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ORTHOGONALITY IN FINSLER C^*-MODULES

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Abstract. In this paper, we introduce some notions of orthogonality in the setting of Finsler C^*-modules and investigate their relations with the Birkhoff-James orthogonality. Suppose that (E,ρ) and (F,ρ') are Finsler modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively, and $\varphi : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism. A map $\Psi : E \to F$ is said to be a φ-morphism of Finsler modules if $\rho'(\Psi(x)) = \varphi(\rho(x))$ and $\Psi(ax) = \varphi(a)\Psi(x)$ for all $a \in \mathcal{A}$ and all $x \in E$. We show that each φ- morphism of Finsler C^*-modules preserves the Birkhoff-James orthogonality and conversely, each surjective linear map between Finsler C^*-modules preserving the Birkhoff-James orthogonality is a φ-morphism under certain conditions. In fact, we state a version of Wigner's theorem in the framework of Finsler C^*-modules.

1. Introduction and preliminaries

The notion of orthogonality is originally associated with inner product spaces. In an inner product space, an element x is orthogonal to y if $\langle x, y\rangle = 0$. Recently, various extensions of this notion have been introduced in the setting of normed spaces. Among them, the Birkhoff-James orthogonality is studied extensively in [3–5]. This notion states that an element x of a normed linear space X is orthogonal to $y \in X$, in short; $x \perp_B y$, if for each $\lambda \in \mathbb{C}$

$$\|x\| \leq \|x + \lambda y\|.$$

The characterizations of the Birkhoff-James orthogonality in C^*-algebras and Hilbert C^*-modules are presented in several papers such as [3, 7, 8].

The notion of Finsler module over a C^*-algebra was introduced by Phillips and Weaver [11]. In fact, Finsler modules over C^*-algebras are generalization of Hilbert C^*-modules [10]. Recently, this theory has been developed by several researchers [1, 2, 9].

Let us recall the definition of a Finlser module. Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{A}^+$ be the set of all positive elements of $\mathcal{A}$. An element $a \in A$ is positive, in short $a \geq 0$, if a is self-adjoint and the spectrum of a sp(a) is a subset of

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$[0, +\infty)$. Let E be a left module over $\mathcal{A}$ and let the map $\rho : E \longrightarrow \mathcal{A}^+$ satisfy the following conditions:

(i) the map $\|x\| : x \mapsto \|\rho(x)\|^{\frac{1}{2}}$ makes E into a Banach space;

(ii) $\rho(ax) = a\rho(x)a^*$ for all $a \in \mathcal{A}$ and $x \in E$.

Then (E, ρ) is called a left Finsler module over $\mathcal{A}$. A right Finsler module can be defined similarly.
A left Finsler module E over $\mathcal{A}$ is said to be full if the linear span $\{\rho(x) : x \in E\}$, denoted by $\mathcal{F}(E)$, is dense in $\mathcal{A}$.

As an example, if E is a left Hilbert C^*-module over $\mathcal{A}$, then E together with $\rho(x) = \langle x, x\rangle$ is a left Finsler module over $\mathcal{A}$, since

$$\rho(ax) = \langle ax, ax\rangle = a\langle x, x\rangle a^* = a\rho(x)a^*.$$

In this paper, we introduce some notions of orthogonality in the setting of Finsler modules over C^*-algebras and investigate their relations with the Birkhoff-James orthogonality. We also show that each φ-morphism of Finsler C^*-modules preserves the Birkhoff-James orthogonality and conversely, each surjective linear map between Finsler C^*-modules which preserves the Birkhoff-James orthogonality is a φ-morphism under certain conditions. We indeed state a version of Wigner's theorem in the framework of Finsler C^*-modules.

2. The Birkhoff-James orthogonality in Finsler C^*-modules

Analogue to the notion of Birkhoff-James orthogonality in the setting Hilbert C^*-modules and the usual Birkhoff-James orthogonality in normed spaces, we present the following notions in Finsler C^*-module content.

