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ON A CLASS OF GENERALIZED FUNCTIONS FOR SOME INTEGRAL TRANSFORM ENFOLDING KERNELS OF MEIJER G FUNCTION TYPE

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Abstract. In this paper, we investigate a modified G^2 transform on a class of Boehmians. We prove the axioms which are necessary for establishing the G^2 class of Boehmians. Addition, scalar multiplication, convolution, differentiation and convergence in the derived spaces have been defined. The extended G^2 transform of a Boehmian is given as a one-to-one onto mapping that is continuous with respect to certain convergence in the defined spaces. The inverse problem is also discussed.

1. Introduction

H functions are generalization of the hypergeometric function ${}_pF_q$ which are utilized for applications in a large variety of problems connected with statistical distribution theory, structures of random variables, generalized distributions, Mathai's pathway models, versatile integrals, reaction, diffusion, reaction diffusion, engineering, communications, fractional differential and integral equations, many areas of theoretical physics and statistical distribution theory as well.

The interest of integral transforms with special function kernels was motivated by the desire to study the corresponding integral equations of the first kind and of the so-called dual and triple equations encountered in various applications.

Some generalization, through a special case of H transforms [8, 10, 11, 13, 14, 21, 22], was proposed as the G transform which partially includes the classical Laplace and Hankel transforms, Riemann-Liouville fractional integral transforms, odd and even Hilbert transforms, integral transforms with Gauss hypergeometric functions, and some others to mention but a few.

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Kilbas and Saigo in [12] have investigated certain types of modified G transforms on the summable space $L_{v,r}, v \in \mathbb{R},\ 1 \leq v < \infty$, of those complex valued Lebesgue measurable functions such that

$$\|\psi\|_{v,r} = \left(\int_0^\infty |\sigma^v \psi(\sigma)|^r \frac{d\sigma}{\sigma}\right)^{\frac{1}{r}} < \infty. \tag{1}$$

In this paper, we consider a modified transform defined by the integral equation [12, 6.2.2]

$$\left(G^2\psi\right)(\xi) = \int_0^\infty G_{p,q}^{m,n}\left[\frac{\sigma}{\xi}\,\middle|\, \begin{matrix}(a_i)_{1,p}\\ (b_j)_{1,q}\end{matrix}\right]\psi(\sigma)\frac{d\sigma}{\xi}, \tag{2}$$

where $G_{p,q}^{m,n}$ is a G function. For sufficiently good function ψ, the modified G^2 transform has a close associate with the ordinary G transform

$$(G\psi)(\xi) = \int_0^\infty G_{p,q}^{m,n}\left[\xi\sigma\,\middle|\, \begin{matrix}(a_i)_{1,p}\\ (b_j)_{1,q}\end{matrix}\right]\psi(\sigma)\,d\sigma \tag{3}$$

noticed as $\left(G^2\psi\right)(\xi) = (RG\psi)(\xi)$, where R has the usual meaning as [12, 3.3.13]

$$(R\psi)(\xi) = \frac{1}{\xi}\psi\left(\frac{1}{\xi}\right).$$

In this article it is of great importance to mention that the cited integral (2) satisfies the equation

$$\left(M_l G^2\psi\right)(\omega) = g_{p,q}^{m,n}\left[1-\omega\,\middle|\, \begin{matrix}(a_i)_{1,p}\\ (b_j)_{1,q}\end{matrix}\right](M_l\psi)(\omega), \tag{4}$$

where

$$g_{p,q}^{m,n}\left[\omega\,\middle|\, \begin{matrix}(a_i)_{1,p}\\ (b_j)_{1,q}\end{matrix}\right] = \frac{\prod_{j=1}^m \Gamma(b_j+\omega)\prod_{i=1}^n \Gamma(1-a_i-\omega)}{\prod_{i=n+1}^p \Gamma(a_i+\omega)\prod_{j=m+1}^q \Gamma(1-b_j-\omega)},$$

$M_l\psi$ being the Mellin transform of $\psi(t)$.

