W_n)_{11} - (s - s^{-1})(W_n)_{12} = s^{2n+1} - u(s - s^{-1})^2 f_n f_{n+1} - u^2 f_n f_{n+1}.$$

Then from $(R_n)_{21} = 0$, Eq. (9) yields Eq. (4). In fact, if $s = \xi_n^k$, then $s^{2n+1} = \epsilon$ and $f_n f_{n+1} = -s/(s+\epsilon)^2 = -1/(s + s^{-1} + 2\epsilon)$. □

For a group G we denote by $r(G)$ the number of irreducible representations to $SL(2, \boldsymbol{C})$ up to conjugate. Then, by Lemmas 3.6 and 3.7 below, we obtain the following.

Proposition 3.5. *For $n > 0$, we have $r(G_n) = 4n$.*

Lemma 3.6. *The nonabelian representations* $\rho : G_n \to SL(2, \boldsymbol{C})$ *defined as above are irreducible.*

Proof. Assume the representation ρ in Eq. (2) is reducible. Then by Lemma 3.2, $u = 4 - p^2$ or $u = 0$. Then Eq. (4) implies $\epsilon p + 2 = 0$, which contradicts Eq. (3). □

Lemma 3.7. *If* $s = \xi_n^k$ $(k = 1, 2, \ldots, 2n, 2n + 2, 2n + 3, \ldots, 4n + 1)$*, then the quadratic equation* (4) *does not have a double root.*

Proof. From Eq. (4) we have

$$\begin{aligned} 2u &= -(p^2 - 4) \pm \sqrt{p^4 - 8p^2 - 4\epsilon p + 8} \\ &= -(p + 2\epsilon)(p - 2\epsilon) \pm \sqrt{(p + 2\epsilon)(p^3 - 2\epsilon p^2 - 4p + 4\epsilon)}. \end{aligned}$$

So, we have only to prove $p^3 - 2\epsilon p^2 - 4p + 4\epsilon \neq 0$. Suppose $p^3 - 2\epsilon p^2 - 4p + 4\epsilon = 0$. Letting $\gamma(t) = t^6 - 2t^5 - t^4 - t^2 - 2t + 1$, we have $p^3 - 2\epsilon p^2 - 4p + 4\epsilon = s^{-3}\gamma(\epsilon s)$, and so $\gamma(\epsilon s) = 0$. Note that ϵs is a primitive dth root of unity for some d, which is a divisor of $4n + 2$. Let $F_d(t)$ be the dth cyclotomic polynomial, which is an irreducible polynomial with integer coefficients. So, $F_d(t)$ is a factor of $\gamma(t)$. Then since $\deg F_d(t) \leq 6$ and $s \neq \pm 1$, we obtain $d \in \{3, 5, 6, 7, 9, 10, 14, 18\}$. For each d, we see that $F_d(t)$ is not a factor of $\gamma(t)$ (see Table 1), a contradiction. □

Table 1. Cyclotomic polynomials.

d	$F_d(t)$
3	$1 + t + t^2$
5	$1 + t + t^2 + t^3 + t^4$
6	$1 - t + t^2$
7	$1 + t + t^2 + t^3 + t^4 + t^5 + t^6$
9	$1 + t^3 + t^6$
10	$1 - t + t^2 - t^3 + t^4$
14	$1 - t + t^2 - t^3 + t^4 - t^5 + t^6$
18	$1 - t^3 + t^6$

Example 3.8. For G_1, we have $p = s + s^{-1} = 2\cos(k\pi/3) = (-1)^{k-1}$ $(k = 1, 2)$, and there are 4 irreducible representations $\rho_j : G_1 \to SL(2, \boldsymbol{C})$ up to conjugate, $1 \leq j \leq 4$; in Table 2 we list the parameters p, u for each ρ_j.

Remark 3.9. Takahashi [12] considered $K_1 = R(1, 1, -2, 1)$ and $R(-2, 1, 1, -2)$; both of which have trivial Alexander polynomial. He has distinguished their knot groups by the representations to $SL(2, \boldsymbol{C})$. In fact, the knot group of

TABLE 2. Parameters for the representations $\rho_j : G_1 \to SL(2,\boldsymbol{C})$.

Representation	p	u
ρ_1	1	$\frac{3+\sqrt{5}}{2}$
ρ_2	1	$\frac{3-\sqrt{5}}{2}$
ρ_3	-1	$\frac{3+\sqrt{5}}{2}$
ρ_4	-1	$\frac{3-\sqrt{5}}{2}$

$R(-2,1,1,-2)$ has infinitely many representations ρ as in Eq. (2) for $s \in \boldsymbol{C} - \{0,\pm 1\}$ and $u = u_0$, where

$$u_0 = \frac{-(1-s^2)^2(1+s^2) \pm \sqrt{(1-s^2-2s^3-s^4+s^6)(1-s^2+2s^3-s^4+s^6)}}{2s^2(1+s^2)}.$$

Note that $R(-2,1,1,-2)$ is positive-amphicheiral.

