

TUBES OF FINITE CHEN-TYPE

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ABSTRACT. In this paper, we consider surfaces in the 3-dimensional Euclidean space \mathbb{E}^3 which are of finite *III*-type, that is, they are of finite type, in the sense of B.-Y. Chen, corresponding to the third fundamental form. We present an important family of surfaces, namely, tubes in \mathbb{E}^3 . We show that tubes are of infinite *III*-type.

1. Introduction

Let M^n be a (connected) submanifold in the m -dimensional Euclidean space \mathbb{E}^m . Let \mathbf{x}, \mathbf{H} be the position vector field and the mean curvature field of M^n respectively. Denote by Δ^I the second Beltrami-Laplace operator corresponding to the first fundamental form I of M^n . Then, it is well known that [7]

$$\Delta^I \mathbf{x} = -n\mathbf{H}.$$

From this formula one can see that M^n is a minimal submanifold if and only if all coordinate functions, restricted to M^n , are eigenfunctions of Δ^I with eigenvalue $\lambda = 0$. Moreover in [17] T. Takahashi showed that the submanifold M^n for which $\Delta^I \mathbf{x} = \lambda \mathbf{x}$, i.e., for which all coordinate functions are eigenfunctions of Δ^I with the same eigenvalue $\lambda \in \mathbb{R}$, are precisely either the minimal submanifold with eigenvalue $\lambda = 0$ or the minimal submanifold of hyperspheres S^{m-1} with eigenvalue $\lambda > 0$. Although the class of finite type submanifolds in an arbitrary dimensional Euclidean spaces is very large, very little is known about surfaces of finite type in the Euclidean 3-space \mathbb{E}^3 . Actually, so far, the only known surfaces of finite type corresponding to the first fundamental form in the Euclidean 3-space are the minimal surfaces, the circular cylinders and the spheres. So in [5] B.-Y. Chen mentions the following problem

Problem 1. Determine all surfaces of finite Chen *I*-type in \mathbb{E}^3 .

In order to provide an answer to the above problem, important families of surfaces were studied by different authors by proving that finite type ruled surfaces [9], finite type quadrics [8], finite type tubes [4], finite type cyclides

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of Dupin [10] and finite type spiral surfaces [2] are surfaces of the only known examples in \mathbb{E}^3 . However, for another classical families of surfaces, such as surfaces of revolution, translation surfaces as well as helicoidal surfaces, the classification of its finite type surfaces is not known yet. For more details, we refer to [6].

Later in [13] O. Garay generalized T. Takahashi's condition studied surfaces in \mathbb{E}^3 for which all coordinate functions (x_1, x_2, x_3) of \mathbf{x} satisfy $\Delta^J \mathbf{x}_i = \lambda_i x_i$, $i = 1, 2, 3$, not necessarily with the same eigenvalue. Another generalization was studied in [11] for which surfaces in \mathbb{E}^3 satisfy the condition $\Delta^J \mathbf{x} = A\mathbf{x} + B$ (†) where $A \in \mathbb{R}^{3 \times 3}$; $B \in \mathbb{R}^{3 \times 1}$. It was shown that a surface S in \mathbb{E}^3 satisfies (†) if and only if it is an open part of a minimal surface, a sphere, or a circular cylinder. Surfaces satisfying (†) are said to be of coordinate finite type.

In the thematic circle of the surfaces of finite type in the Euclidean space \mathbb{E}^3 , S. Stamatakis and H. Al-Zoubi in [15] restored attention to this theme by introducing the notion of surfaces of finite type corresponding to the second or the third fundamental forms of S in the following way:

A surface S is said to be of finite type corresponding to the fundamental form J , or briefly of finite J -type, $J = II, III$, if the position vector \mathbf{x} of S can be written as a finite sum of nonconstant eigenvectors of the operator Δ^J , that is, if

$$(1.1) \quad \mathbf{x} = \mathbf{x}_0 + \sum_{i=1}^k \mathbf{x}_i, \quad \Delta^J \mathbf{x}_i = \lambda_i \mathbf{x}_i, \quad i = 1, \dots, k,$$

where \mathbf{x}_0 is a fixed vector and $\mathbf{x}_1, \dots, \mathbf{x}_k$ are nonconstant maps such that $\Delta^J \mathbf{x}_i = \lambda_i \mathbf{x}_i$, $i = 1, \dots, k$. If, in particular, all eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ are mutually distinct, then S is said to be of J -type k , otherwise S is said to be of infinite type. When $\lambda_i = 0$ for some $i = 1, \dots, k$, then S is said to be of null J -type k .