The Parseval formula for G^2 transform was expressed in terms of the identities:

$$\int_0^\infty \psi(\xi)\left(G^2\delta\right)(\xi)\,d\xi = \int_0^\infty \left(G^1\psi\right)(\xi)\,\delta(\xi)\,d\xi \tag{5}$$

and

$$\int_0^\infty \psi(\xi)\left(G^1\delta\right)(\xi)\,d\xi = \int_0^\infty \left(G^2\psi\right)(\xi)\,\delta(\xi)\,d\xi, \tag{6}$$

where G^1 is the modified transform defined by [12, 6.2.1]

$$\left(G^1\psi\right)(\xi) = \int_0^\infty G_{p,q}^{m,n}\left[\xi\sigma\,\middle|\, \begin{matrix}(a_i)_{1,p}\\ (b_j)_{1,q}\end{matrix}\right]\psi(\sigma)\frac{d\sigma}{\sigma}.$$

The numbers $a^*, \Delta, a_1^*, a_2^*, \alpha$ and β wherever they appear have the benefit of:

$$
\left\{
\begin{array}{l}
a^* = 2\,(m+n) - p - q; \\
\Delta = q - p; \\
a_1^* = m + n - p; \\
a_2^* = m + n - q; \\
\alpha = \left\{ \begin{array}{ll} -\min_{1\le j\le m} \operatorname{Re}(b_j) & , m > 0 \\ -\infty & , m = 0 \end{array} \right. \\
\beta = \left\{ \begin{array}{ll} 1 - \max_{1\le i\le n} \operatorname{Re}(a_i) & , n > 0 \\ \infty & , n = 0 \end{array} \right. \\
\text{and} \\
\mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_j + \dfrac{p-q}{2}.
\end{array}
\right. \tag{7}
$$

Following theorem is due to Kilbas and Saigo [12, Theorem 6.26(i)].

Theorem 1. *We suppose* (i) $\alpha < 1 - v < \beta$ *and that either of the conditions* (ii) $a^* > 0$ *or* (iii) $a^* = 0,\ \Delta(1-v) + \operatorname{Re}(\mu) \le 0$ *holds. Then, we have the following results*:

(a) *There is a one-to-one transform* $G^2 \in [L_{v,2}, L_{v,2}]$ *and there holds Equation* (4) *for* $\operatorname{Re}(\omega) = v$ *and* $\psi \in L_{v,2}$.

(b) *If* $a^* = 0, \Delta(1-v) + \operatorname{Re}(\mu) = 0$ *and* $v \notin \varepsilon_g$, *then the transform* G^2 *maps* $L_{v,2}$ *onto* $L_{v,2}$, *where* ε_g *is the exceptional set of*

$$
g(\omega) = G_{p,q}^{m,n}\left[\omega \,\middle|\, \begin{array}{l} (a_i)_{1,p} \\ (b_j)_{1,q} \end{array}\right]
$$

of real numbers v *such that* $\alpha < 1 - v < \beta$ *and* $g(\omega)$ *has a zero on the line* $\operatorname{Re}(\omega) = 1 - v$.

For a somehow much more detailed account of several significant results on the modified G transforms, we refer to the monograph [12].

Integral transforms of the generalized space of Boehmians were defined in many papers once the topic started. In the survey articles [1, 3, 5, 17] and [2, 4, 6, 7, 9, 15, 16, 18, 19], various integral transforms were extended to various spaces of Boehmians. Throughout this paper, we aim to extend the so-called modified G^2 transform to some class of Boehmians and discuss some related results. In the following section we present some fundamental convolution products for generating the generalized spaces and give the definition of the extended transform on the generalized spaces. In Section 3 we derive some properties of our transform in a generalized sense.

2. Generalized spaces of Boehmians

The concept of Boehmians is motivated by regular operators introduced by Boehme [9]. Boehmians have an algebraic character of Mikusinski operators and at the same time do not have restriction on the support. Applying the general construction to various function spaces yields various spaces of Boehmians.

General Boehmians contain the Schwartz space of distributions , Roumieu ultradistributions, regular operators and tempered distributions as well.

