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PSEUDO PROJECTIVE RICCI SYMMETRIC SPACETIMES

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Abstract. The object of the present paper is to prove the non-existence of pseudo projective Ricci symmetric spacetimes $(PWRS)_4$ with different types of energy momentum tensor. We also discuss whether a fluid $(PWRS)_4$ spacetime with the basic vector field as the velocity vector field of the fluid can admit heat flux. Next we consider perfect fluid and dust fluid $(PWRS)_4$ spacetimes respectively. Finally we construct an example of a $(PWRS)_4$ spacetime.

1. Introduction

General relativity flows from the Einstein equation which implies that the energy-momentum tensor is of vanishing divergence. This requirement of the energy-momentum tensor is satisfied if this tensor is covariant constant, that is, $\nabla T = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. In the general theory of relativity, energy-momentum tensor plays an important role and the condition on energy-momentum tensor for a perfect fluid spacetime changes the nature of spacetime [20]. In a recent paper [3] Chaki and Roy studied general relavistic spacetime with covariant constant energy-momentum tensor. The spacetime of general relativity and cosmology is regarded as a connected 4-dimensional semi-Riemannian manifold (M^4, g) with Lorentzian metric g with signature $(-, +, +, +)$. The geometry of Lorentz manifold begins with the study of causal character of vectors of the manifold. It is due to this causality that Lorentz manifold becomes a convenient choice for the study of general relativity. Indeed by basing its study on Lorentzian manifold the general theory of relativity opens the way to the study of global questions about it ([2], [5], [8], [11], [12]) and many others. Also several authors studied spacetimes in different way such as ([1], [13], [7], [16], [22]) and many others.

Einstein's field equation without cosmological constant is given by

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa T(X,Y), \tag{1.1}$$

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where r is the scalar curvature of the manifold and $\kappa \neq 0$. The equation (1.1) of Einstein implies that matter determines the geometry of spacetime and conversely that the motion of matter is determined by the metric tensor of the space which is not flat.

In general relativity the matter content of the spacetime is described by the energy momentum tensor. The matter content is assumed to be a fluid having density and pressure and possessing dynamical and kinematical quantities like velocity, acceleration, vorticity, shear and expansion.

In a perfect fluid spacetime, the energy momentum tensor T of type $(0,2)$ is of the form ([18]):

$$T(X,Y) = pg(X,Y) + (\sigma + p)A(X)A(Y), \tag{1.2}$$

where σ and p are the energy density and the isotropic pressure respectively. The velocity vector field ρ metrically equivalent to the non-zero 1-form A is a time-like vector, that is, $g(\rho,\rho) = -1$. The fluid is called perfect because of the absence of heat conduction terms and stress terms corresponding to viscosity [11]. In addition, p and σ are related by an equation of state governing the particular sort of perfect fluid under consideration. In general, this is an equation of the form $p = p(\sigma, T_0)$, where T_0 is the absolute temperature. However, we shall only be concerned with situations in which T_0 is effectively constant so that the equation of state reduces to $p = p(\sigma)$. In this case, the perfect fluid is called isentropic [11]. Moreover, if $p = \sigma$, then the perfect fluid is termed as stiff matter (see [20, page 66]). Recently, De et al. ([6], [7]) studied conformally flat almost pseudo-Ricci symmetric spacetimes and spacetimes with semisymmetric energy momentum tensor respectively. Also in [14] Mallick, Suh and De studied spacetime with pseudo-projective curvature tensor. Moreover in [17] Mantica and Molinari studied weakly Z symmetric manifolds. Also several authors studied spacetimes in different way such as ([7,9,10,13,16,22]) and many others. In [3] Chaki and Ray studied spacetimes with covariant constant energy momentum tensor. Motivated by the above studies in the present paper we characterize pseudo projective Ricci symmetric spacetimes.

Apart from conformal curvature tensor, the projective curvature tensor is another important tensor from the differential geometric point of view. Let M be an n-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 3$, M is locally projectively flat if and only if the well known Projective curvature tensor W vanishes at each point of the manifold.