In general when S is of finite type k , it follows from (1.1) that there exists a monic polynomial, say $R(x) \neq 0$, such that $R(\Delta^J)(\mathbf{x} - \mathbf{c}) = \mathbf{0}$. Suppose that if $R(x) = x^k + \sigma_1 x^{k-1} + \dots + \sigma_{k-1} x + \sigma_k$, then coefficients σ_i are given by

$$\begin{aligned} \sigma_1 &= -(\lambda_1 + \lambda_2 + \dots + \lambda_k), \\ \sigma_2 &= (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \dots + \lambda_1 \lambda_k + \lambda_2 \lambda_3 + \dots + \lambda_2 \lambda_k + \dots + \lambda_{k-1} \lambda_k), \\ \sigma_3 &= -(\lambda_1 \lambda_2 \lambda_3 + \dots + \lambda_{k-2} \lambda_{k-1} \lambda_k), \\ &\dots\dots\dots \\ \sigma_k &= (-1)^k \lambda_1 \lambda_2 \dots \lambda_k. \end{aligned}$$

Therefore the position vector \mathbf{x} satisfies the following equation, (see [7])

$$(1.2) \quad (\Delta^J)^k \mathbf{x} + \sigma_1 (\Delta^J)^{k-1} \mathbf{x} + \dots + \sigma_k (\mathbf{x} - \mathbf{c}) = \mathbf{0}.$$

In this paper we will pay attention to surfaces of finite III -type. Firstly, we will establish a formula for $\Delta^{III} \mathbf{x}$ by using Cartan's method of the moving

frame. Further, we continue our study by proving finite type surfaces for an important class of surfaces, namely, tubes in \mathbb{E}^3 .

2. Preliminaries

Let S be a (connected) surface in the Euclidean 3-space \mathbb{E}^3 , whose Gaussian curvature K never vanishes. Let $\varphi = \{\varepsilon_1(u, v), \varepsilon_2(u, v), \varepsilon_3(u, v)\}$ be a moving frame of S , $\varepsilon_3 = \mathbf{n}$ be the Gauss map of S and $\det(\varepsilon_1, \varepsilon_2, \varepsilon_3) = 1$. Then it is well known that there are linear differential forms $\omega_1, \omega_2, \omega_{31}, \omega_{32}$ and ω_{12} , such that [12]

$$\begin{aligned} d\mathbf{x} &= \omega_1\varepsilon_1 + \omega_2\varepsilon_2, & d\mathbf{n} &= \omega_{31}\varepsilon_1 + \omega_{32}\varepsilon_2, \\ d\varepsilon_1 &= \omega_{12}\varepsilon_2 - \omega_{31}\varepsilon_3, & d\varepsilon_2 &= -\omega_{12}\varepsilon_1 - \omega_{32}\varepsilon_3, \end{aligned}$$

and functions a, b, c, q_1, q_2 of u, v such that

$$\omega_{31} = -a\omega_1 - b\omega_2, \quad \omega_{32} = -b\omega_1 - c\omega_2, \quad \omega_{12} = q_1\omega_1 + q_2\omega_2.$$

We can choose the moving frame of S , such that the vectors $\varepsilon_1, \varepsilon_2$ are the principle directions of S . Then a, c are the principle curvatures of S and $b = 0$, so the differential forms ω_1 and ω_2 become

$$\omega_1 = -\frac{1}{a}\omega_{31}, \quad \omega_2 = -\frac{1}{c}\omega_{32}.$$