As our main product, we shall make a free use of the product $\bullet$ that we define here as

$$(\psi \bullet \delta)(\xi) = \int_0^\infty \psi\left(\varsigma^{-1}\xi\right)\varsigma^{-1}\delta(\varsigma)\,d\varsigma, \tag{8}$$

when the integral exists. On the other hand, the second integral, which is identical to our definition (8) , is the Mellin type convolution product defined by [20]

$$(\psi \times \delta)(\xi) = \int_0^\infty \zeta^{-1}\psi\left(\xi\zeta^{-1}\right)\delta(\zeta)\,d\zeta, \tag{9}$$

whose integral properties are as follows identities:

(i) $\psi_1 \times \psi_2 = \psi_2 \times \psi_1$;
(ii) $(\psi_1 \times \psi_2) \times \psi_3 = \psi_1 \times (\psi_2 \times \psi_3)$;
(iii) $(\alpha\psi_1) \times \psi_2 = \alpha(\psi_1 \times \psi_2)$;
(iv) $\psi_1 \times (\psi_2 + \psi_3) = \psi_1 \times \psi_2 + \psi_1 \times \psi_3$.

By $\mathcal{C}_c^\infty$ we denote the standard notation of the set of smooth functions of compact supports on $(0,\infty)$, and by Δ we denote the subset of $\mathcal{C}_c^\infty$ of delta sequences satisfying Conditions (10)-(12),

$$\int_0^\infty \delta_n(\xi)\,d\xi = 1 \text{ for every } n \in \mathbb{N}. \tag{10}$$

$$|\delta_n(\xi)| < M \text{ for every } n \in \mathbb{N}, \text{ where } M \in \mathbb{R}, M > 0. \tag{11}$$

$$\operatorname{supp}\delta_n(\xi) \subset (a_n, b_n), \text{ where } a_n, b_n \to 0 \text{ as } n \to \infty. \tag{12}$$

For the modified transform, we are generating some generalized function spaces of Boehmians, $\beta(L_{v,2}, \times, \bullet)$ and $\beta(L_{v,2}, \times)$, with fairly cited products.

It worthwhile that we establish the following lemma.

Lemma 2. *Given that $\psi \in L_{v,2}$ and $\delta \in \mathcal{C}_c^\infty$. Then, we have $G^2(\psi \times \delta)(\xi) = \left(\left(G^2\psi\right) \bullet \delta\right)(\xi)$.*

Proof. Assume $\psi \in L_{v,2}$ and $\delta \in \mathcal{C}_c^\infty$ be given. Then, by (9) we write

$$G^2(\psi \times \delta)(\xi) = \int_0^\infty G_{p,q}^{m,n}\left[\frac{\xi}{\sigma}\left|\begin{array}{c}(a_i)_{1,p}\\(b_j)_{1,q}\end{array}\right.\right]\int_0^\infty \zeta^{-1}\psi\left(\sigma\zeta^{-1}\right)\delta(\zeta)\,d\zeta\frac{d\sigma}{\xi}.$$

Setting variables, $\sigma^{-1}\zeta = \rho$, gives

$$G^2(\psi \times \delta)(\xi) = \int_0^\infty \delta(\zeta)\zeta^{-1}\int_0^\infty G_{p,q}^{m,n}\left[\frac{\zeta\rho}{\xi}\left|\begin{array}{c}(a_i)_{1,p}\\(b_j)_{1,q}\end{array}\right.\right]\psi(\rho)\frac{d\rho}{\xi}. \tag{13}$$

Modifying (13) yields

$$G^2(\psi \times \delta)(\xi) = \int_0^\infty \delta(\zeta)\zeta^{-1}\int_0^\infty G_{p,q}^{m,n}\left[\frac{\rho}{\zeta^{-1}\xi}\left|\begin{array}{c}(a_i)_{1,p}\\(b_j)_{1,q}\end{array}\right.\right]\psi(\rho)\frac{d\rho}{\zeta^{-1}\xi}.$$

Hence, we have obtained

$$G^2(\psi \times \delta)(\xi) = \int_0^\infty \left(G^2\psi\right)\left(\zeta^{-1}\xi\right)\zeta^{-1}\delta(\zeta)\,d\zeta.$$

This reveals

$$G^2(\psi \times \delta)(\xi) = \left(G^2\psi \bullet \delta\right)(\xi).$$

Hence, the proof of this theorem has been finished. □

Let us first consider several axioms that are in charge of defining the space $\beta(L_{v,2}, \curlyvee, \bullet)$.

