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# ON FRAMES FOR COUNTABLY GENERATED HILBERT MODULES OVER LOCALLY $C^*$-ALGEBRAS

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Abstract. Let $\mathcal{X}$ be a countably generated Hilbert module over a locally $C^*$-algebra $\mathcal{A}$ in multiplier module $M(\mathcal{X})$ of $\mathcal{X}$. We propose the necessary and sufficient condition such that a sequence $\{h_n : n \in \mathbb{N}\}$ in $M(\mathcal{X})$ is a standard frame of multipliers in $\mathcal{X}$. We also show that if $T$ in $b(L_{\mathcal{A}}(\mathcal{X}))$, the space of bounded maps in set of all adjointable maps on $\mathcal{X}$, is surjective and $\{h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in $\mathcal{X}$, then $\{T \circ h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in $\mathcal{X}$, too.

## 1. Introduction and preliminaries

Locally $C^*$-algebras are generalizations of $C^*$-algebras. Locally $C^*$-algebras were first introduced by A. Inoue [5] and were also studied more by N. C. Phillips (under the name of pro-$C^*$-algebra) [10].

A locally $C^*$-algebra is a complete Hausdorff complex topological $*$-algebra $\mathcal{A}$, whose topology is determined by its continuous $C^*$-seminorms in the sense that the net $\{a_i\}_{i\in I}$ converges to 0 in $\mathcal{A}$ if and only if the net $\{p(a_i)\}_{i\in I}$ converges to 0 for every continuous $C^*$-seminorm $p$ in set $S(\mathcal{A})$ of all continuous $C^*$-seminorms on $\mathcal{A}$.

Hilbert modules are essentially objects like Hilbert spaces by allowing the inner product to take values in a (locally) $C^*$-algebra rather than the field of complex numbers. The notion of Hilbert module over locally $C^*$-algebras generalize the notion of Hilbert $C^*$-module. Hilbert modules over locally $C^*$-algebras were first considered by N. C. Phillips [10]. He showed that many properties of the Hilbert $C^*$-modules are valid for Hilbert modules over locally $C^*$-algebras. But the main body of the work on Hilbert modules over locally $C^*$-algebras is due to M. Joita, all her work on the subject can be found in her book, under the title "Hilbert modules over locally $C^*$-algebras" (See [7]).

Here we recall some results about Hilbert modules over locally $C^*$-algebras from [7] and [10].

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A (right) pre-Hilbert module over a locally $C^*$-algebra $\mathcal{A}$ (or a pre-Hilbert $\mathcal{A}$-module) is a complex vector space $\mathcal{X}$ which is also a right $\mathcal{A}$-module, compatible with the complex algebra structure, equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{A}$ which is $\mathbb{C}$- and $\mathcal{A}$-linear in its second variable and satisfies the following relations:

(1) $\langle y, x\rangle = \langle x, y\rangle^*$ for every $x, y \in \mathcal{X}$;
(2) $\langle x, x\rangle \geq 0$ for every $x \in \mathcal{X}$;
(3) $\langle x, x\rangle = 0$ if and only if $x = 0$.

A pre-Hilbert $\mathcal{A}$-module $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module if $\mathcal{X}$ is complete with respect to the topology determined by the family of seminorms $\{\bar{p}_{\mathcal{X}}\}_{p\in S(\mathcal{A})}$ where $\bar{p}_{\mathcal{X}}(x) = \sqrt{p(\langle x, x\rangle)}$, $x \in \mathcal{X}$.

If $\mathcal{A}$ is a locally $C^*$-algebra, then $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module with $\langle a, b\rangle = a^*b$, and the set $\mathcal{H}_{\mathcal{A}}$ of all sequences $(a_n)_n$ with $a_n \in \mathcal{A}$ such that $\Sigma_n a_n^* a_n$ converges in $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module with the action of $\mathcal{A}$ on $\mathcal{H}_{\mathcal{A}}$ defined by $(a_n)_n b = (a_n b)_n$ and the inner product defined by $\langle (a_n)_n, (b_n)_n\rangle = \Sigma_n a_n^* b_n$.