The Gauss and mean curvature are respectively

$$K = ac, \quad H = \frac{a+c}{2}.$$

Let $\nabla_1 f, \nabla_2 f$ be the derivatives of Pfaff of a function $f(u, v) \in C^1$ along the curves $\omega_2 = 0, \omega_1 = 0$ respectively. Then we have the following well known relations [3]

$$\begin{aligned} \nabla_1 \mathbf{x} &= \varepsilon_1, & \nabla_2 \mathbf{x} &= \varepsilon_2, \\ \nabla_1 \varepsilon_1 &= q_1 \varepsilon_2 + a\mathbf{n}, & \nabla_2 \varepsilon_1 &= q_2 \varepsilon_2, \\ \nabla_1 \varepsilon_2 &= -q_1 \varepsilon_1, & \nabla_2 \varepsilon_2 &= -q_2 \varepsilon_1 + c\mathbf{n}, \\ \nabla_1 \mathbf{n} &= -a\varepsilon_1, & \nabla_2 \mathbf{n} &= -c\varepsilon_2. \end{aligned}$$

We denote by $\tilde{\nabla}_1 f$ and $\tilde{\nabla}_2 f$ the derivatives of Pfaff of f along the curves $\omega_{32} = 0, \omega_{31} = 0$ respectively. One finds

$$\tilde{\nabla}_1 f = -\frac{1}{a}\nabla_1 f, \quad \tilde{\nabla}_2 f = -\frac{1}{c}\nabla_2 f.$$

It follows that

$$(2.1) \quad \tilde{\nabla}_1 \mathbf{x} = -\frac{1}{a}\varepsilon_1, \quad \tilde{\nabla}_2 \mathbf{x} = -\frac{1}{c}\varepsilon_2,$$

$$(2.2) \quad \tilde{\nabla}_1 \varepsilon_1 = p_1 \varepsilon_2 - \mathbf{n}, \quad \tilde{\nabla}_2 \varepsilon_1 = p_2 \varepsilon_2,$$

$$(2.3) \quad \begin{aligned} \tilde{\nabla}_1 \varepsilon_2 &= -p_1 \varepsilon_1, & \tilde{\nabla}_2 \varepsilon_2 &= -p_2 \varepsilon_1 - \mathbf{n}, \\ \tilde{\nabla}_1 \mathbf{n} &= \varepsilon_1, & \tilde{\nabla}_2 \mathbf{n} &= \varepsilon_2, \end{aligned}$$

where $p_1 = -\frac{1}{a}q_1, p_2 = -\frac{1}{c}q_2$ are the geodesic curvatures of the spherical curves $\omega_{32} = 0$ and $\omega_{31} = 0$ respectively. The Mainardi-Codazzi equations have the following forms

$$(2.4) \quad \tilde{\nabla}_1 \frac{1}{c} = p_2 \left(\frac{1}{a} - \frac{1}{c} \right), \quad \tilde{\nabla}_2 \frac{1}{a} = p_1 \left(\frac{1}{a} - \frac{1}{c} \right).$$

Let f be a sufficient differentiable function on S . Then the second differential parameter of Beltrami corresponding to the fundamental form III of S is defined by

$$(2.5) \quad \Delta^{III} f = -\tilde{\nabla}_1 \tilde{\nabla}_1 f - \tilde{\nabla}_2 \tilde{\nabla}_2 f - p_2 \tilde{\nabla}_1 f + p_1 \tilde{\nabla}_2 f.$$

Applying (2.5) to the position vector \mathbf{x} gives

$$\Delta^{III} \mathbf{x} = -\tilde{\nabla}_1 \tilde{\nabla}_1 \mathbf{x} - \tilde{\nabla}_2 \tilde{\nabla}_2 \mathbf{x} - p_2 \tilde{\nabla}_1 \mathbf{x} + p_1 \tilde{\nabla}_2 \mathbf{x}.$$

From (2.1) we obtain

$$(2.6) \quad \Delta^{III} \mathbf{x} = \tilde{\nabla}_1 \left(\frac{1}{a} \boldsymbol{\varepsilon}_1 \right) + \tilde{\nabla}_2 \left(\frac{1}{c} \boldsymbol{\varepsilon}_2 \right) + \frac{1}{a} p_2 \boldsymbol{\varepsilon}_1 - \frac{1}{c} p_1 \boldsymbol{\varepsilon}_2.$$