Theorem 3. *Given that $\psi \in L_{v,2}$ and $\delta \in \mathcal{C}_c^\infty$. Then, we have $\psi \bullet \delta \in L_{v,2}$.*

Proof. By aid of Equation (1) and Equation (8), we write

$$\|\psi \bullet \delta\|_{v,2}^2 = \int_0^\infty \left|\sigma^v \int_0^\infty \psi\left(\zeta^{-1}\sigma\right)\zeta^{-1}\delta(\zeta)\,d\zeta\right|^2 \frac{d\sigma}{\sigma}. \tag{14}$$

Appealing to Fubini's theorem and Jensen's inequality, (14) reduces to

$$\begin{aligned}\|\psi \bullet \delta\|_{v,2}^2 &\le \|\psi\|_{v,2}^2 \int_a^b \left|\zeta^{-1}\delta(\zeta)\right| d\zeta \\ &< A\,\|\psi\|_{v,2}^2,\end{aligned}$$

where $[a, b]$ is a real bounded set containing the support of δ, $A > 0$, $A \in \mathbb{R}$. Hence, we have reached to the conclusion that

$$\psi \bullet \delta \in L_{v,2}.$$

The proof is therefore finished. □

Theorem 4. *Given that $(\delta_n), (\varepsilon_n) \in \Delta$. Then, we have $(\delta_n \times \varepsilon_n) \in \Delta$.*

Proof of this theorem follows from the properties of the product $\curlyvee$. Details are deleted.

Proof of the following theorems is straightforward.

Theorem 5. *Given that $(\psi_n), \psi \in L_{v,2}$ and $\delta \in \mathcal{C}_c^\infty$. Then, we have*

$$\psi_n \bullet \delta \to \theta \bullet \delta \text{ and } (r^*\psi_n) \bullet \delta = r^*(\psi_n \bullet \delta),\ r^* \in \mathbb{C}.$$

Theorem 6. *Given that $\psi_1, \psi_2 \in L_{v,2}$ and $\delta \in \mathcal{C}_c^\infty$. Then, we have*

$$(\psi_1 + \psi_2) \bullet \delta = \psi_1 \bullet \delta + \psi_2 \bullet \delta.$$

Theorem 7. *Given that $\psi \in L_{v,2}$ and $\delta_1, \delta_2 \in \mathcal{C}_c^\infty$. Then, the following hold*

(i) $\psi \bullet (\delta_1 \times \delta_2) = (\psi \bullet \delta_1) \bullet \delta_2$.

(ii) *If $(\delta_n) \in \Delta$ and $\psi \in L_{v,2}$, then $\psi \bullet \delta_n \to \psi$ as $n \to \infty$.*

Proof. Proof of (`i`) is a straightforward consequence of the properties of Mellin convolution products. For more details, let $\psi \in L_{v,2}$ and $\delta_1, \delta_2 \in \mathcal{C}_c^{\infty}$. Then, by Equation (8), we have

$$(\psi \bullet (\delta_1 \times \delta_2))(\xi) = \int_0^{\infty} \psi\left(\zeta^{-1}\xi\right)\zeta^{-1}(\delta_1 \times \delta_2)(\zeta)\,d\zeta.$$

Equation (9) reveals that

$$(\psi \bullet (\delta_1 \times \delta_2))(\xi) = \int_0^{\infty} \psi\left(\zeta^{-1}\xi\right)\zeta^{-1}\int_0^{\infty} \sigma^{-1}\delta\left(\zeta\sigma^{-1}\right)\delta_2(\sigma)\,d\sigma d\zeta.$$

Change of variables, $\zeta\sigma^{-1} = \rho$, and Fubini's theorem imply

$$\begin{aligned}(\psi \bullet (\delta_1 \times \delta_2))(\xi) &= \int_0^{\infty} \delta_2(\sigma)\,\sigma^{-1}\int_0^{\infty} \psi\left(y^{-1}\xi\right)y^{-1}\delta_1\left(y\sigma^{-1}\right)d\zeta d\sigma \\ &= \int_0^{\infty} \delta_2(\sigma)\,\sigma^{-1}(\psi \bullet \delta_1)\left(\sigma^{-1}\xi\right)d\sigma. \qquad (15)\end{aligned}$$

We have therefore finished the proof of Part (`i`) of the theorem.