Example 3.10. For G_2, we have $p = s + s^{-1} = 2\cos(k\pi/5)$ $(k = 1,2,3,4)$ $= \frac{1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}$, and there are 8 irreducible representations $\rho_j : G_2 \to SL(2,\boldsymbol{C})$ up to conjugate, $1 \le j \le 8$; in Table 3 we list the parameters p, u for each ρ_j.

TABLE 3. Parameters for the representations $\rho_j : G_2 \to SL(2,\boldsymbol{C})$.

Representation	p	u
ρ_1	$\frac{1+\sqrt{5}}{2}$	1
ρ_2	$\frac{1+\sqrt{5}}{2}$	$\frac{3-\sqrt{5}}{2}$
ρ_3	$\frac{-1+\sqrt{5}}{2}$	1
ρ_4	$\frac{-1+\sqrt{5}}{2}$	$\frac{3+\sqrt{5}}{2}$
ρ_5	$\frac{1-\sqrt{5}}{2}$	1
ρ_6	$\frac{1-\sqrt{5}}{2}$	$\frac{3+\sqrt{5}}{2}$
ρ_7	$\frac{-1-\sqrt{5}}{2}$	1
ρ_8	$\frac{-1-\sqrt{5}}{2}$	$\frac{3-\sqrt{5}}{2}$

4. Twisted Alexander polynomial of K_n

Let $\alpha : G_n \to \langle t \rangle \cong \boldsymbol{Z}$ be an abelianization defined by $\alpha(x) = \alpha(y) = t$, which induces the ring homomorphism $\tilde{\alpha} : \boldsymbol{Z}G_n \to \boldsymbol{Z}[t, t^{-1}]$. For an $SL(2, \boldsymbol{C})$ representation of G_n $\rho : G_n \to SL(2, \boldsymbol{C})$ the ring homomorphism $\tilde{\rho} : \boldsymbol{Z}G_n \to M(2, \boldsymbol{C})$ is brought out from ρ. For the free group $\langle x, y \rangle$ with free basis $\{x, y\}$ let $\phi : \langle x, y \rangle \to G_n$ be the canonical homomorphism, which induces the ring homomorphism $\tilde{\phi} : \boldsymbol{Z}\langle x, y \rangle \to \boldsymbol{Z}G_n$. Now, we define a ring homomorphism $\Phi = (\tilde{\rho} \otimes \tilde{\alpha}) \circ \tilde{\phi}$ as follows.

$$\Phi : \boldsymbol{Z}\langle x, y \rangle \xrightarrow{\tilde{\phi}} \boldsymbol{Z}G_n \xrightarrow{\tilde{\rho}\otimes\tilde{\alpha}} M(2, \boldsymbol{C}[t, t^{-1}])$$

$$\frac{\partial r_n}{\partial y} \longmapsto \sum \nu_g g \longmapsto \sum \nu_g \rho(g) \alpha(g),$$

where $r_n = w_n x - y w_n$, $\partial/\partial y$ denotes the Fox derivation, $g \in G_n$, and $\nu_g \in \boldsymbol{Z}$. Let $A_{\rho,y} = \Phi(\partial r_n/\partial y)$. Then the twisted Alexander polynomial of G_n associated to the representation ρ [13] is defined to be a rational function

$$\Delta_{G_n,\rho}(t) = \frac{\det A_{\rho,y}}{\det \Phi(x-1)}. \tag{11}$$

Note that if two representations ρ, ρ' are conjugate, then $\Delta_{G_n,\rho}(t) = \Delta_{G_n,\rho'}(t)$.

The remainder of this section will be devoted to the proof of the following proposition, where the *breadth* of a Laurent polynomial is the difference between the highest and lowest degrees.