Definition 2.1. Let (E, ρ) be a Finsler module over a unital C^*-algebra $\mathcal{A}$, with unit I and $x, y \in E$. We say that x is strongly Birkhoff-James orthogonal to y with respect to ρ, in short; $x \perp^s_{B\rho} y$, if for each $a \in \mathcal{A}$

$$\rho(x) \leq \|\rho(x + ay)\| I.$$

In the definition above, if the role of the elements of the underlying C^*-algebra is played by the scalars, we say that x is Birkhoff-James orthogonal to y with respect to ρ, in short; $x \perp_{B\rho} y$, where for each $\lambda \in \mathbb{C}$

$$\rho(x) \leq \|\rho(x + \lambda y)\| I.$$

We also say that x is strongly Birkhoff-James orthogonal to y, in short; $x \perp^s_B y$, if for each a $\in \mathcal{A}$

$$\|x\| \leq \|x + ay\|.$$

Remark 2.2. It is clear that $x \perp_{B\rho} y$ if and only if $x \perp_B y$, as well as $x \perp^s_{B\rho} y$ if and only if $x \perp^s_B y$. These assertions are deduced from the fact that if $a \in \mathcal{A}$ and $a \geq 0$, then $a \leq MI$ if and only if $\|a\| \leq M$ for some $M \geq 0$.

It is obvious that $x \perp^s_{B\rho} y$ ensures $x \perp_{B\rho} y$.

Definition 2.3. Let (E, ρ) be a Finsler module over a C^*-algebra and $x, y \in E$. We say that x is ρ-orthogonal to y, in short; $x \perp_\rho y$, if

$$\rho(x) \leq \rho(x + \lambda y) \text{ for each } \lambda \in \mathbb{C}.$$

Evidently, $x \perp_\rho y$ ensures $x \perp_B y$. The converse of this fact dose not hold in general, as shown in the following example.

Example 2.4. Suppose that $\mathcal{A} = \mathbb{M}_2(\mathbb{C})$ as a Finsler module over itself with $\rho(A) = AA^*$. Let $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Then for each $\lambda \in \mathbb{C}$

$$\|A + \lambda B\| = \left\| \begin{bmatrix} 1+\lambda & 0 \\ 0 & i \end{bmatrix} \right\| = \max\{|1+\lambda|, |\mathrm{i}|\} \geq 1 = \|\mathrm{A}\|,$$

whence $A \perp_B B$. We, however, have $A \not\perp_\rho B$, since

$$\rho(A) = I \ \text{ and } \ \rho(A + \lambda B) = \begin{bmatrix} 1+\lambda & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \overline{1+\lambda} & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} |1+\lambda|^2 & 0 \\ 0 & 1 \end{bmatrix}.$$

If $\lambda = -1$, then

$$\rho(A + \lambda B) - \rho(A) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \ngeq 0,$$

because $\mathrm{sp}\left(\left[\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\right) = \{-1, 0\} \nsubseteq \mathbb{R}^+$.

Proposition 2.5. *Let (E, ρ) be a Finsler module over a C^*-algebra $\mathcal{A}$ and $x, y \in E$.*

(i) *If $x \perp_\rho y$, then $ax \perp_\rho ay$ for each $a \in \mathcal{A}$.*

(ii) *If $ux \perp_\rho uy$ for each unitary element $u \in \mathcal{A}$, then $x \perp_\rho y$.*

Proof. (i) If $x \perp_\rho y$, then $\rho(x) \leq \rho(x + \lambda y)$ for each $\lambda \in \mathbb{C}$. Thus

$$\rho(ax) = a\rho(x)a^* \leq a\rho(x + \lambda y)a^* = \rho(ax + \lambda ay).$$

Hence $ax \perp_\rho ay$.