Proof of Part (`ii`). Assume $(\delta_n) \in \Delta$ and $\psi \in L_{v,2}$ be given, then by Equation (11) we have

$$|\delta_n(\xi)| < M \qquad (16)$$

for some positive real number M. By Equation (12) we also have

$$\operatorname{supp}\delta_n(\xi) \subseteq (a_n, b_n),\ (a_n, b_n) \to 0 \text{ as } n \to \infty. \qquad (17)$$

Hence, using the set of norms of $L_{v,2}$ gives

$$\|(\psi \bullet \delta_n - \psi)(\sigma)\|_{v,2}^2 = \int_0^{\infty} |\sigma^v(\psi \bullet \delta_n - \psi)(\sigma)|^2 \frac{d\sigma}{\sigma}$$

which can written by (10) in the form

$$\|(\psi \bullet \delta_n - \psi)(\sigma)\|_{v,2}^2 = \int_0^{\infty} \left|\sigma^v\left(\int_0^{\infty}\left(\psi\left(\zeta\sigma^{-1}\right)\zeta^{-1} - \psi(\sigma)\right)\delta_n(\zeta)\,d\zeta\right)\right|^2 \frac{d\sigma}{\sigma}.$$

By Equation (16) and Equation (17) , we finally get

$$\|(\psi \bullet \delta_n - \psi)(\sigma)\|_{v,2}^2 \leq AM(a_n, b_n), \qquad (18)$$

where A and M are positive constants.
Hence, considering the limit as $n \to \infty$, we from (18) obtain

$$\psi \bullet \delta_n \to \psi \text{ as } n \to \infty.$$

This finishes the proof of the theorem. □

Theorem 8. *Given $\psi \bullet \delta_n = \psi_1 \bullet \delta_n$, where $\psi, \psi_1 \in L_{v,2}$ and $(\delta_n) \in \Delta$. Then $\psi = \psi_1$.*

Proof of this theorem follows from Theorem 7(ii). Proof is therefore omitted.

The axioms of our space are established. Hence, the space under consideration is well-defined.

Addition in $\beta(L_{v,2},\times,\bullet)$ is defined as $\frac{\varphi_n}{\delta_n}+\frac{\psi_n}{\varepsilon_n}=\frac{\varphi_n\bullet\delta_n+\psi_n\bullet\delta_n}{\delta_n\times\varepsilon_n}$. Scalar multiplication in $\beta(L_{v,2},\times,\bullet)$ is defined as $\Omega\frac{\varphi_n}{\delta_n}=\Omega\frac{\varphi_n}{\delta_n}=\frac{\Omega\varphi_n}{\delta_n}$, $\Omega\in\mathbb{C}$. We define the convolution $\bullet$ in $\beta(L_{v,2},\times,\bullet)$ as $\frac{\varphi_n}{\delta_n}\bullet\frac{\psi_n}{\varepsilon_n}=\frac{\varphi_n\bullet\psi_n}{\delta_n\times\varepsilon_n}$. Differentiation in $\beta(L_{v,2},\times,\bullet)$ is defined as $\mathcal{D}^\alpha\frac{\varphi_n}{\delta_n}=\frac{\mathcal{D}^\alpha\varphi_n}{\delta_n}$, α is a real number. The product $\bullet$ for $\beta(L_{v,2},\times,\bullet)\times L_{v,2}$ we define as $\frac{\varphi_n}{\delta_n}\bullet\varphi=\frac{\varphi_n\bullet\varphi}{\delta_n}$, where $\frac{\varphi_n}{\delta_n}\in\beta(L_{v,2},\times,\bullet)$ and $\varphi\in L_{v,2}$. A sequence (β_n) of $\beta(L_{v,2},\times,\bullet)$ is δ convergent to β in $\beta(L_{v,2},\times,\bullet)$ $\left(\beta_n\xrightarrow{\delta}\beta\right)$ if there exists a delta sequence (δ_n) such that

$$(\beta_n\bullet\delta_k),(\beta\bullet\delta_k)\in L_{v,2}\ \ (\forall k,n\in\mathbb{N}),$$

and

$$(\beta_n\bullet\delta_k)\to(\beta\bullet\delta_k)\ \text{ as } n\to\infty,\ \text{ in } L_{v,2} \text{ for every } k\in\mathbb{N}.$$