Let $\mathcal{A}$ be a locally $C^*$-algebra and let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module. A subset $\mathcal{Y}$ of $\mathcal{X}$ is a generating set for $\mathcal{X}$ if the closed submodule of $\mathcal{X}$ generated by $\mathcal{Y}$ is the whole of $\mathcal{X}$. We say that $\mathcal{X}$ is countably generated if it has a countable generating set.

Let $\mathcal{A}$ be a locally $C^*$-algebra and let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules. An $\mathcal{A}$-module map $T : \mathcal{X} \longrightarrow \mathcal{Y}$ is called bounded if for all $p \in S(\mathcal{A})$, there is $M_p > 0$ such that $\bar{p}_{\mathcal{Y}}(Tx) \leq M_p \bar{p}_{\mathcal{X}}(x)$ for all $x \in \mathcal{X}$, and it is adjointable if there is a map $T^* : \mathcal{Y} \longrightarrow \mathcal{X}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$. It is easy to see that every adjointable map is a bounded $\mathcal{A}$-module map. The set of all adjointable maps from $\mathcal{X}$ into $\mathcal{Y}$ is denoted by $L_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ and we write $L_{\mathcal{A}}(\mathcal{X})$ for $L_{\mathcal{A}}(\mathcal{X}, \mathcal{X})$. The vector space $L_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ is a complete locally convex space with respect to the topology defined by the family of seminorms $\{\hat{p}_{\mathcal{X},\mathcal{Y}}\}_{p\in S(\mathcal{A})}$, where $\hat{p}_{\mathcal{X},\mathcal{Y}}$ defined by $\hat{p}_{\mathcal{X},\mathcal{Y}}(T) = \sup\{\bar{p}_{\mathcal{Y}}(Tx) : x \in \mathcal{X} \text{ and } \bar{p}_{\mathcal{X}}(x) \leq 1\}$. In particular, $L_{\mathcal{A}}(\mathcal{X})$ becomes a locally $C^*$-algebra with respect to the topology defined by the family of seminorms $\{\hat{p}_{\mathcal{X}}\}_{p\in S(\mathcal{A})}$.

We say that an element $T$ of $L_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ is bounded in $L_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ if there is $M > 0$ such that $\hat{p}_{\mathcal{X},\mathcal{Y}}(T) \leq M$ for all $p \in S(\mathcal{A})$. The set of all bounded elements in $L_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ is denoted by $b\,(L_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}))$. It is clear that the map $\|\cdot\|_\infty$ defined by $\|T\|_\infty = \sup\{\hat{p}_{\mathcal{X},\mathcal{Y}}(T) : p \in S(\mathcal{A})\}$ is a norm on $b\,(L_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}))$. And $b\,(L_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}))$ is a Banach space with respect to the norm $\|\cdot\|_\infty$. So $b\,(L_{\mathcal{A}}(\mathcal{X}))$ is a $C^*$-algebra with respect to the norm $\|\cdot\|_\infty$.

Now we recall some fact about multiplier modules from [8] and [9].