(ii) Let $ux \perp_\rho uy$ for each unitary element $u \in \mathcal{A}$. Then $\rho(ux) \leqslant \rho(ux + \lambda uy)$ for each $\lambda \in \mathbb{C}$. Thus

$$\rho(x) = u^*u\rho(x)u^*u \leqslant u^*u\rho(x + \lambda y)u^*u = \rho(x + \lambda y).$$

It ensures that $x \perp_\rho y$. □

Note that by the above proposition, we may deduce $ux \perp_\rho uy$ for some and hence for each unitary element $u \in \mathcal{A}$ if and only if $x \perp_\rho y$.

The following definition is a generalization of Definition 2.3.

Definition 2.6. Let (E, ρ) be a Finsler module over a C^*-algebra $\mathcal{A}$ and $x, y \in E$. We say that x is strongly ρ-orthogonal to y, in short $x \perp_\rho^s y$, if for all $a \in \mathcal{A}$,

$$\rho(x) \leq \rho(x + ay).$$

If $\mathcal{A}$ is unital, then $x \perp_\rho^s y$ implies $x \perp_\rho y$. In fact, if $x \perp_\rho^s y$, then $\rho(x) \leq \rho(x + (1.\lambda)y) = \rho(x + \lambda y)$ for each $\lambda \in \mathbb{C}$. Hence $x \perp_\rho y$.

Lemma 2.7. *Let (E, ρ) be a Finsler module over a unital C^*-algebra $\mathcal{A}$ and $x, y \in E$.*

(i) *$x \perp_\rho^s y$ if and only if $bx \perp_\rho^s by$ for each $b \in \mathscr{Z}(\mathcal{A})$.*

(ii) *If $x \perp_\rho^s y$, then $x \perp_B^s y$.*

Proof. (i) Let $x \perp_\rho^s y$. Then $\rho(x) \leq \rho(x + ay)$ for each $a \in \mathcal{A}$. In addition, $\rho(bx) = b\rho(x)b^* \leq b\rho(x+ay)b^* = \rho(bx+bay) = \rho(bx+aby)$ for each $b \in \mathscr{Z}(\mathcal{A})$, whence $bx \perp_\rho^s by$.

Conversely, let $bx \perp_\rho^s by$ for each $b \in \mathscr{Z}(\mathcal{A})$. Taking $b = 1$ we deduce that $x \perp_\rho^s y$.

(ii) Let $x \perp_\rho^s y$. Then $\rho(x) \leq \rho(x + ay)$ for each $a \in \mathcal{A}$, so $\|\rho(x)\| \leq \|\rho(x + ay)\|$. Hence $\|x\| \leq \|x + ay\|$. Thus $x \perp_B^s y$. □

In the following example, we show that the converse of (ii) in Lemma 2.7 does not hold in general.

Example 2.8. Suppose that $\mathcal{A} = \mathbb{M}_2(\mathbb{C})$ as a Finsler module over itself via $\rho(A) = AA^*$. Let $A = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. For any $B = \left[\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right]$ we have

$$\|I + BA\| = \left\| \begin{bmatrix} 1 & b_2 \\ 0 & b_4 + 1 \end{bmatrix} \right\| \geq 1 = \|I\|.$$

Therefore $I \perp_B^s A$, we, however, have $I \not\perp_\rho^s A$ since if $B = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, then

$$\rho(I + BA) - \rho(I) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \not\geq 0,$$

because

$$\mathrm{sp}\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right) = \{\frac{1 - \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2}\} \not\subseteq \mathbb{R}^+.$$

Now we obtain some characterizations of the strongly ρ-orthogonality in the framework of Finsler C^*-modules.

Proposition 2.9. *Let E be a Finsler module over a unital C^*-algebra $\mathcal{A}$ and $x, y \in E$. Then $x \perp_\rho^s y$ if and only if $x \perp_\rho ay$ for all $a \in \mathcal{A}$.*

Proof. If $x \perp_\rho^s y$, then for each $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$ we have $\lambda a \in \mathcal{A}$ and

$$\rho(x) \leq \rho(x + \lambda.ay).$$

Hence $x \perp_\rho ay$.