The equivalent argument of δ convergence is that : $\beta_n\xrightarrow{\delta}\beta$ $(n\to\infty)$ in $\beta(L_{v,2},\times,\bullet)$ if and only if there is $\varphi_{n,k},\varphi_k\in L_{v,2}$ and $\delta_k\in\Delta$ such that $\beta_n=\frac{\varphi_{n,k}}{\delta_k},\beta=\frac{\varphi_k}{\delta_k}$ and for each $k\in\mathbb{N}$, $\varphi_{n,k}\to\varphi_k$ as $n\to\infty$ in $L_{v,2}$.

A sequence (β_n) of $\beta(L_{v,2},\times,\bullet)$ is Δ convergent to β in $\beta(L_{v,2},\times,\bullet)$ $\left(\text{or } \beta_n\xrightarrow{\Delta}\beta\right)$ if there exists a $(\delta_n)\in\Delta$ such that $(\beta_n-\beta)\bullet\delta_n\in L_{v,2},\forall n\in\mathbb{N}$, and $(\beta_n-\beta)\bullet\delta_n\to0$ as $n\to\infty$ in $L_{v,2}$.

Generating the space $\beta(L_{v,2},\times)$ follows from similar proofs to above and in taking into account properties of the product $\times$ recited above.

The operations: addition, multiplication by a scalar and convergence in $\beta(L_{v,2},\times)$ can be extended to $\beta(L_{v,2},\times,\bullet)$ similarly. Hence we avoid repeating same arguments.

With the help of Lemma 2, and the preceding debates we are lead to the following definition.

Definition 9. Given $\alpha<1-v<\beta$ and that either of the conditions (ii) or (iii) of Theorem 1 holds. We consider to define the extended G^2 transform of $\frac{\psi_n}{\delta_n}\in\beta(L_{v,2},\times)$ as

$$G^{2,b}_{ex.}\frac{\psi_n}{\delta_n}=\frac{G^2\psi_n}{\delta_n} \tag{19}$$

in $\beta(L_{v,2},\times,\bullet)$.

The numerator of the right hand side of Equation (19) belongs to $L_{v,2}$. Hence, the equivalence class (19) defines a Boehmian in the space $\beta(L_{v,2},\times,\bullet)$ by Lemma 2.

3. Abelian theorems for $G^{2,b}_{ex.}$ transform

Theorem 10. *Given $\alpha < 1 - v < \beta$ and that either of the conditions* (ii) *or* (iii) *of Theorem 1 holds. Then, the operator $G^{2,b}_{ex.}$ is well-defined and linear from $\beta(L_{v,2}, \times)$ into $\beta(L_{v,2}, \times, \bullet)$.*

Proof. Assume $\frac{\psi_n}{\delta_n} = \frac{\varphi_n}{\epsilon_n} \in \beta(L_{v,2}, \times)$ and the hypothesis of the theorem satisfies. Then by the notion of equivalence classes of $\beta(L_{v,2}, \times)$ it holds that

$$\varphi_n \times \delta_m = \psi_m \times \epsilon_n = \psi_n \times \epsilon_m \ \ (\forall n, m \in \mathbb{N}).$$

Employing G^2 transform for both sides of the previous equation and investing Lemma 2 suggest to write

$$G^2\varphi_n \bullet \delta_m = G^2\psi_n \bullet \epsilon_m \ \ (\forall n, m \in \mathbb{N}).$$

Thus, the quotients $\frac{G^2\varphi_n}{\epsilon_n}$ and $\frac{G^2\psi_n}{\delta_n}$ are equivalent in the sense of $\beta(L_{v,2}, \times, \bullet)$ and, consequently,

$$\frac{G^2\varphi_n}{\epsilon_n} = \frac{G^2\psi_n}{\delta_n} \ \ (\forall n \in \mathbb{N}).$$