Projective curvature tensor W in a semi-Riemannian manifold $(M^n, g)(n \geq 2)$ is defined by [21]

$$W(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{1.3}$$

where R is the Riemannian curvature tensor and S is the Ricci tensor of type $(0,2)$. From this tensor W a symmetric tensor P of type (0,2) can be obtained as follows:

$$P(X,Y) = \tilde{W}(X,e_i,e_i,Y), \tag{1.4}$$

where $\tilde{W}(X,Y,Z,W) = g(W(X,Y)Z,W)$ and $\{e_i\}$, $i = 1,2,3,\ldots,n$ is an orthonormal basis of the tangent space at each point, i being summed in (1.4) for $1 \leq i \leq n$. This tensor P is called the projective Ricci tensor.

In the present paper we consider a special type of spacetime which is called pseudo projective Ricci symmetric spacetime. The notion of pseudo projective Ricci symmetric manifold was introduced by Chaki and Saha [4]. A non-flat semi-Riemannian manifold $(M^n,g)(n>2)$ is said to be pseudo projective Ricci symmetric manifold if its projective Ricci tensor $P(\neq 0)$ satisfies the condition

$$(\nabla_X P)(Y,Z) = 2A(X)P(Y,Z) + A(Y)P(X,Z) + A(Z)P(Y,X), \tag{1.5}$$

where A is a non-zero 1-form given by

$$g(X,\rho) = A(X) \tag{1.6}$$

for every vector field X and ∇ denotes the operator of covariant differentiation with respect to g. Such an n-dimensional manifold of this kind shall be denoted by $(PWRS)_n$. From (1.3) and (1.4) we have

$$P(X,Y) = \frac{n}{n-1}S(X,Y) - \frac{r}{n-1}g(X,Y), \tag{1.7}$$

where r is the scalar curvature of the manifold. A Lorentzian manifold (M^4,g) is said to be a pseudo projective Ricci symmetric spacetime $(PWRS)_4$ if the projective Ricci tensor satisfies (1.5), where the vector field ρ metrically equivalent to the 1-form A is a time-like vector.

In [15] the authors introduced the symmetric tensor $Z_{kl} = R_{kl} + \phi g_{kl}$, where ϕ is an arbitrary scalar function: they studied then pseudo Z symmetric manifolds defined by

$$\nabla_k Z_{jl} = 2A_k Z_{jl} + A_j Z_{kl} + A_l Z_{jk}.$$

It is worth to note that from equation (1.7) we have $P_{kl} = \frac{n}{n-1}Z_{kl}$ with $\phi = -\frac{R}{n}$. Thus pseudo projective Ricci symmetric manifolds are a particular case of pseudo Z symmetric manifolds.

Let l and L denote respectively the symmetric endomorphisms of the tangent space at each point corresponding to the tensors P and S. Then

$$g(lX,Y) = P(X,Y), \tag{1.8}$$

and

$$g(LX,Y) = S(X,Y). \tag{1.9}$$

From (1.5) it follows that

$$(\nabla_X l)(Y) = 2A(X)lY + A(Y)lX + P(X,Y)\rho. \tag{1.10}$$

Using (1.8), (1.9), (1.10) and (1.7) we obtain

$$S(X,\rho)=\frac{r}{4}g(X,\rho), \tag{1.11}$$

which implies that ρ is an eigenvector corresponding to the eigenvalue $\frac{r}{4}$.

The paper is organized as follows: After introduction, in Section 2, we prove the non-existence of a $(PWRS)_4$ spacetime with covariant constant energy momentum tensor. Similarly, in the next two sections we prove the non-existence $(PWRS)_4$ spacetimes with Cyclic parallel energy momentum tensor and Codazzi type of energy momentum tensor respectively. Section 5 is concerned with the possibility of a fluid $(PWRS)_4$ spacetime to admit heat flux. In Sections 6 and 7 we consider perfect fluid and dust fluid $(PWRS)_4$ spacetimes respectively. Finally, we construct an example of a $(PWRS)_4$ spacetime.