Conversely, let $x \perp_\rho ay$ for each $a \in \mathcal{A}$. Then for each scalar $\lambda \in \mathbb{C}$ we have $\rho(x) \leq \rho(x + \lambda ay)$. Putting $\lambda = 1$ we conclude that $\rho(x) \leq \rho(x + ay)$. Hence $x \perp_\rho^s y$. □

Note that if $\mathcal{A} \cong \mathbb{C}$, then by a simple computation we infer that $x \perp_\rho y$ if and only if $x \perp_\rho^s y$. It is an interesting problem whether the converse is true or not.

It is evident that $x \perp_\rho y$ does not imply $x \perp_\rho^s y$ in general. It however holds under certain conditions. For example if $\mathcal{A} = \mathsf{B}(\mathcal{H})$, regarded as a Finsler module over itself via $\rho(T) = TT^*$ and $\mathscr{Z}(\mathsf{B}(\mathcal{H}))$ denotes the center of $\mathsf{B}(\mathcal{H})$, then $\mathscr{Z}(\mathsf{B}(\mathcal{H})) = \mathbb{C}I = \{\lambda I : \lambda \in \mathbb{C}\} \simeq \mathbb{C}$.

Let $T, S \in \mathsf{B}(\mathcal{H})$ such that $T \perp_\rho S$ and $U \in \mathscr{Z}(\mathsf{B}(\mathcal{H}))$. Then there is $\lambda \in \mathbb{C}$ such that $U = \lambda I$. Hence $\rho(T) \leq \rho(T + \lambda S) = \rho(T + \lambda IS)$. Therefore $\rho(T) \leq \rho(T + US)$ for each $U \in \mathscr{Z}(\mathsf{B}(\mathcal{H}))$.

3. Relation between φ-morphisms of Finsler C^*-modules and orthogonality

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $T : H \to H$ be a surjective map, which satisfies $|\langle Tx, Ty\rangle| = |\langle x, y\rangle|$. The Wigner theorem states that T is of the form $Tx = \varphi(x)Ux$ for each $x \in H$, where $U : H \to H$ is either a unitary or an antiunitary operator and $\varphi : H \to \mathbb{C}$ is a phase function (i.e., its values are of modulus one) [8].

In this section, we introduce the notion of φ-morphism of Finsler C^*-modules and try to construct a version of Wigner's theorem in the framework of Finsler C^*-modules. Indeed we replace the above condition by that of preserving Birkhoff-James orthogonality and show that under certain conditions each surjective linear map between Finsler C^*-modules, which preserves the Birkhoff-James orthogonality is a φ- morphism.

Definition 3.1. Suppose that (E, ρ) and (F, ρ') are Finsler modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively, and $\varphi : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism of C^*-algebras. A linear map $\Psi : E \to F$ is said to be a φ-morphism of Finsler modules if for each $x \in E$ and $a \in \mathcal{A}$ the following conditions are satisfied:
(i) $\rho'(\Psi(x)) = \varphi(\rho(x))$;
(ii) $\Psi(ax) = \varphi(a)\Psi(x)$.

By [1, Theorem 3.2], let Ψ be a φ-morphism between full Finsler modules. If Ψ(or φ) is injective, then φ and also Ψ are isometry.