Using (2.2) and (2.3), equation (2.6) becomes

$$(2.7) \quad \Delta^{III} \mathbf{x} = \left(\tilde{\nabla}_1 \frac{1}{a} \right) \boldsymbol{\varepsilon}_1 + \left(\frac{1}{a} - \frac{1}{c} \right) p_2 \boldsymbol{\varepsilon}_1 + \left(\tilde{\nabla}_2 \frac{1}{c} \right) \boldsymbol{\varepsilon}_2 + \left(\frac{1}{a} - \frac{1}{c} \right) p_1 \boldsymbol{\varepsilon}_2 - \left(\frac{1}{a} + \frac{1}{c} \right) \mathbf{n}.$$

Taking into account the Mainardi-Codazzi equations (2.4), so equation (2.7) reduces to

$$\Delta^{III} \mathbf{x} = \left(\tilde{\nabla}_1 \left(\frac{1}{a} + \frac{1}{c} \right) \right) \boldsymbol{\varepsilon}_1 + \left(\tilde{\nabla}_2 \left(\frac{1}{a} + \frac{1}{c} \right) \right) \boldsymbol{\varepsilon}_2 - \left(\frac{1}{a} + \frac{1}{c} \right) \mathbf{n}$$

or equivalently, (see [15])

$$(2.8) \quad \Delta^{III} \mathbf{x} = \text{grad}^{III} \left(\frac{2H}{K} \right) - \left(\frac{2H}{K} \right) \mathbf{n}.$$

Remark 1. S. Stamatakis and H. Al-Zoubi proved in [15] relation (2.8) by using tensors calculus.

From (2.8) the following results were proved in [15].

Theorem 1. *A surface S in \mathbb{E}^3 is of 0-type 1 corresponding to the third fundamental form if and only if S is minimal.*

Theorem 2. *A surface S in \mathbb{E}^3 is of III-type 1 if and only if S is part of a sphere.*

Corollary 1. *The Gauss map of every surface S in \mathbb{E}^3 is of III-type 1. The corresponding eigenvalue is $\lambda = 2$.*

Up to now, the only known surfaces of finite III-type in \mathbb{E}^3 are parts of spheres, the minimal surfaces and the parallel of the minimal surfaces which are of null III-type 2. So the following question seems to be interesting:

Problem 2. Other than the surfaces mentioned above, which surfaces in \mathbb{E}^3 are of finite *III*-type?

Another generalization of the above problem is to study surfaces in \mathbb{E}^3 with the position vector \mathbf{x} satisfying

$$(2.9) \quad \Delta^{III} \mathbf{x} = A\mathbf{x},$$

where $A \in \mathbb{R}^{3 \times 3}$.

From this point of view, we also pose the following problem.

Problem 3. Classify all surfaces in \mathbb{E}^3 with the position vector \mathbf{x} satisfying relation (2.9).

Concerning this problem, in [16] S. Stamatakis and H. Al-Zoubi studied the class of surfaces of revolution and they proved that: A surface of revolution S satisfies (2.9) if and only if S is a catenoid or part of a sphere. Recently, the same authors in [1] studied the class of ruled surfaces and the class of quadric surfaces. In particular, they proved that helicoids and spheres are the only ruled and quadric surfaces satisfying (2.9) respectively.

This paper provides the first attempt at the study of finite type families of surfaces in \mathbb{E}^3 corresponding to the third fundamental form. Our main result is the following.

Theorem 3. *All tubes in \mathbb{E}^3 are of infinite III-type.*

Our discussion is local, which means that we show in fact that any open part of a tube is of infinite Chen type.