Proposition 4.1. *Suppose $n > 0$. For the irreducible representation ρ defined in Sect. 3 the twisted Alexander polynomial of G_n, $\Delta_{G_n,\rho}(t)$ in Eq. (11), is a Laurent polynomial of breadth $2n$ such that the coefficients of the highest degree term and lowest degree term are 1 and $u/(\epsilon p + 2)$, respectively.*

Since

$$\frac{\partial r_n}{\partial y} = \frac{\partial w_n}{\partial y} - y\frac{\partial w_n}{\partial y} - 1,$$

we have

$$\tilde{\alpha} \circ \tilde{\phi}\left(\frac{\partial r_n}{\partial y}\right) = (1-t)\left(\tilde{\alpha} \circ \tilde{\phi}\left(\frac{\partial w_n}{\partial y}\right)\right) - 1.$$

For $w_n = xy^n x^{-n-1} y$ we have

$$\frac{\partial w_n}{\partial y} = x + xy + xy^2 + \cdots + xy^{n-1} + w_n y^{-1}.$$

Thus, we obtain

$$\begin{aligned} A_{\rho,y} &= \Phi\left(\frac{\partial r_n}{\partial y}\right) \\ &= (E - tY)\left(tX(E + tY + t^2Y^2 + \cdots + t^{n-1}Y^{n-1}) + W_n Y^{-1}\right) - E. \end{aligned}$$

On the other hand,

$$\det\Phi(x-1) = \det(tX-E)t^2 - t(s+s^{-1}) + 1 = (t-s)(t-s^{-1}). \tag{12}$$

We can prove the following by induction.

Lemma 4.2.

$$E + tY + t^2Y^2 + \cdots + t^{n-1}Y^{n-1} = \begin{pmatrix} g_n & 0 \\ \dfrac{u}{s-s^{-1}}(g_n - h_n) & h_n \end{pmatrix},$$

where

$$g_n = \frac{1-(st)^n}{1-st}, \quad h_n = \frac{1-(s^{-1}t)^n}{1-s^{-1}t}.$$

Put

$$\det A_{\rho,y} = \varphi_0 + \varphi_1 u + \varphi_2 u^2,$$

where $\varphi_i \in \boldsymbol{C}[t, t^{-1}]$.

Then,

$$\begin{aligned}
\varphi_0 &= t^{2n+2};\\
(s^2-1)^2\varphi_1 &= -t^2 s^{-2n-1}\left(s^{n+1}t^n - s^{n+3}t^n - s^{3n+3}t^n + s^{3n+5}t^n\right.\\
&\qquad \left.+2s^{2n+3} - s^{4n+5} - s\right) - s^{-2n-1}\left(2s^{2n+3} - s^{4n+3} - s^3\right)\\
&\qquad - ts^{-2n-1}\left(-s^{n+2}t^n + s^{n+4}t^n + s^{3n+2}t^n - s^{3n+4}t^n\right.\\
&\qquad \left.-s^{2n+2} - s^{2n+4} - s^{2n+6} + s^{4n+2} + s^{4n+6} - s^{2n} + s^4 + 1\right);\\
(s^2-1)^2\varphi_2 &= -ts^{-2n-1}\left(-s^{2n+2} - s^{2n+4} + s^{4n+4} + s^2\right).
\end{aligned}$$

Substituting $s^{2n+1} = \epsilon = (-1)^k$, we obtain:

$$\begin{aligned}
(s^2-1)^2\varphi_1 &= -\epsilon t^2\left(s^{n+1}t^n - s^{n+3}t^n - \epsilon s^{n+2}t^n + \epsilon s^{n+4}t^n + 2\epsilon s^2 - s^3 - s\right)\\
&\qquad - \epsilon\left(2\epsilon s^2 - s - s^3\right) - \epsilon t\left(-s^{n+2}t^n + s^{n+4}t^n\right.\\
&\qquad \left.+ \epsilon s^{n+1}t^n - \epsilon s^{n+3}t^n - \epsilon s - \epsilon s^3 - \epsilon s^5 + 2 + 2s^4 - \epsilon s^{-1}\right)\\
&= -\epsilon t^2\left((1 - s^2 - \epsilon s + \epsilon s^3)s^{n+1}t^n - s(\epsilon - s)^2\right) + \epsilon s(\epsilon - s)^2\\
&\qquad - \epsilon t\left((-s + s^3 + \epsilon - \epsilon s^2)s^{n+1}t^n\right.\\
&\qquad \left.-\epsilon s^{-1}(s^2 + s^4 + s^6 - 2\epsilon s - 2\epsilon s^5 + 1\right)\\
&= -\epsilon t^2\left(\epsilon(\epsilon - s)(1 - s^2)s^{n+1}t^n - s(\epsilon - s)^2\right) + \epsilon s(\epsilon - s)^2\\
&\qquad - \epsilon t\left((\epsilon - s)(1 - s^2)s^{n+1}t^n - \epsilon s^{-1}(\epsilon - s)^2(1 + s^4)\right);\\
(s^2-1)^2\varphi_2 &= -\epsilon t\left(-\epsilon s - \epsilon s^3 + 2s^2\right) = st(\epsilon - s)^2.
\end{aligned}$$

