

## THE TOEPLITZNESS OF WEIGHTED COMPOSITION OPERATORS

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*Dedicated to the memory of Takahiko Nakazi*

ABSTRACT. We will consider the asymptotic toeplitzness associated with weighted composition operators on the Hardy-Hilbert space  $H^2$ .

### 1. Introduction

Throughout let  $H^2$  be the Hardy-Hilbert space of all analytic functions on the open unit disk  $\mathbb{D}$  with square-summable Taylor coefficients. For  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  and  $g(z) = \sum_{n=0}^{\infty} b_n z^n$  in  $H^2$ , identifying functions in  $H^2$  with their boundary functions, the standard inner product is defined as

$$\begin{aligned}\langle f, g \rangle &= \sum_{n=0}^{\infty} a_n \overline{b_n} \\ &= \int_{\partial \mathbb{D}} f(e^{i\theta}) \overline{g(e^{i\theta})} dm(\theta),\end{aligned}$$

where  $m$  is the normalized Lebesgue measure on the boundary  $\partial \mathbb{D}$  of  $\mathbb{D}$ . Refer to [8, 15] for the basic properties of the classical Hardy spaces.

Let  $T$  be a bounded linear operator on  $H^2$ . Then  $T$  is a Toeplitz operator if and only if  $S^*TS = T$ , where  $S$  is the forward shift defined by  $Sf(z) = zf(z)$  for  $z \in \partial \mathbb{D}$  and  $f \in H^2$  and  $S^*$  is the backward shift on  $H^2$ . In the natural way, for a bounded measurable function  $u \in L^\infty(\partial \mathbb{D})$ , a Toeplitz operator  $T_u$  on  $H^2$  is defined as  $T_u f = P(uf)$  for  $f \in H^2$ , where  $P$  is the orthogonal projection from  $L^2(\partial \mathbb{D})$  to  $H^2$ . Recall that the compact Toeplitz operator on  $H^2$  is only the zero operator. See [6, 13] for operator theory on  $H^2$ .

In [1], Barría and Halmos firstly called an operator  $T$  on  $H^2$  asymptotically Toeplitz if the sequence of operators  $\{S^{*n}TS^n\}$  converges strongly on  $H^2$ . Then Feintuch [9] suggested the analogous conditions relative to either weak or norm

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operator convergence. So there are actually three different kinds of asymptotic toepplitzness.

**Definition.** Let  $T$  be a bounded linear operator on  $H^2$ .

(i)  $T$  is said to be *uniformly asymptotically Toeplitz* if there is a bounded linear operator  $A$  on  $H^2$  such that  $\|S^{*n}TS^n - A\| \rightarrow 0$  as  $n \rightarrow \infty$ .

(ii)  $T$  is said to be *strongly asymptotically Toeplitz* if there is an operator  $A$  on  $H^2$  such that  $\|(S^{*n}TS^n - A)f\| \rightarrow 0$  as  $n \rightarrow \infty$  for any  $f \in H^2$ .

(iii)  $T$  is said to be *weakly asymptotically Toeplitz* if there is an operator  $A$  on  $H^2$  such that  $\langle (S^{*n}TS^n - A)f, g \rangle \rightarrow 0$  as  $n \rightarrow \infty$  for all  $f, g \in H^2$ .

Feintuch [9] showed the following result.

**Theorem of Feintuch.** *A bounded linear operator on  $H^2$  is uniformly asymptotically Toeplitz if and only if it is the sum of a Toeplitz operator and a compact operator.*

The asymptotic toepplitzness of composition operators originally was considered by Shapiro. For an analytic self-map  $\varphi$  of  $\mathbb{D}$ , the composition operator  $C_\varphi$  is defined by  $C_\varphi f = f \circ \varphi$ . It has been known for a long time that such operators are bounded linear operators on  $H^2$ . See [3, 16, 19] for the study of composition operators. Nazarov and Shapiro [14] investigated properties of the asymptotic toepplitzness of composition operators and adjoints. Also, refer to [17, 18] for a survey of early results on the toepplitzness of composition operators. Recently the toepplitzness of products of composition operators and their adjoints is independently investigated in [4, 7].

The concept of composition operators has been generalized to weighted composition operators. Let  $u$  be a non-zero bounded analytic function on  $\mathbb{D}$  and  $\varphi$  an analytic self-map of  $\mathbb{D}$ . We define the weighted composition operator  $M_u C_\varphi$  by

$$M_u C_\varphi f = u \cdot (f \circ \varphi)$$

for  $f \in H^2$ . Then  $M_u C_\varphi$  is a bounded linear operator on  $H^2$ .

In this article we would consider the asymptotic toepplitzness associated with weighted composition operators on  $H^2$ .

## 2. Toeplitzness of weighted composition operators

First we consider the condition for the weighted composition operator to be a Toeplitz operator.