2. $(PWRS)_4$ spacetime with covariant constant energy momentum tensor

In this section we consider a $(PWRS)_n$ spacetime of non-zero scalar curvature with covariant constant energy momentum tensor, i.e.,

$$(\nabla_X T)(Y,Z)=0. \tag{2.1}$$

In a paper [4] Chaki and Saha prove that the scalar curvature r of a $(PWRS)_n$ is constant. We suppose that the $(PWRS)_n$ spacetime obeys Einstein's field equation without cosmological constant. Then[18]

$$S(Y,Z)-\frac{r}{2}g(Y,Z)=\kappa T(Y,Z), \tag{2.2}$$

where κ is the gravitational constant. Differentiating (2.2), we get

$$(\nabla_X S)(Y,Z)-\frac{1}{2}g(Y,Z)dr(X)=\kappa(\nabla_X T)(Y,Z). \tag{2.3}$$

Now from (1.7), we obtain

$$(\nabla_X P)(Y,Z)=\frac{4}{3}(\nabla_X S)(Y,Z)-\frac{1}{3}g(Y,Z)dr(X). \tag{2.4}$$

Since r is constant and T is covariant constant, we obtain from (2.3) and (2.4) that

$$(\nabla_X S)(Y,Z)=0$$

and

$$(\nabla_X P)(Y,Z)=0.$$

Hence from (1.5) and (1.7) we obtain

$$\begin{aligned}2A(X)[\frac{4}{3}S(Y,Z)-\frac{r}{3}g(Y,Z)]+A(Y)[\frac{4}{3}S(X,Z)-\frac{r}{3}g(X,Z)]\\ +A(Z)[\frac{4}{3}S(X,Y)-\frac{r}{3}g(X,Y)]=0.\end{aligned} \tag{2.5}$$

Putting $Z = \rho$ in (2.5) and using (1.11), we have

$$S(X,Y) = \frac{r}{4} g(X,Y). \tag{2.6}$$

But, in virtue of (1.7) and (2.6) we obtain that $P(X,Y) = 0$ which is a contradiction.

Thus we can state the following:

Theorem 2.1. *There does not exist a pseudo projective Ricci symmetric spacetime with covariant constant energy momentum tensor satisfying Einstein's field equation without cosmological constant.*

3. $(PWRS)_4$ spacetime with cyclic parallel energy momentum tensor

In this section we consider a $(PWRS)_4$ spacetime of non-zero scalar curvature satisfying cyclic parallel energy momentum tensor, i.e.,

$$(\nabla_X T)(Y,Z) + (\nabla_Y T)(X,Z) + (\nabla_Z T)(X,Y) = 0. \tag{3.1}$$

It has been proved by Chaki and Saha [4] that the scalar curvature r of a $(PWRS)_4$ spacetime is constant. We suppose that the $(PWRS)_4$ spacetime obey's Einstein's field equation without cosmological constant. Then equation (2.2) holds. Since T is cyclic parallel and r is constant, we obtain from (3.1) that

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(X,Z) + (\nabla_Z S)(X,Y) = 0. \tag{3.2}$$

Using (1.7) and (3.2) we have

$$(\nabla_X P)(Y,Z) + (\nabla_Y P)(X,Z) + (\nabla_Z P)(X,Y) = 0. \tag{3.3}$$

Now using (1.6) and (1.7) in (3.3) we obtain

$$A(X)[\frac{4}{3}S(Y,Z) - \frac{r}{3}g(Y,Z)] + A(Y)[\frac{4}{3}S(X,Z) - \frac{r}{3}g(X,Z)] + A(Z)[\frac{4}{3}S(X,Y) - \frac{r}{3}g(X,Y)] = 0. \tag{3.4}$$

Putting $Z = \rho$ in (3.4) and using (1.11) we finally obtain

$$S(X,Y) = \frac{r}{4} g(X,Y). \tag{3.5}$$

But, in virtue of (1.7) and (3.5) we obtain that $P(X,Y) = 0$ which is a contradiction.

Thus we can state the following:

Theorem 3.1. *There does not exist a pseudo projective Ricci symmetric spacetime with cyclic parallel energy momentum tensor satisfying Einstein's field equation without cosmological constant.*