Let $\mathcal{A}$ be a locally $C^*$-algebra and let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module. It is not difficult to check that $L_{\mathcal{A}}(\mathcal{A}, \mathcal{X})$ is a Hilbert $L_{\mathcal{A}}(\mathcal{A})$-module with the action of $L_{\mathcal{A}}(\mathcal{A})$ on $L_{\mathcal{A}}(\mathcal{A}, \mathcal{X})$ defined by $t.s = t \circ s$, $t \in L_{\mathcal{A}}(\mathcal{A}, \mathcal{X})$ and $s \in L_{\mathcal{A}}(\mathcal{A})$ and the $L_{\mathcal{A}}(\mathcal{A})$-valued inner-product defined by $\langle t, s\rangle = t^* \circ s$. Moreover, $\bar{p}_{L_{\mathcal{A}}(\mathcal{A},\mathcal{X})}(s) = \hat{p}_{L_{\mathcal{A}}(\mathcal{A},\mathcal{X})}(s)$ for all $s \in L_{\mathcal{A}}(\mathcal{A}, \mathcal{X})$ and for all $p \in S(\mathcal{A})$, the topology on $L_{\mathcal{A}}(\mathcal{A}, \mathcal{X})$ induced by the inner product coincides with the topology determined by the family of seminorms $\{\bar{p}_{L_{\mathcal{A}}(\mathcal{A},\mathcal{X})}\}_{p\in S(\mathcal{A})}$. Therefore $L_{\mathcal{A}}(\mathcal{A},\mathcal{X})$ is a Hilbert $L_{\mathcal{A}}(\mathcal{A})$-module and since $L_{\mathcal{A}}(\mathcal{A})$ can be identified with the multiplier algebra $M(\mathcal{A})$ of $\mathcal{A}$ (see [7] and [10]), $L_{\mathcal{A}}(\mathcal{A},\mathcal{X})$ becomes a Hilbert $M(\mathcal{A})$-module. The Hilbert $M(\mathcal{A})$-module $L_{\mathcal{A}}(\mathcal{A},\mathcal{X})$ is called the multiplier module of $\mathcal{X}$, and it is denoted by $M(\mathcal{X})$.

The map $i_{\mathcal{X}} : \mathcal{X} \longrightarrow M(\mathcal{X})$ defined by $i_{\mathcal{X}}(x)(a) = xa$, $x \in \mathcal{X}$ and $a \in \mathcal{A}$ embeds $\mathcal{X}$ as a closed submodule of $M(\mathcal{X})$. Moreover, if $t \in M(\mathcal{X})$, then $t.a = t(a)$ for all $a \in \mathcal{A}$ and $\langle t, x\rangle = t^*(x)$ for all $x \in \mathcal{X}$.

A Hilbert $\mathcal{A}$-module $\mathcal{X}$ is countably generated in $M(\mathcal{X})$ if there is a countable set $\{h_n : h_n \in M(\mathcal{X}),\ n = 1, 2, \ldots\}$ such that the closed submodule of $M(\mathcal{X})$ generated by $\{h_n.a : a \in \mathcal{A},\ n = 1, 2, \ldots\}$ is the whole of $\mathcal{X}$.

If $\mathcal{X}$ is a countably generated Hilbert $\mathcal{A}$-module, then $\mathcal{X}$ is countably generated in $M(\mathcal{X})$. In general, $\mathcal{X}$ is not always countably generated when $\mathcal{X}$ is countably generated in $M(\mathcal{X})$.

Frames for Hilbert spaces were introduced by R. J. Duffin and A. C. Schaeffer [3] in 1952 as part of their research in non-harmonic Fourier series. They were reintroduced and developed by I. Daubechies, A. Grossmann and Y. Meyer [2] in 1986. Many generalizations of frames were introduced, meanwhile, M. Frank and D. R. Larson [4] presented a general approach to the frame theory in Hilbert $C^*$-modules. A frame for a countably generated Hilbert $C^*$-module $\mathcal{X}$ is a sequence $\{x_n : n \in \mathbb{N}\}$ for which there are constants $C, D > 0$ such that

$$C\langle x, x\rangle \le \sum_n \langle x, x_n\rangle\langle x_n, x\rangle \le D\langle x, x\rangle,\ x \in \mathcal{X}.$$

M. Joita [8] generalized this definition to the situation Hilbert modules over locally $C^*$-algebras. A frame of multipliers for a countably generated Hilbert module $\mathcal{X}$ over a locally $C^*$-algebra $\mathcal{A}$ is a sequence $\{h_n : n \in \mathbb{N}\}$ in $M(\mathcal{X})$ for which there are constants $C, D > 0$ such that

$$C\langle x, x\rangle \le \sum_n \langle x, h_n\rangle\langle h_n, x\rangle \le D\langle x, x\rangle,\ x \in \mathcal{X}. \tag{1.1}$$

The numbers $C$ and $D$ are called lower and upper frame bounds, respectively. We consider standard frames of multipliers for which the sum in the middle of (1.1) converges in $\mathcal{A}$ for every $x \in \mathcal{X}$.