**Theorem 2.1.** *Let  $u$  be a non-zero bounded analytic function on  $\mathbb{D}$  and  $\varphi$  a non-constant analytic self-map of  $\mathbb{D}$ . Then  $M_u C_\varphi$  is Toeplitz if and only if  $\varphi$  is the identity.*

*Proof.* By the definition,  $M_u C_\varphi$  is Toeplitz if and only if  $S^* M_u C_\varphi S = M_u C_\varphi$ . Then, taking  $f \equiv 1$ ,  $S^* M_u C_\varphi S 1 = M_u C_\varphi 1$  and  $S^*(u\varphi) = u$ . So

$$\frac{u(z)\varphi(z) - u(0)\varphi(0)}{z} = u(z) \quad \text{and} \quad u(z)(\varphi(z) - z) = u(0)\varphi(0).$$

Next, taking  $f(z) \equiv z$ ,  $S^*M_uC_\varphi S z = M_uC_\varphi z$  and  $S^*(u\varphi^2) = u\varphi$ . Thus

$$u(z)\varphi(z)(\varphi(z) - z) = u(0)\varphi^2(0).$$

Consequently it holds that  $\varphi(z)u(0)\varphi(0) = u(0)\varphi^2(0)$ . If  $u(0)\varphi(0) \neq 0$ ,  $\varphi(z) = \varphi(0) = \text{constant}$  and this is a contradiction. If  $u(0)\varphi(0) = 0$ , then  $u(z)(\varphi(z) - z) = 0$ . By the analyticity,  $\varphi(z) \equiv z$ .  $\square$

In [12], toeplitzness of weighted composition operators on  $H^2$  is considered from another viewpoint.

Due to Feintuch's theorem, we can show the following.

**Theorem 2.2.** *Let  $u$  be a non-zero bounded analytic function on  $\mathbb{D}$  and  $\varphi$  a non-constant analytic self-map of  $\mathbb{D}$ . Then  $M_uC_\varphi$  is uniformly asymptotically Toeplitz if and only if  $M_uC_\varphi$  is compact or  $\varphi$  is the identity.*

*Proof.* Suppose that  $M_uC_\varphi$  is uniformly asymptotically Toeplitz. By Theorem of Feintuch, it holds that  $M_uC_\varphi - T_f = K$ , where  $f \in L^\infty(\partial\mathbb{D})$  and  $K$  is a compact operator. Moreover, assume that  $M_uC_\varphi$  is not compact and  $\varphi$  is not the identity. Let  $K_\lambda(z) = 1/(1 - \bar{\lambda}z)$  for each  $\lambda \in \mathbb{D}$  and  $k_\lambda(z) = \sqrt{1 - |\lambda|^2}K_\lambda(z)$ . By the compactness of  $K$ ,  $\langle (M_uC_\varphi - T_f)^*k_\lambda, k_\lambda \rangle \rightarrow 0$  as  $|\lambda| \rightarrow 1$ .

$$\begin{aligned} \langle (M_uC_\varphi - T_f)^*k_\lambda, k_\lambda \rangle &= (1 - |\lambda|^2) \langle \overline{u(\lambda)}K_{\varphi(\lambda)}, K_\lambda \rangle - \langle \bar{f}k_\lambda, k_\lambda \rangle \\ &= \frac{(1 - |\lambda|^2)\overline{u(\lambda)}}{1 - \overline{\varphi(\lambda)}\lambda} - P[\bar{f}](\lambda), \end{aligned}$$

where  $P[\bar{f}]$  is the Poisson integral of  $\bar{f}$ .

As  $\varphi$  is not the identity,  $\varphi(e^{i\theta}) \neq e^{i\theta}$  a.e. on  $\partial\mathbb{D}$ . Whenever  $\lambda$  tends to  $e^{i\theta}$ , then

$$\langle (M_uC_\varphi - T_f)^*k_\lambda, k_\lambda \rangle \rightarrow -\overline{f(e^{i\theta})} = 0 \quad \text{a.e. on } \partial\mathbb{D}.$$

So  $M_uC_\varphi = K$  and this contradicts that  $M_uC_\varphi$  is not compact.  $\square$

The compactness of  $M_uC_\varphi$  on  $H^2$  is an interesting problem and was characterized in [2, 5] in terms of Carleson measures, but there would not be the function-theoretic characterization. Gunatillake [10] characterized some sufficient conditions for the compactness, assuming the continuity of  $u$  and  $\varphi$ . We here would present the condition independently of the continuity hypothesis, using the similar notion as in [11].