3. Tubes in \mathbb{E}^3

Let $C : \boldsymbol{\alpha} = \boldsymbol{\alpha}(t), t \in (a, b)$ be a regular unit speed curve of finite length which is topologically imbedded in \mathbb{E}^3 . The total space $N_{\boldsymbol{\alpha}}$ of the normal bundle of $\boldsymbol{\alpha}((a, b))$ in \mathbb{E}^3 is naturally diffeomorphic to the direct product $(a, b) \times \mathbb{E}^2$ via the translation along $\boldsymbol{\alpha}$ with respect to the induced normal connection. For a sufficiently small $r > 0$ the tube of radius r about the curve $\boldsymbol{\alpha}$ is the set:

$$T_r(\boldsymbol{\alpha}) = \{\exp_{\boldsymbol{\alpha}(t)} \mathbf{u} \mid \mathbf{u} \in N_{\boldsymbol{\alpha}}, \|\mathbf{u}\| = r, t \in (a, b)\}.$$

Assume that $\mathbf{t}, \mathbf{h}, \mathbf{b}$ is the Frenet frame and κ the curvature of the unit speed curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(t)$. For a small real number r satisfies $0 < r < \min \frac{1}{|\kappa|}$, the tube $T_r(\boldsymbol{\alpha})$ is a smooth surface in \mathbb{E}^3 , [14]. Then a parametric representation of the tube $T_r(\boldsymbol{\alpha})$ is given by

$$(3.1) \quad F : \mathbf{x}(t, \varphi) = \boldsymbol{\alpha} + r \cos \varphi \mathbf{h} + r \sin \varphi \mathbf{b}.$$

It is easily verified that the first and the second fundamental forms of F are given by

$$I = (\delta^2 + r^2 \tau^2) dt^2 + 2r^2 \tau dt d\varphi + r^2 d\varphi^2,$$

$$II = (-\kappa \delta \cos \varphi + r \tau^2) dt^2 + 2r \tau dt d\varphi + r d\varphi^2,$$

where $\delta := (1 - r\kappa \cos \varphi)$ and τ is the torsion of the curve α . The Gauss curvature of F is given by

$$(3.2) \quad K = -\frac{\kappa \cos \varphi}{r\delta}.$$

Notice that $\kappa \neq 0$ since the Gauss curvature vanishes. The Beltrami operator corresponding to the third fundamental form of F can be expressed as follows

$$(3.3) \quad \Delta^{III} = \frac{1}{(\kappa \cos \varphi)^2} \left[-\frac{\partial^2}{\partial t^2} + 2\tau \frac{\partial^2}{\partial t \partial \varphi} - (\tau^2 + \kappa^2 \cos^2 \varphi) \frac{\partial^2}{\partial \varphi^2} + \frac{\beta}{\kappa \cos \varphi} \frac{\partial}{\partial t} + \left(\tau' + \kappa^2 \cos \varphi \sin \varphi - \frac{\tau\beta}{\kappa \cos \varphi} \right) \frac{\partial}{\partial \varphi} \right],$$

where $\beta := \kappa' \cos \varphi + \kappa \tau \sin \varphi$ and $' := \frac{d}{dt}$.

Before we start of proving our main result, we mention and prove the following special case of tubular surfaces for later use.

3.1. Anchor rings

A tube in the Euclidean 3-space is called an anchor ring if the curve C is a plane circle (or is an open portion of a plane circle). In this case, the torsion τ of α vanishes identically and the curvature κ of α is a nonzero constant. Then the position vector \mathbf{x} of the anchor ring can be expressed as

$$(3.4) \quad F : \mathbf{x}(t, \varphi) = \{(a + r \cos t) \cos \varphi, (a + r \cos t) \sin \varphi, r \sin t\},$$

$$a > r, \quad a \in \mathbb{R}.$$

The first fundamental form is

$$I = r^2 dt^2 + (a + r \cos t)^2 d\varphi^2,$$

while the second is

$$II = r dt^2 + (a + r \cos t) \cos t d\varphi^2.$$

Hence, the Beltrami operator is given by

$$(3.5) \quad \Delta^{III} = -\frac{\partial^2}{\partial t^2} + \frac{\sin t}{\cos t} \frac{\partial}{\partial t} - \frac{1}{\cos^2 t} \frac{\partial^2}{\partial \varphi^2}.$$