Any countably generated Hilbert $\mathcal{A}$-module $\mathcal{X}$ in $M(\mathcal{X})$ admits a standard frame of multipliers [8, Proposition 3.6].

In this paper we extend some results from [1] in the context of Hilbert modules over locally $C^*$-algebras.

## 2. Main results

First, we investigate some properties of bounded $\mathcal{A}$-linear maps between Hilbert $\mathcal{A}$-modules.

**Definition 2.1.** An element $T$ of $L_{\mathcal{A}}(\mathcal{X},\mathcal{Y})$ is bounded below in $L_{\mathcal{A}}(\mathcal{X},\mathcal{Y})$ if there is $M > 0$ such that $M\bar{p}_{\mathcal{X}}(x) \le \bar{p}_{\mathcal{Y}}(Tx)$ for all $p \in S(\mathcal{A})$ and $x \in \mathcal{X}$.

Our first result is a generalization of [1, Proposition 2.1].

**Proposition 2.2.** *Let $\mathcal{A}$ be a locally $C^*$-algebra, $\mathcal{X}$ and $\mathcal{Y}$ Hilbert $\mathcal{A}$-modules and $T \in b(L_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}))$. The following statements are mutually equivalent*:

(1) *$T$ is surjective*;
(2) *$T^*$ is bounded below in $L_{\mathcal{A}}(\mathcal{Y}, \mathcal{X})$, i.e., there is $m > 0$ such that $m\bar{p}_{\mathcal{Y}}(x) \leq \bar{p}_{\mathcal{X}}(T^*x)$ for all $p \in S(\mathcal{A})$ and $x \in \mathcal{Y}$*;
(3) *$T^*$ is bounded below with respect to inner product, i.e., there is $m' > 0$ such that $m'\langle x, x\rangle \leq \langle T^*x, T^*x\rangle$ for all $x \in \mathcal{Y}$.*

*Proof.* If (1) holds, then $\mathrm{Im}T = \mathcal{Y}$ is closed. It follows from [7, Remark 3.2.5] that $\mathrm{Im}T^*$ is also closed, $\mathrm{Ker}T \oplus \mathrm{Im}T^* = \mathcal{X}$ and $\mathrm{Ker}T^* \oplus \mathrm{Im}T = \mathcal{Y}$. We shall prove that $TT^*$ is bijective.

If $TT^*x = 0$ for some $x \in \mathcal{Y}$, then $T^*x \in \mathrm{Ker}T \cap \mathrm{Im}T^* = \{0\}$, hence $T^*x = 0$. Now $x \in \mathrm{Ker}T^* = (\mathrm{Im}T)^{\perp} = \mathcal{Y}^{\perp} = \{0\}$, implies $x = 0$. This proves that $TT^*$ is injective.

Let $z$ be an arbitrarily chosen element of $\mathcal{Y}$. $T$ is surjective, then we have $z = Ty$ for some $y \in \mathcal{X}$. There are $y_1 \in \mathrm{Ker}T$ and $x \in \mathcal{Y}$ such that $y = y_1 \oplus T^*x$. Then $z = Ty = T(y_1 \oplus T^*x) = Ty_1 + TT^*x = TT^*x$; therefore $TT^*$ is surjective.