Theorem 3.2. *Suppose that (E, ρ) and (F, ρ') are Finsler modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively, $\varphi : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism of C^*-algebras and Ψ is a φ-morphism of Finsler modules. If $x \perp_\rho y$, then $\Psi(x) \perp_{\rho'} \Psi(y)$.*

Proof. Let $x \perp_\rho y$. Then $\rho(x) \leq \rho(x + \lambda y)$. Thus

$$\varphi(\rho(x)) \leq \varphi(\rho(x + \lambda y)),$$

since $\varphi : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism. Further,

$$\rho'(\Psi(x)) \leq \rho'(\Psi(x + \lambda y)) = \rho'(\Psi(x) + \lambda\Psi(y)),$$

since Ψ is a φ-morphism. Hence $\Psi(x) \perp_{\rho'} \Psi(y)$. □

Theorem 3.3. *Suppose that (E, ρ) and (F, ρ') are Finsler modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively, $\varphi : \mathcal{A} \to \mathcal{B}$ is an injective $*$-homomorphism of C^*-algebras and Ψ is a φ-morphism of Finsler modules. If $x \perp_B y$, then $\Psi(x) \perp_B \Psi(y)$.*

Proof. Let $x \perp_{BJ} y$. Then $\|x\| \leq \|x+\lambda y\|$ for each $\lambda \in \mathbb{C}$. Hence $\|\rho(x)\| \leq \|\rho(x+\lambda y)\|$ for each $\lambda \in \mathbb{C}$. Since φ is injective, it is an isometry. Hence $\|\varphi(\rho(x))\| \leq \|\varphi(\rho(x+\lambda y))\|$. Since Ψ is a φ-morphism, we have $\|\rho'(\Psi(x))\| \leq \|\rho'(\Psi(x+\lambda y))\|$. Thus

$$\|\Psi(x)\| \leq \|\Psi(x+\lambda y)\| = \|\Psi(x)+\lambda\Psi(y)\|.$$

Therefore $\Psi(x) \perp_B \Psi(y)$. □

Lemma 3.4. *Suppose that (E,ρ) and (F,ρ') are full Finsler modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively, $\varphi : \mathcal{A} \to \mathcal{B}$ is a map and Ψ is a surjective linear operator of Finsler modules such that $\Psi(ax) = \varphi(a)\Psi(x)$ for each $x \in E$ and $a \in \mathcal{A}$. Then φ is a homomorphism. Moreover, if φ is continuous and $\rho'(\Psi(x)) = \varphi(\rho(x))$ for each $x \in E$, then φ is a $*$-homomorphism and Ψ is a φ-morphism.*

Proof. Let $a, b \in \mathcal{A}, x \in E$ and $\lambda \in \mathbb{C}$. Then

$$\begin{aligned}(\varphi(a+b)-\varphi(a)-\varphi(b))\Psi(x) &= \varphi(a+b)\Psi(x)-\varphi(a)\Psi(x)-\varphi(b)\Psi(x)\\ &= \Psi((a+b)x)-\Psi(ax)-\Psi(bx)\\ &= \Psi(ax)+\Psi(bx)-\Psi(ax)-\Psi(bx)\\ &= 0\end{aligned}$$

and

$$\begin{aligned}(\varphi(ab)-\varphi(a)\varphi(b))\Psi(x) &= \varphi(ab)\Psi(x)-(\varphi(a)\varphi(b))\Psi(x)\\ &= \Psi((ab)x)-\varphi(a)\Psi(bx)\\ &= \Psi(abx)-\Psi(abx)\\ &= 0.\end{aligned}$$

Similarly, $(\varphi(\lambda a)-\lambda\varphi(a))\Psi(x) = 0$. Since Ψ is surjective and F is full, the map φ is a homomorphism by [1, Lemma 1.2].