Let x_1 be the first coordinate function of \mathbf{x} . By virtue of (3.5) one can find

$$(3.6) \quad \Delta^{III} x_1 = \frac{1}{\cos^2 t} a \cos \varphi + 2r \cos t \cos \varphi.$$

Moreover, by a direct computation, we obtain

$$(3.7) \quad (\Delta^{III})^2 x_1 = \left(\frac{2}{\cos^2 t} - \frac{3}{\cos^4 t} \right) a \cos \varphi + 4r \cos t \cos \varphi,$$

$$(3.8) \quad (\Delta^{III})^3 x_1 = \left(\frac{4}{\cos^2 t} - \frac{42}{\cos^4 t} + \frac{45}{\cos^6 t} \right) a \cos \varphi + 8r \cos t \cos \varphi.$$

It can be seen that $\Delta^{III}(\cos t \cos \varphi) = 2 \cos t \cos \varphi$, and for each integer $k > 0$, it is easy to see that

$$(3.9) \quad \Delta^{III} \frac{\cos \varphi}{\cos^k t} = \left(k^2 - k - \frac{k^2 - 1}{\cos^2 t} \right) \frac{\cos \varphi}{\cos^k t}.$$

Thus, by induction, one finds

$$(3.10) \quad (\Delta^{III})^m x_1 = \left(\frac{d_{0,m}}{\cos^2 t} - \frac{d_{1,m}}{\cos^4 t} + \dots + \frac{d_{m-1,m}}{\cos^{2m} t} \right) a \cos \varphi + 2^m r \cos t \cos \varphi,$$

where $d_{j,m}$ are constants, $j = 1, 2, \dots, m - 1$, and

$$d_{0,m} = 2^{m-1}, \quad d_{m-1,m} = (-1)^{m-1} (2m - 1) \prod_{j=1}^m (2j - 3)^2.$$

Notice that $d_{m-1,m} \neq 0$, for each integer $m \geq 1$. Now, if F is of finite type, then there exist real numbers, c_1, c_2, \dots, c_m such that

$$(3.11) \quad (\Delta^{III})^m \mathbf{x} + c_1 (\Delta^{III})^{m-1} \mathbf{x} + \dots + c_{m-1} \Delta^{III} \mathbf{x} + c_m \mathbf{x} = \mathbf{0}.$$

Since $x_1 = (a + r \cos t) \cos \varphi$ is the first coordinate of \mathbf{x} , (3.11), one gets

$$(3.12) \quad (\Delta^{III})^m x_1 + c_1 (\Delta^{III})^{m-1} x_1 + \dots + c_{m-1} \Delta^{III} x_1 + c_m x_1 = 0.$$

From (3.6-3.8), (3.10) and (3.12) we obtain that

$$\begin{aligned} & 2^m r \cos t \cos \varphi + a \cos \varphi \sum_{j=1}^m \frac{d_{j-1,m}}{\cos^{2j} t} + 2^{m-1} c_1 r \cos t \cos \varphi \\ & + c_1 a \cos \varphi \sum_{j=1}^{m-1} \frac{d_{j-1,m-1}}{\cos^{2j} t} + \dots + 2c_{m-1} r \cos t \cos \varphi \\ & + c_{m-1} \frac{a \cos \varphi}{\cos^2 t} + c_m (a + r \cos t) \cos \varphi = 0 \end{aligned}$$

which can be rewritten as

$$(3.13) \quad \frac{d_{m-1,m}}{\cos^{2m} t} + \frac{1}{\cos^{2m-2} t} F(\cos t) = 0,$$

where $F(u)$ is a polynomial in $u = \cos t$ of degree $2m - 2$.