Since $TT^*$ is a positive invertible element of the $C^*$-algebra $b(L_{\mathcal{A}}(\mathcal{Y}))$, then $0 \leq (TT^*)^{-1} \leq ||(TT^*)^{-1}||_{\infty} id_{\mathcal{Y}}$, so $TT^* \geq ||(TT^*)^{-1}||_{\infty}^{-1} id_{\mathcal{Y}}$, where $id_{\mathcal{Y}}$ stands for the identity operator on $\mathcal{Y}$. Denoting $m' = ||(TT^*)^{-1}||_{\infty}^{-1}$ we get $TT^* - m' id_{\mathcal{Y}} \geq 0$. This is equivalent to $\langle (TT^* - m' id_{\mathcal{Y}})x, x\rangle \geq 0$ for all $x \in \mathcal{Y}$, i.e., $\langle T^*x, T^*x\rangle \geq m'\langle x, x\rangle$ for all $x \in \mathcal{Y}$. So (3) holds.

The implication (3) $\Rightarrow$ (2) is trivial.

Suppose that (2) holds. Then $T^*$ is clearly injective, and it is easy to see that $\mathrm{Im}T^*$ is closed. Then $T$ has the closed range, again by [7, Remark 3.2.5], and $\mathcal{Y} = \mathrm{Ker}T^* \oplus \mathrm{Im}T = \{0\} \oplus \mathrm{Im}T = \mathrm{Im}T$. This gives (1). ☐

In view of [6, Theorem 3.7], we notice that if $T \in b(L_{\mathcal{A}}(\mathcal{X}))$, $\langle Tx, Tx\rangle \leq \|T\|_{\infty}^2 \langle x, x\rangle$ for all $x \in \mathcal{X}$. So we have the following corollary:

**Corollary 2.3.** *Let $\mathcal{A}$ be a locally $C^*$-algebra, $\mathcal{X}$ a Hilbert $\mathcal{A}$-module and $T \in b(L_{\mathcal{A}}(\mathcal{X}))$ such that $T^* = T$. The following statements are mutually equivalent*:

(1) *$T$ is surjective*;
(2) *There are $m, M > 0$ such that $m\bar{p}_{\mathcal{X}}(x) \leq \bar{p}_{\mathcal{X}}(Tx) \leq M\bar{p}_{\mathcal{X}}(x)$ for all $p \in S(\mathcal{A})$ and $x \in \mathcal{X}$*;
(3) *There are $m', M' > 0$ such that $m'\langle x, x\rangle \leq \langle Tx, Tx\rangle \leq M'\langle x, x\rangle$ for all $x \in \mathcal{X}$.*

*Remark* 2.4. Let $\mathcal{X}$ be a countably generated Hilbert $\mathcal{A}$-module in $M(\mathcal{X})$ and let $\{h_n\}_n$ be a standard frame of multipliers in $\mathcal{X}$. The module morphism $\theta : \mathcal{X} \longrightarrow \mathcal{H}_{\mathcal{A}}$ defined by $\theta(x) = (\langle h_n, x\rangle)_n$ is called the frame transform for $\{h_n\}_n$. The frame transform $\theta$ is an injective adjointable module morphism from $\mathcal{X}$ to $\mathcal{H}_{\mathcal{A}}$ with closed range. Moreover, $\theta \in b(L_{\mathcal{A}}(\mathcal{X}, \mathcal{H}_{\mathcal{A}}))$ which realizes an embedding of $\mathcal{X}$ onto an orthogonal summand of $\mathcal{H}_{\mathcal{A}}$. The adjoint operator

$\theta^*$ is surjective. Moreover, $\theta^* \circ \theta$ is an invertible element in $b(L_{\mathcal{A}}(\mathcal{X}))$. For details we refer the reader to [8].

**Theorem 2.5.** *Let $\mathcal{A}$ be a locally $C^*$-algebra, $\mathcal{X}$ a countably generated Hilbert $\mathcal{A}$-module in $M(\mathcal{X})$, $\{h_n : n \in \mathbb{N}\}$ a sequence in $M(\mathcal{X})$ and $\theta(x) = (\langle h_n, x\rangle)_{n\in\mathbb{N}}$ for $x \in \mathcal{X}$. The following statements are mutually equivalent*:

(1) *$\{h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in $\mathcal{X}$.*
(2) *$\theta \in b(L_{\mathcal{A}}(\mathcal{X}, \mathcal{H}_{\mathcal{A}}))$ and $\theta$ is bounded below in $L_{\mathcal{A}}(\mathcal{X}, \mathcal{H}_{\mathcal{A}})$.*
(3) *$\theta \in b(L_{\mathcal{A}}(\mathcal{X}, \mathcal{H}_{\mathcal{A}}))$ and $\theta^*$ is surjective.*

*Proof.* (1) $\Rightarrow$ (3): It was proved in [8, Theorem 3.11].