Since $s^2 - 1 = (s-\epsilon)(s+\epsilon)$, we have:

$$\begin{aligned}
(\epsilon + s)^2\varphi_1 &= -\epsilon t^2\left(\epsilon(\epsilon + s)s^{n+1}t^n - s\right) + \epsilon s - \epsilon t\left((\epsilon + s)s^{n+1}t^n - \epsilon s^{-1}(1 + s^4)\right)\\
&= -(\epsilon + s)s^{n+1}(\epsilon + t)t^{n+1} + \epsilon s t^2 + \epsilon s + s^{-1}(1 + s^4)t\\
&= -(s^{-2n-1} + s)s^{n+1}(\epsilon + t)t^{n+1} + \epsilon s t^2 + \epsilon s + s^{-1}(1 + s^4)t;
\end{aligned}$$

$(\epsilon+s)^2\varphi_2 = st.$

Since $(\epsilon+s)^2 = s(s+s^{-1}+2\epsilon)$, we have:

$(s+s^{-1}+2\epsilon)\varphi_1 = -(s^{-n-1}+s^{n+1})(\epsilon+t)t^{n+1}+\epsilon t^2+\epsilon+((s+s^{-1})^2-2)t;$

$(s+s^{-1}+2\epsilon)\varphi_2 = t.$

Putting $p = s+s^{-1}$ and $\psi_n(p) = s^{-n-1}+s^{n+1} \in \boldsymbol{Z}[p]$, we obtain:

$$(p+2\epsilon)\varphi_1 = -\psi_n(p)(\epsilon+t)t^{n+1}+\epsilon t^2+\epsilon+(p^2-2)t;$$
$$(p+2\epsilon)\varphi_2 = t.$$

Thus, we have:

$$\begin{aligned}&(p+2\epsilon)\det A_{\rho,y}\\ &= (p+2\epsilon)t^{2n+2}+\left(-\psi_n(p)(\epsilon+t)t^{n+1}+\epsilon t^2+\epsilon+(p^2-2)t\right)u+u^2t\\ &= \epsilon u+\left((p^2-2)u+u^2\right)t+\epsilon ut^2-\psi_n(p)u(\epsilon+t)t^{n+1}+(p+2\epsilon)t^{2n+2}.\end{aligned}$$

Since $u^2+(p^2-4)u+\epsilon p+2=0$ from Eq. (4), this becomes:

$$\begin{aligned}&(p+2\epsilon)\det A_{\rho,y}\\ &= \epsilon u+(2u-\epsilon p-2)t+\epsilon ut^2-\psi_n(p)u(\epsilon+t)t^{n+1}+(p+2\epsilon)t^{2n+2}.\end{aligned} \tag{13}$$

Lemma 4.3. *For the irreducible representation ρ defined in Sect. 3 the twisted Alexander polynomial of G_n, $\Delta_{G_n,\rho}(t)$ in Eq.* (11)*, is a Laurent polynomial.*

Proof. Let $P(t)$ be the right-hand side polynomial of Eq. (13). Then by Eq. (12) the result follows from $P(s) = P(s^{-1}) = 0$. In fact,

$$\begin{aligned}P(s) =&\epsilon u+(2u-\epsilon p-2)s+\epsilon us^2-\psi_n(p)u(\epsilon+s)s^{n+1}+(p+2\epsilon)s^{2n+2}\\ =&\epsilon u+(2u-\epsilon p-2)s+\epsilon us^2-(\epsilon s+1)u(\epsilon+s)+(p+2\epsilon)\epsilon s\\ =&\epsilon u+(2u)s+\epsilon us^2-u(2s+\epsilon+\epsilon s^2)=0;\end{aligned}$$

$P(s^{-1})=0$ is similar. □

Remark 4.4. It is known [5] that the twisted Alexander polynomial of a knot in S^3 for any nonabelian representation into $SL(2,\boldsymbol{F})$ over a field $\boldsymbol{F}$ is always a Laurent polynomial. For a reducible representation $\rho : \pi K \to SL(2,\boldsymbol{C})$ and for a representation $\rho : \pi K \to SL(2,\boldsymbol{F}_p)$ over a prime field $\boldsymbol{F}_p$ there are ribbon 2-knots K of 1-fusion whose twisted Alexander polynomial are not Laurent polynomials (see [3]).