Let $a \in \mathcal{A}$. Since E is full, there is a sequence $\{u_n\}$ in $\mathcal{F}(E)$ such that $a = \lim_n u_n$, where $u_n = \sum_{i=1}^{k_n}\lambda_{i,n}\rho(x_{i,n})$ for some $\lambda_{i,n} \in \mathbb{C}$ and $x_{i,n} \in E$. If φ is continuous, then

$$\begin{aligned}\varphi(a^*) &= \lim_n \varphi\left(\sum_{i=1}^{k_n}\lambda_{i,n}\rho(x_{i,n})\right)^* = \lim_n \sum_{i=1}^{k_n}\overline{\lambda_{i,n}}\varphi(\rho(x_{i,n}))\\ &= \lim_n \sum_{i=1}^{k_n}\overline{\lambda_{i,n}}\rho'(\Psi(x_{i,n})) = \left(\lim_n \sum_{i=1}^{k_n}\lambda_{i,n}\rho'(\Psi(x_{i,n}))\right)^*\\ &= \left(\varphi(\lim_n \sum_{i=1}^{k_n}\lambda_{i,n}\rho(x_{i,n}))\right)^* = (\varphi(a))^*.\end{aligned}$$

Hence φ is a $*$-homomorphism and Ψ is a φ-morphism. □

Theorem 3.5. *Suppose that (E,ρ) and (F,ρ') are full Finsler modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively, $\Psi : E \to F$ is a surjective linear operator that preserves $\perp_B$ and $\Psi(\rho(x)y) = \rho'(\Psi(x))\Psi(y)$ for each $x, y \in E$. Then there exists a $*$-isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ such that Ψ is a φ-morphism of Finsler modules.*

Proof. Since Ψ preserves Birkhoff-James orthogonality, by [6, Theorem 3.1], there is a constant $k > 0$ such that $\|\Psi(x)\| = k\|x\|$. Hence Ψ is injective and continuous.

Let us define $\varphi : \mathcal{F}(E) \to \mathcal{F}(F)$ by

$$\varphi\left(\sum_{i=1}^{n}\lambda_i\rho(x_i)\right) = \sum_{i=1}^{n}\lambda_i\rho'(\Psi(x_i))$$

for $\lambda_i \in \mathbb{C}$ and $x_i \in E$.

If $\sum_{i=1}^{n}\lambda_i\rho(x_i) = 0$, then $\sum_{i=1}^{n}\lambda_i\rho(x_i)z = 0$ for each $z \in E$. Hence $\Psi\left(\sum_{i=1}^{n}\lambda_i\rho(x_i)z\right) = \sum_{i=1}^{n}\lambda_i\Psi(\rho(x_i)z) = 0$, since Ψ is linear. By the assumption, $\sum_{i=1}^{n}\lambda_i\rho'(\Psi(x_i))\Psi(z) = 0$. Hence $\sum_{i=1}^{n}\lambda_i\rho'(\Psi(x_i)) = 0$, since Ψ is surjective. Thus φ is well-define and $\rho'(\Psi(x)) = \varphi(\rho(x))$ for each $x \in E$.

Let $u = \sum_{i=1}^{n}\lambda_i\rho(x_i)$ be an arbitrary element of $\mathcal{F}(E)$. Then

$$\varphi(u)\Psi(z) = \sum_{i=1}^{n}\lambda_i\rho'(\Psi(x_i))\Psi(z) = \Psi\left(\sum_{i=1}^{n}\lambda_i\rho(x_i)z\right) = \Psi(uz).$$

By Lemma 3.4, φ is linear on $\mathcal{F}(E)$.

Let $\{u_n\}$ be a sequence in $\mathcal{F}(E)$ such that $u_n \to u$. Then $u_n z \to uz$ for all $z \in E$. In view of the continuity of Ψ, we have $\lim_n\Psi(u_n z) = \Psi(uz)$. On the other hand $\varphi(u)\Psi(z) = \Psi(uz)$. Hence, $\lim_n\varphi(u_n)\Psi(z) = \varphi(u)\Psi(z)$, whence $\lim_n(\varphi(u_n) - \varphi(u))\Psi(z) = 0$. Since Ψ is surjective, $\lim_n(b_n)w = 0$ for each $w \in F$, where $b_n = \varphi(u_n) - \varphi(u)$. From the continuity of ρ' we deduce that $\lim_n\rho'(b_n w) = \rho'(\lim_n(b_n)w) = 0$. Therefore $\lim_n(b_n\rho'(w)b_n^*) = 0$. Due to F is full, $\lim_n(b_n bb^* b_n^*) = 0$ for all $b \in B$. Thus $\lim_n\|b_n b\|^2 = \lim_n\|b^* b_n^*\|^2 = 0$, whence we get $\lim_n b^* b_n^* b_n b = 0$ for all $b \in B$.