4. $(PWRS)_4$ spacetime with Codazzi type energy momentum tensor

In this section we consider a $(PWRS)_4$ spacetime of non-zero scalar curvature with Codazzi type energy momentum tensor, i.e.,

$$(\nabla_X T)(Y,Z) = (\nabla_Z T)(Y,X). \tag{4.1}$$

It has been proved by Chaki and Saha [4] that the scalar curvature r of a $(PWRS)_4$ spacetime is constant. We suppose that the $(PWRS)_4$ spacetime obey's Einstein's field equation without cosmological constant. Then equation (2.2) holds. Since T is of Codazzi type and r is constant, we obtain from (4.1) that

$$(\nabla_X S)(Y,Z) = (\nabla_Z S)(Y,X). \tag{4.2}$$

Using (1.7) and (4.2) we have

$$(\nabla_X P)(Y,Z) = (\nabla_Z P)(Y,X). \tag{4.3}$$

Now using (1.6) and (1.7) in (4.3) we obtain

$$A(X)[\frac{4}{3}S(Y,Z) - \frac{r}{3}g(Y,Z)] - A(Z)[\frac{4}{3}S(X,Y) - \frac{r}{3}g(X,Y)] = 0. \tag{4.4}$$

Putting $Z = \rho$ in (4.4) and using (1.11) we finally obtain

$$S(X,Y) = \frac{r}{4}g(X,Y). \tag{4.5}$$

But, in virtue of (1.7) and (4.5) we obtain that $P(X,Y) = 0$ which is a contradiction.

Thus we can state the following:

Theorem 4.1. *There does not exist a pseudo projective Ricci symmetric spacetime with Codazzi type of energy momentum tensor satisfying Einstein's field equation without cosmological constant.*

5. Possibility of a fluid $(PWRS)_4$ spacetime to admit heat flux

In this section we discuss whether a fluid $(PWRS)_4$ spacetime with the basic vector field as the velocity vector field of the fluid can admit heat flux.

If possible, let us suppose that the energy momentum tensor be of the following form:

$$T(X,Y) = (\sigma + p)A(X)A(Y) + pg(X,Y) + A(X)B(Y) + A(Y)B(X), \tag{5.1}$$

where $g(X,\xi) = B(X)$, for all X, ξ being the heat flux vector field. Then, since ξ is spacelike, $g(\rho,\xi) = 0$, i.e.,

$$B(\rho) = 0. \tag{5.2}$$

In this case, Einstein's field equation with cosmological constant can be written as follows:

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = \kappa[(\sigma + p)A(X)A(Y) + pg(X,Y)$$

$$+ A(X)B(Y) + A(Y)B(X)]. \tag{5.3}$$

Now putting $Y = \rho$ in (5.3) and using equations (1.11) and (5.3) we obtain

$$\kappa B(X) = (\frac{r}{4} - \sigma\kappa - \lambda)A(X). \tag{5.4}$$

Again putting $X = \rho$ in (5.4) we have

$$\frac{r}{4} - \sigma\kappa - \lambda = 0. \tag{5.5}$$

Hence from (5.4), it follows that $B(X) = 0$.

Therefore we can state the following:

Theorem 5.1. *If in a $(PWRS)_4$ spacetime of non-zero scalar curvature the matter distribution is a fluid with the basic vector field as the velocity vector field of the fluid, then such a fluid can not admit heat flux.*

6. Perfect fluid $(PWRS)_4$ spacetimes

Now we consider the matter distribution in perfect fluid whose velocity vector field is the vector field ρ corresponding to the 1-form A of the spacetime. Therefore the energy momentum tensor T of type $(0, 2)$ is of the form ([18]):

$$T(X, Y) = pg(X, Y) + (\sigma + p)A(X)A(Y), \tag{6.1}$$

where σ and p are the energy density and the isotropic pressure respectively. Hence from the Einstein's field equation we get

$$S(X, Y) - \frac{r}{2}g(X, Y) = \kappa[pg(X, Y) + (\sigma + p)A(X)A(Y)]. \tag{6.2}$$

Putting $Y = \rho$ in (6.2) and then using (1.11) we obtain that

$$r = 4\kappa\sigma. \tag{6.3}$$

Also taking a frame field and contracting X and Y in (6.2) we get

$$r = \kappa(\sigma - 3p). \tag{6.4}$$

Since r is constant in a perfect fluid $(PWRS)_4$ spacetime, then the energy density and the isotropic pressure are constants. Also equations (6.3) and (6.4) yields

$$\sigma + p = 0. \tag{6.5}$$

Thus in view of the above we can state the following:

Theorem 6.1. *A perfect fluid $(PWRS)_4$ spacetime obeying Einstein's field equation without cosmological constant represents the matter contents of the spacetime satisfy the vacuum like state equation and the energy density and the isotropic pressure are constants.*

Remark 1. In this case $p = -\sigma$, that is, $p = p(\sigma)$. Hence we conclude that the fluid is isentropic [11].