(2) $\Leftrightarrow$ (3): It follows from Proposition 2.2.

(2) $\Rightarrow$ (1): Since

$$\langle \theta x, \theta x\rangle = \sum_n \langle x, h_n\rangle\langle h_n, x\rangle,\ x \in \mathcal{X},$$

from (2) it follows that $\{h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in $\mathcal{X}$. □

Another direct consequence of Proposition 2.2 is that if $T \in b(L_{\mathcal{A}}(\mathcal{X}))$ is surjective and $\{h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in $\mathcal{X}$, then $\{T \circ h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in $\mathcal{X}$, too.

**Theorem 2.6.** *Let $\mathcal{A}$ be a locally $C^*$-algebra, $\mathcal{X}$ a countably generated Hilbert $\mathcal{A}$-module in $M(\mathcal{X})$, and $T \in b(L_{\mathcal{A}}(\mathcal{X}))$ surjective. If $\{h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in $\mathcal{X}$ with frame bounds $C$ and $D$, then $\{T \circ h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in $\mathcal{X}$ with frame bounds $C||(TT^*)^{-1}||_\infty^{-1}$ and $D||T||_\infty^2$.*

*Proof.* Let $x \in \mathcal{X}$. Since

$$\sum_{k=1}^{n} \langle x, T \circ h_k\rangle\langle T \circ h_k, x\rangle = \sum_{k=1}^{n} \langle T^*x, h_k\rangle\langle h_k, T^*x\rangle$$

and since $\{h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in $\mathcal{X}$ and $T^*x \in \mathcal{X}$, $\sum_n \langle x, T \circ h_n\rangle\langle T \circ h_n, x\rangle$ converges in $\mathcal{A}$, and

$$C\langle T^*x, T^*x\rangle \le \sum_n \langle T^*x, h_n\rangle\langle h_n, T^*x\rangle \le D\langle T^*x, T^*x\rangle.$$

From the proof of Proposition 2.2 we have $\langle T^*x, T^*x\rangle \ge ||(TT^*)^{-1}||_\infty^{-1}\langle x, x\rangle$, since $T$ is surjective. It follows that

$$C||(TT^*)^{-1}||_\infty^{-1}\langle x, x\rangle \le \sum_n \langle x, T \circ h_n\rangle\langle T \circ h_n, x\rangle \le D||T||_\infty^2\langle x, x\rangle.$$

From these facts we conclude that $\{T \circ h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in $\mathcal{X}$ with frame bounds $C||(TT^*)^{-1}||_\infty^{-1}$ and $D||T||_\infty^2$. □

The next result shows that the condition (1.1) from the definition of standard frames can be replaced with a weaker one.

**Theorem 2.7.** *Let $\mathcal{A}$ be a locally $C^*$-algebra, $\mathcal{X}$ a countably generated Hilbert $\mathcal{A}$-module in $M(\mathcal{X})$, and $\{h_n : n \in \mathbb{N}\}$ a sequence in $M(\mathcal{X})$ such that $\sum_n \langle x, h_n\rangle\langle h_n, x\rangle$ converges in $\mathcal{A}$ for every $x \in \mathcal{X}$. Then $\{h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in $\mathcal{X}$ if and only if there are constants $C, D > 0$ such that*

$$C\bar{p}_{\mathcal{X}}(x)^2 \leq p\left(\sum_n \langle x, h_n\rangle\langle h_n, x\rangle\right) \leq D\bar{p}_{\mathcal{X}}(x)^2,\ x \in \mathcal{X},\ p \in S(\mathcal{A}). \tag{2.1}$$

*Proof.* Obviously, every standard frame of multipliers in $\mathcal{X}$ satisfies (2.1).