This is impossible for any $m \geq 1$ since $d_{m-1,m} \neq 0$. Consequently, we have the following

Corollary 2. *Every anchor ring in the Euclidean 3-space is of infinite type.*

4. Proof of the main theorem

Applying relation (3.3) on the position vector \mathbf{x} of (3.1) gives

$$\Delta^{III} \mathbf{x} = \frac{\beta}{\kappa^3 \cos^3 \varphi} \mathbf{t} + \left(2r \cos \varphi - \frac{1}{\kappa \cos^2 \varphi} \right) \mathbf{h} + 2r \sin \varphi \mathbf{b},$$

which can be rewritten as

$$(4.1) \quad \Delta^{III} \mathbf{x} = \frac{\beta}{\kappa^3 \cos^3 \varphi} \mathbf{t} + \frac{1}{\kappa^2 \cos^2 \varphi} \mathbf{P}_1(\cos \varphi, \sin \varphi),$$

where $\mathbf{P}_1(u, v)$ is a vector valued polynomial in u, v of degree 3 with functions in t as coefficients. Moreover, by a long computation, we obtain

$$(4.2) \quad (\Delta^{III})^2 \mathbf{x} = \frac{\beta^3}{\kappa^7 \cos^7 \varphi} \mathbf{t} + \frac{1}{\kappa^6 \cos^6 \varphi} \mathbf{P}_2(\cos \varphi, \sin \varphi),$$

where $\mathbf{P}_2(u, v)$ is a vector valued polynomial in u, v of degree 7 with functions in t as coefficients.

We need the following lemma which can be proved directly by using (3.3).

Lemma 1. *For any natural numbers m and n we have*

$$\left(\Delta^{III} \frac{\beta^m}{(\kappa \cos \varphi)^n} \right) = -\frac{n(n+2)\beta^{m+2}}{(\kappa \cos \varphi)^{n+4}} + \frac{1}{(\kappa \cos \varphi)^{n+3}} P(\cos \varphi, \sin \varphi),$$

where P is a polynomial in u, v of degree $n+4$ with functions in t as coefficients.

Using Lemma 1 and relation (3.3) one finds

$$(4.3) \quad (\Delta^{III})^\lambda \mathbf{x} = d_\lambda \frac{\beta^{2\lambda-1}}{(\kappa \cos \varphi)^{4\lambda-1}} \mathbf{t} + \frac{1}{(\kappa \cos \varphi)^{4\lambda-2}} \mathbf{P}_\lambda(\cos \varphi, \sin \varphi),$$

where

$$d_\lambda = (-1)^{\lambda-1} \prod_{j=1}^{2\lambda-1} (2j-1).$$

It can be seen that $d_\lambda \neq 0$, for each natural number λ . Moreover, we have

$$(4.4) \quad (\Delta^{III})^{\lambda+1} \mathbf{x} = d_{\lambda+1} \frac{\beta^{2\lambda+1}}{(\kappa \cos \varphi)^{4\lambda+3}} \mathbf{t} + \frac{1}{(\kappa \cos \varphi)^{4\lambda+2}} \mathbf{P}_{\lambda+1}(\cos \varphi, \sin \varphi).$$

Let F be of finite type. Then there exist real numbers, $c_1, c_2, \dots, c_\lambda$ such that

$$(4.5) \quad (\Delta^{III})^{\lambda+1} \mathbf{x} + c_1 (\Delta^{III})^\lambda \mathbf{x} + \dots + c_\lambda \Delta^{III} \mathbf{x} = \mathbf{0}.$$

Using (4.1)-(4.4), one has

$$(4.6) \quad d_{\lambda+1} \frac{\beta^{2\lambda+1}}{\kappa \cos \varphi} \mathbf{t} = Q_1 \mathbf{t} + Q_2 \mathbf{h} + Q_3 \mathbf{b},$$

where $Q_i, i = 1, 2, 3$, are polynomials in u, v with functions in t as coefficients.

Now, if $\beta \neq 0$. From (4.6) we find

$$(4.7) \quad d_{\lambda+1} \frac{\beta^{2\lambda+1}}{\kappa \cos \varphi} = Q_1(\cos \varphi, \sin \varphi).$$

This is impossible, since Q_1 is polynomial in $\cos \varphi$ and $\sin \varphi$. Assume now $\beta = 0$. Then $\kappa' = 0$ and $\kappa\tau = 0$ so $\kappa = \text{const.} \neq 0$ and $\tau = 0$. Therefore the curve C is a circle, and so F is anchor ring. Hence, F is of infinite type according to Corollary (2). This completes our proof.

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