7. Dust fluid $(PWRS)_4$ spacetimes

In a dust or pressureless fluid spacetime, the energy momentum tensor is of the form [19]

$$T(X,Y) = \sigma A(X)A(Y), \tag{7.1}$$

where σ is the energy density of the dust-like matter and A is a non-zero 1-form such that $g(X,\rho) = A(X)$ for all X, A being the velocity vector field of the flow, that is, $g(\rho,\rho) = -1$.

Using (1.1) and (7.1) we obtain

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa\sigma A(X)A(Y). \tag{7.2}$$

A frame field after contraction over X and Y leads to

$$r = \kappa\sigma. \tag{7.3}$$

Putting $Y = \rho$ in (7.2) and then using (1.11) we obtain that

$$r = 4\kappa\sigma. \tag{7.4}$$

Thus combining the equations (7.3)and (7.4), we finally obtain that

$$\sigma = 0. \tag{7.5}$$

Thus from (7.1) and (7.5) we conclude that

$$T(X,Y) = 0.$$

This means that the spacetime is devoid of matter. Thus we can state the following:

Theorem 7.1. *A dust fluid $(PWRS)_4$ spacetime satisfying Einstein's field equation without cosmological constant is vacuum.*

8. Example of a $(PWRS)_4$ spacetime

In this section we prove the existence of a $(PWRS)_4$ spacetime by constructing a non-trivial concrete example.

We consider a Lorentzian manifold (M^4, g) endowed with the Lorentzian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^2)^2(dx^3)^2 - (dx^4)^2, \tag{8.1}$$

where $i, j = 1, 2, 3, 4$.

The only non-vanishing components of the Christoffel symbols, the curvature tensor, the Ricci tensor, the projective Ricci tensor and the derivatives of the components of projective Ricci tensors are

$$\Gamma^1_{22} = -x^1, \ \Gamma^2_{33} = -\frac{x^2}{(x^1)^2}, \ \Gamma^2_{12} = \frac{1}{x^1}, \ \Gamma^3_{23} = \frac{1}{x^2}, \ R_{1332} = -\frac{x^2}{x^1},$$

$$S_{12} = -\frac{1}{x^1x^2}, \ P_{12} = -\frac{4}{3x^1x^2}, \ P_{12,1} = \frac{8}{3(x^1)^2x^2}, \ P_{12,2} = \frac{4}{3(x^2)^2x^1}.$$

It can be easily shown that the scalar curvature of the manifold is zero. Therefore R^4 with the considered metric is a Lorentzian manifold (M^4, g) of vanishing scalar curvature. We shall now show that this M^4 is a $(PWRS)_4$ spacetime i.e., it satisfies the defining relation (1.5).

We choose the associated 1-form as follows:

$$A_i(x) = \begin{cases} -\frac{2}{3x^1} & \text{for } \ i = 1 \\ -\frac{1}{3x^2} & \text{for } \ i = 2, \\ 0, & \text{otherwise} \end{cases}$$

at any point x $\in \mathbb{R}^4$.

Now equation (1.5) reduces to

$$P_{12,1} = 2A_1 P_{12} + A_1 P_{12} + A_2 P_{11}, \tag{8.2}$$

$$P_{12,2} = 2A_2 P_{12} + A_2 P_{12} + A_1 P_{22}. \tag{8.3}$$

Clearly, equations (8.2) and (8.3) are all true. So the manifold under consideration is $(PWRS)_4$ spacetime.

Thus we can state the following:

Theorem 8.1. *Let* $(\mathbb{R}^4, g)$ *be a* 4*-dimensional Lorentzian manifold with the Lorentzian metric g given by*

$$ds^2 = g_{ij} dx^i dx^j = (dx^1)^2 + (x^1)^2 (dx^2)^2 + (x^2)^2 (dx^3)^2 - (dx^4)^2,$$

where $i,\ j = 1, 2, 3, 4$. *Then* $(\mathbb{R}^4, g)$ *is a* $(PWRS)_4$ *spacetime.*

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