Proof of Proposition 4.1. By Eqs. (12), (13) and Lemma 4.3 we obtain Proposition 4.1. □

Example 4.5. For $n=1$, we give explicit forms of the twisted Alexander polynomials $\Delta_{G_1,\rho}(t)$. Since $p=-\epsilon$ and $\psi_1(p)=-1$, Eqs. (12) and (13) become

$$\det\Phi(x-1) = 1+\epsilon t+t^2;$$

$$\begin{aligned}\det A_{\rho,y} &= u + \epsilon(2u-1)t + 2ut^2 + \epsilon ut^3 + t^4 \\ &= (1+\epsilon t + t^2)(u + \epsilon(u-1)t + t^2),\end{aligned}$$

from which we obtain

$$\begin{aligned}\Delta_{G_1,\rho}(t) &= u + \epsilon(u-1)t + t^2 \\ &= (\epsilon u - t)(\epsilon - t).\end{aligned}$$

For each representation ρ_j we list the polynomial in Table 4.

Table 4. Twisted Alexander polynomials of G_1.

Representation	$\Delta_{G_1,\rho}(t)$
ρ_1	$\frac{3+\sqrt{5}}{2} + \frac{1+\sqrt{5}}{2}t + t^2$
ρ_2	$\frac{3-\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2}t + t^2$
ρ_3	$\frac{3+\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}t + t^2$
ρ_4	$\frac{3-\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}t + t^2$

Remark 4.6. The twisted Alexander polynomial of $R(-2,1,1,-2)$ associated to the representation ρ given in Remark 3.9 is $u_0(1+t^2)$.

Example 4.7. For $n=2$, we give explicit forms of the twisted Alexander polynomials $\Delta_{G_2,\rho}(t)$ in Table 5.

Proof of Theorem 1.1. Part (i) follows from Proposition 2.2. Since the mirror image of K_n is isotopic to $R(1,-n-1,n,1)$, which is K_{-n-1}; this implies Part (ii). By Lemma 3.7 (or also Proposition 4.1), the knot groups G_m and G_n are isomorphic if and only if either $m=n$ or $m+n=-1$. This implies Part (iii) since K_0 and K_{-1} are trivial.

In order to prove Part (iv) we prove K_n and K_{-n-1} are not isotopic. Suppose $n>0$. By Proposition 4.1 the coefficients of the highest degree term and lowest degree term of the twisted Alexander polynomials of K_n, $\Delta_{G_n,\rho}(t)$, are 1 and $u/(\epsilon p+2)$, respectively. Since K_{-n-1} is the mirror image of K_n, the set of the twisted Alexander polynomials of K_{-n-1} consists of $\Delta_{G_n,\rho}(t^{-1})$, and so the coefficients of their highest degree terms are $u/(\epsilon p+2)$, where $p = 2\cos(k\pi/(2n+1))$ and u is a root of Eq. (4). For $p=p_0$ there are double roots $u=u_1$, u_2 for Eq. (4) by Lemma 3.7, and so at least one of $u_1/(\epsilon p_0+2)$ and $u_2/(\epsilon p_0+2)$ does not equal to 1. Thus, K_n and K_{-n-1} have different twisted Alexander polynomials. □

Remark 4.8. Part (iii) of Theorem 1.1, the non-triviality of K_n ($n \neq 0, -1$), also follows from [7].

TABLE 5. Twisted Alexander polynomials of G_2.

Representation	$\Delta_{G_2,\rho}(t)$
ρ_1	$\frac{3+\sqrt{5}}{2}+\frac{1+\sqrt{5}}{2}t^3+t^4$
ρ_2	$1+\frac{-1+\sqrt{5}}{2}t+t^2+\frac{1+\sqrt{5}}{2}t^3+t^4$
ρ_3	$\frac{3-\sqrt{5}}{2}+\frac{-1+\sqrt{5}}{2}t^3+t^4$
ρ_4	$1+\frac{1+\sqrt{5}}{2}t+t^2+\frac{-1+\sqrt{5}}{2}t^3+t^4$
ρ_5	$\frac{3-\sqrt{5}}{2}+\frac{1-\sqrt{5}}{2}t^3+t^4$
ρ_6	$1+\frac{-1-\sqrt{5}}{2}t+t^2+\frac{1-\sqrt{5}}{2}t^3+t^4$
ρ_7	$\frac{3+\sqrt{5}}{2}+\frac{-1-\sqrt{5}}{2}t^3+t^4$
ρ_8	$1+\frac{1-\sqrt{5}}{2}t+t^2+\frac{-1-\sqrt{5}}{2}t^3+t^4$

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Taizo Kanenobu
Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan
Email address: kanenobu@sci.osaka-cu.ac.jp

Toshio Sumi
Faculty of Arts and Science
Kyushu University
Motooka 744, Nishi-ku, Fukuoka, 819-0395, Japan
Email address: sumi@artsci.kyushu-u.ac.jp