Proof of linearity of $G^{2,b}_{ex.}$ is as follows. Let $\frac{\varphi_n}{\epsilon_n}, \frac{\psi_n}{\delta_n} \in \beta(L_{v,2}, \times) \ (\forall n \in \mathbb{N})$. Then, by Lemma 2 and for all $n \in \mathbb{N}$ we have

$$\begin{aligned} G^{2,b}_{ex.}\left(\frac{\varphi_n}{\epsilon_n} + \frac{\psi_n}{\delta_n}\right) &= G^{2,b}_{ex.}\left(\frac{\varphi_n \times \delta_n + \psi_n \times \epsilon_n}{\epsilon_n \times \delta_n}\right) \\ &= \frac{G^2(\varphi_n \times \delta_n + \psi_n \times \epsilon_n)}{\epsilon_n \times \delta_n} \\ &= \frac{G^2\varphi_n \bullet \delta_n + G^2\psi_n \bullet \epsilon_n}{\epsilon_n \times \delta_n}. \end{aligned}$$

This is expressed to give $G^{2,b}_{ex.}\left(\frac{\varphi_n}{\epsilon_n} + \frac{\psi_n}{\delta_n}\right) = G^{2,b}_{ex.}\frac{\varphi_n}{\epsilon_n} + G^{2,b}_{ex.}\frac{\psi_n}{\delta_n} \ (\forall n \in \mathbb{N})$. Let $\Omega \in \mathbb{C}$, then of course $\Omega G^{2,b}_{ex.}\frac{\varphi_n}{\epsilon_n} = \Omega\frac{G^2\varphi_n}{\epsilon_n} = \frac{G^2\Omega\varphi_n}{\epsilon_n}$. Hence, we lead to write

$$\Omega G^{2,b}_{ex.}\frac{\varphi_n}{\epsilon_n} = G^{2,b}_{ex.}\left(\Omega\frac{\varphi_n}{\epsilon_n}\right) \ \ (\forall n \in \mathbb{N}).$$

We have finished the proof of the theorem. □

Theorem 11. *Under Conditions* (i) *and* (ii) *of Theorem 1, the operator $G^{2,b}_{ex.} : \beta(L_{v,2}, \times) \to \beta(L_{v,2}, \times, \bullet)$ is injective.*

Proof. Assume $G^{2,b}_{ex.}\frac{\psi_n}{\delta_n} = G^{2,b}_{ex.}\frac{\varphi_n}{\epsilon_n} \ (\forall n \in \mathbb{N})$. Using the concept of quotients in $\beta(L_{v,2}, \times, \bullet)$ implies

$$G^2\psi_n \bullet \epsilon_m = G^2\varphi_m \bullet \delta_n \ \ (\forall m, n \in \mathbb{N}).$$

Lemma 2 also implies $G^2(\psi_n \times \epsilon_m) = G^2(\varphi_m \times \delta_n) \ \ (\forall m, n \in \mathbb{N})$. Hence $\psi_n \times \epsilon_m = \varphi_m \times \delta_n$. Therefore

$$\frac{\psi_n}{\delta_n} = \frac{\varphi_n}{\epsilon_n} \ \ (\forall n \in \mathbb{N}).$$

This finishes the proof of our theorem. □

Theorem 12. *Given* $a^* = 0$, $\Delta(1-v) + \mathrm{Re}(\mu) = 0$ *and* $v \notin \varepsilon_g$. *Then, the operator* $G^{2,b}_{ex.} : \beta(L_{v,2}, \times) \to \beta(L_{v,2}, \times, \bullet)$ *is surjective ,* ε_g *has usual meaning.*

Proof of this theorem is obvious by Theorem 1. Hence, we omit details.