Now, in contrary, assume that $\lim_n b_n \neq 0$. Then there would exist $\varepsilon > 0$ and a subsequence $\{b_{n_k}\}$ of $\{b_n\}$ such that $\varepsilon \leq \|b_{n_k}\|$, or equivalently, $\varepsilon^2 \leq b_{n_k}^* b_{n_k}$. Hence $\varepsilon^2 b^* b \leq b^* b_{n_k}^* b_{n_k} b$ for all $b \in B$. It follows that $b = 0$ for all $b \in B$ giving a contradiction. Hence $\lim_n b_n = 0$ and so $\lim_n\varphi(u_n) = \varphi(u)$. Thus φ is continuous. It should be noted that $\mathcal{B}$ is a Banach space. We can extend φ to a linear map $\overline{\varphi}$ from $\mathcal{A} = \overline{\mathcal{F}(E)}$ into $\overline{\mathcal{F}(F)} = \mathcal{B}$ and denote it by the same φ.

Let $a \in \mathcal{A}$. Then $a = \lim_n\sum_{i=1}^{k_n}\lambda_{i,n}\rho(x_{i,n})$ for some $\lambda_{i,n} \in \mathbb{C}$ and $x_{i,n} \in E$. It follows from continuity of φ that

$$\varphi(a) = \varphi\left(\lim_n\sum_{i=1}^{k_n}\lambda_{i,n}\rho(x_{i,n})\right) = \lim_n\sum_{i=1}^{k_n}\lambda_{i,n}\rho'(\Psi(x_{i,n})).$$

Therefore, for each $x \in E$

$$\begin{aligned}\varphi(a)\Psi(x) &= \left(\lim_n \sum_{i=1}^{k_n} \lambda_{i,n}\rho'(\Psi(x_{i,n}))\right)\Psi(x) \\ &= \lim_n \sum_{i=1}^{k_n} (\lambda_{i,n}\Psi(\rho(x_{i,n})x)) \\ &= \Psi(\lim_n \sum_{i=1}^{k_n} \lambda_{i,n}\rho(x_{i,n})x) \\ &= \Psi(ax).\end{aligned}$$

Employing Lemma 3.4, we observe that φ is a $*$-homomorphism and Ψ is a φ-morphism of Finsler modules. By [2, Theorem 3.2(iii)], φ is an injective $*$-homomorphism, since E is a full Finsler module over $\mathcal{A}$ and Ψ is injective. Due to Ψ is surjective and F is a full Finsler module over $\mathcal{B}$, from [2, Theorem 3.4(iv)], we deduce that φ is surjective. Thus $\varphi : \mathcal{A} \to \mathcal{B}$ is a $*$-isomorphism of C^*-algebras. □

Remark 3.6. In Theorem 3.5, we can replace the assumption $\perp_B$ by both the continuity and the injectivity of Ψ.

Recall that if $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, E and F are Finsler module over $\mathcal{A}$ and $\mathcal{B}$, respectively, then a linear operator $\Psi : E \to F$ is said to be a unitary operator if there exists an injective $*$-homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ such that Ψ is a surjective φ-homomorphism.

Remark 3.7. If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, E and F are Finsler module over $\mathcal{A}$ and $\mathcal{B}$, respectively, and $\Psi : E \to F$ is a unitary operator, then there exists an injective $*$-homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ such that Ψ is a surjective φ-homomorphism. Hence, it preserves the Birkhoff-James orthogonality. It follows from [2, Theorem 3.2(i)] that Ψ is an isometry.

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