For the converse we suppose that a sequence $\{h_n : n \in \mathbb{N}\}$ in $M(\mathcal{X})$ fulfills (2.1). For an arbitrary $x \in \mathcal{X}$ and a finite $J \subseteq \mathbb{N}$ we define $x_J = \sum_{n\in J} h_n\langle h_n, x\rangle$. Then

$$\begin{aligned}
\bar{p}_{\mathcal{X}}(x_J)^4 = p(\langle x_J, x_J\rangle)^2 &= p\left(\langle x_J, \sum_{n\in J} h_n\langle h_n, x\rangle\rangle\right)^2 \\
&= p\left(\sum_{n\in J}\langle x_J, h_n\rangle\langle h_n, x\rangle\right)^2 \\
&\leq p\left(\sum_{n\in J}\langle x_J, h_n\rangle\langle h_n, x_J\rangle\right) p\left(\sum_{n\in J}\langle x, h_n\rangle\langle h_n, x\rangle\right) \\
&\leq D\bar{p}_{\mathcal{X}}(x_J)^2 p\left(\sum_{n\in J}\langle x, h_n\rangle\langle h_n, x\rangle\right),
\end{aligned}$$

therefore

$$\bar{p}_{\mathcal{X}}\left(\sum_{n\in J} h_n\langle h_n, x\rangle\right)^2 = \bar{p}_{\mathcal{X}}(x_J)^2 \leq Dp\left(\sum_{n\in J}\langle x, h_n\rangle\langle h_n, x\rangle\right).$$

Since $J$ is arbitrary, the series $\sum_n h_n\langle h_n, x\rangle$ converges in $\mathcal{X}$ and since

$$\bar{p}_{\mathcal{X}}\left(\sum_{n\in J} h_n\langle h_n, x\rangle\right)^2 \leq Dp\left(\sum_{n\in J}\langle x, h_n\rangle\langle h_n, x\rangle\right) \leq D^2\bar{p}_{\mathcal{X}}(x)^2,$$

$$\bar{p}_{\mathcal{X}}\left(\sum_{n\in J} h_n\langle h_n, x\rangle\right) \leq D\bar{p}_{\mathcal{X}}(x).$$

Since $x \in \mathcal{X}$ is arbitrarily chosen, the operator

$$T : \mathcal{X} \longrightarrow \mathcal{X},\ x \longmapsto \sum_n h_n\langle h_n, x\rangle$$

is well defined, bounded and $\mathcal{A}$-linear. It is easy to check that $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in \mathcal{X}$, so $T \in L_{\mathcal{A}}(\mathcal{X})$ and $T = T^*$. From $\langle Tx, x\rangle = \sum_n \langle x, h_n\rangle\langle h_n, x\rangle \geq 0$ for all $x \in \mathcal{X}$, it follows that $T \geq 0$. Now (2.1) and

$\langle T^{\frac{1}{2}}x, T^{\frac{1}{2}}x\rangle = \sum_n \langle x, h_n\rangle\langle h_n, x\rangle$ imply $C^{\frac{1}{2}}\bar{p}_{\mathcal{X}}(x) \le \bar{p}_{\mathcal{X}}(T^{\frac{1}{2}}x) \le D^{\frac{1}{2}}\bar{p}_{\mathcal{X}}(x)$ for all $x \in \mathcal{X}$. By Corollary 2.3, there are constants $C', D' > 0$ such that

$$C'\langle x, x\rangle \le \langle T^{\frac{1}{2}}x, T^{\frac{1}{2}}x\rangle = \sum_n \langle x, h_n\rangle\langle h_n, x\rangle \le D'\langle x, x\rangle,\ x \in \mathcal{X}.$$

This proves that $\{h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in $\mathcal{X}$. □

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