Definition 13. Given that Conditions (i) and (ii) of Theorem 1 satisfy and that $\frac{\hat{\psi}_n}{\delta_n} \in \beta(L_{v,2}, \times, \bullet)$. Then, for each $(\delta_n) \in \Delta$ and some $(\psi_n) \in L_{v,2}$ we define the inverse G^2 transform as

$$\left(G^{2,b}_{ex.}\right)^{-1} \frac{\hat{\psi}_n}{\delta_n} = \frac{\psi_n}{\delta_n}. \tag{20}$$

Theorem 14. *Let the identities* (i) *and* (ii) *of Theorem 1 satisfy. Then, the mapping* $\left(G^{2,b}_{ex.}\right)^{-1}$ *is well-defined.*

Proof. Assume $\frac{\hat{\psi}_n}{\delta_n} = \frac{\hat{\varphi}_n}{\epsilon_n}$ in $\beta(L_{v,2}, \times, \bullet)$. Then, $\hat{\psi}_n \times \epsilon_m = \hat{\varphi}_m \times \delta_n$ $(\forall m, n \in \mathbb{N})$ in the sense of $\beta(L_{v,2}, \times, \bullet)$. Therefore, applying the inverse G^2 transform and investing Lemma 2 yield

$$\left(G^2\right)^{-1} \hat{\psi}_n \bullet \epsilon_m = \left(G^2\right)^{-1} \hat{\varphi}_m \bullet \delta_n.$$

In $\beta(L_{v,2}, \times)$, it means $\frac{\left(G^2\right)^{-1} \hat{\psi}_n}{\delta_n} = \frac{\left(G^2\right)^{-1} \hat{\varphi}_n}{\epsilon_n}$ $(\forall n \in \mathbb{N})$. This finishes the proof of the theorem. □

Theorem 15. *Let* $a^* = 0, \Delta(1-v) + \mathrm{Re}(\mu) = 0$ *and* $v \notin \varepsilon_g$. *Then the mapping* $\left(G^{2,b}_{ex.}\right)^{-1}$ *is linear from* $\beta(L_{v,2}, \times, \bullet)$ *onto* $\beta(L_{v,2}, \times)$.

Proof. Assume the hypothesis satisfies for two Boehmians $\frac{\hat{\psi}_n}{\delta_n}$ and $\frac{\hat{\varphi}_n}{\delta_n}$ in $\beta(L_{v,2}, \times, \bullet)$. Then, for all $n \in \mathbb{N}$, we have

$$\frac{\hat{\psi}_n}{\delta_n} + \frac{\hat{\varphi}_n}{\epsilon_n} = \frac{\hat{\psi}_n \times \epsilon_n + \hat{\varphi}_n \times \epsilon_n}{\delta_n \times \epsilon_n}.$$

Applying Equation (20) to above yields

$$\left(G^{2,b}_{ex.}\right)^{-1} \left(\frac{\hat{\psi}_n}{\delta_n} + \frac{\hat{\varphi}_n}{\epsilon_n}\right) = \frac{\left(G^2\right)^{-1}\left(\hat{\psi}_n \times \epsilon_n + \hat{\varphi}_n \times \epsilon_n\right)}{\delta_n \times \epsilon_n}.$$

On account of Lemma 2 we get

$$\left(G^{2,b}_{ex.}\right)^{-1} \left(\frac{\hat{\psi}_n}{\delta_n} + \frac{\hat{\varphi}_n}{\epsilon_n}\right) = \frac{\left(G^2\right)^{-1}\left(G^2\left(\psi_n \bullet \epsilon_n\right) + G^2\left(\varphi_n \bullet \delta_n\right)\right)}{\delta_n \times \epsilon_n}.$$

Notion of addition in $\beta(L_{v,2}, \times)$ implies

$$\left(G^{2,b}_{ex.}\right)^{-1} \left(\frac{\hat{\psi}_n}{\delta_n} + \frac{\hat{\varphi}_n}{\epsilon_n}\right) = \frac{\psi_n}{\delta_n} + \frac{\varphi_n}{\epsilon_n} \ (\forall n \in \mathbb{N}).$$

Finally, for some $\eta \in \mathbb{C}$ and all $n \in \mathbb{N}$, we have

$$\left(G_{ex.}^{2,b}\right)^{-1}\left(\eta \frac{\hat{\psi}_n}{\delta_n}\right) = \eta \left(G_{ex.}^{2,b}\right)^{-1} \frac{\hat{\psi}_n}{\delta_n}.$$

This finishes the proof of the theorem. □

Theorem 16. *$G_{ex.}^{2,b}$ is continuous with respect to δ and Δ convergence.*

Proof of this theorem can be followed similarly as in the citations of the same author. Hence, we avoid to repeat the similar proofs.

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