For a non-constant analytic self-map  $\varphi$  of  $\mathbb{D}$ , denote  $\Gamma(\varphi) = \{e^{i\theta} \in \partial\mathbb{D} : |\varphi(e^{i\theta})| = 1\}$ , where we are identifying  $\varphi$  with its boundary function. For each  $r, 0 < r < 1$ , let

$$\{|\varphi| > r\} = \{e^{i\theta} \in \partial\mathbb{D} : |\varphi(e^{i\theta})| > r\}$$

and  $\|f\|_\infty$  the essential supremum norm of a function  $f$  on  $\partial\mathbb{D}$ .

**Theorem 2.3.** *Let  $u$  be a non-zero bounded analytic function on  $\mathbb{D}$  and  $\varphi$  a non-constant analytic self-map of  $\mathbb{D}$ . If  $M_u C_\varphi$  is compact on  $H^2$ , then  $|\varphi| < 1$  a.e. on  $\partial\mathbb{D}$ .*

*Moreover, if  $\|u\chi_{\{|\varphi|>r\}}\|_\infty \rightarrow 0$  as  $r \rightarrow 1$ , then  $M_u C_\varphi$  is compact on  $H^2$ .*

*Proof.* Let  $\{z^n\}$  be an orthogonal basis in  $H^2$ . By the compactness,

$$\|(M_u C_\varphi)z^n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Suppose  $m(\Gamma(\varphi)) > 0$ . Then

$$\begin{aligned} \|(M_u C_\varphi)z^n\|^2 &= \int_{\partial\mathbb{D}} |u(e^{i\theta})|^2 |\varphi(e^{i\theta})|^{2n} dm(\theta) \\ &\geq \int_{\Gamma(\varphi)} |u(e^{i\theta})|^2 dm(\theta). \end{aligned}$$

So we obtain  $u = 0$  on  $\Gamma(\varphi)$ . By the analyticity of  $u$ ,  $u \equiv 0$ . This is a contradiction.

Moreover we assume that  $\|u\chi_{\{|\varphi|>r\}}\|_\infty \rightarrow 0$  as  $r \rightarrow 1$ . For any  $\varepsilon > 0$ , there exists a constant  $\delta$ ,  $0 < \delta < 1$  such that  $\|u\chi_{\{|\varphi|>r\}}\|_\infty < \varepsilon$  for  $\delta < r < 1$ .

Let  $\{f_n\}$  in  $H^2$  with  $\|f_n\| \leq 1$  such that  $f_n$  converges to 0 uniformly on compact subsets of  $\mathbb{D}$ . Then we have

$$\begin{aligned} &\|(M_u C_\varphi)f_n\|^2 \\ &= \int_{\{|\varphi|>r\}} |u(e^{i\theta})f_n(\varphi(e^{i\theta}))|^2 dm(\theta) + \int_{\{|\varphi|\leq r\}} |u(e^{i\theta})f_n(\varphi(e^{i\theta}))|^2 dm(\theta) \\ &\leq \varepsilon^2 \|C_\varphi f_n\|^2 + \sup_{|\varphi(e^{i\theta})|\leq r} |f_n(\varphi(e^{i\theta}))|^2 \|u\|_\infty^2. \end{aligned}$$

Taking  $n \rightarrow \infty$ ,  $\|(M_u C_\varphi)f_n\|^2 \leq \varepsilon^2 \|C_\varphi\|^2$ . As  $\varepsilon$  is arbitrary,  $M_u C_\varphi$  is compact on  $H^2$ .  $\square$

For example,  $u(z) = 1 - z$  and  $\varphi(z) = (1 + z)/2$  satisfy this condition.

Indeed,  $|\varphi(e^{i\theta})| = \left| \cos \frac{\theta}{2} \right|$  and

$$\sup_{\{|\varphi|>r\}} |1 - e^{i\theta}| = \sup_{\{|\varphi|>r\}} 2 \left| \sin \frac{\theta}{2} \right| \leq 2\sqrt{1 - r^2}.$$

As  $r \rightarrow 1$ ,  $\sup_{\{|\varphi|>r\}} |1 - e^{i\theta}| \rightarrow 0$ . So  $M_u C_\varphi$  is compact on  $H^2$ .

Let  $u(z) = \exp\left(\frac{z+1}{z-1}\right)$  be a singular inner function and  $\varphi(z) = (1 + z)/2$  as in [10]. Then  $\|u\chi_{\{|\varphi|>r\}}\|_\infty \not\rightarrow 0$  as  $r \rightarrow 1$ . In fact,  $M_u C_\varphi$  is not compact on  $H^2$ .

Next we consider the strongly asymptotically toepplitzness. If  $M_u C_\varphi$  is compact, then  $M_u C_\varphi$  is uniformly asymptotically Toeplitz and so strongly (weakly) asymptotically Toeplitz.

**Theorem 2.4.** *Let  $u$  be a non-zero bounded analytic function on  $\mathbb{D}$  and  $\varphi$  a non-constant analytic self-map of  $\mathbb{D}$  such that  $M_u C_\varphi$  is not compact. If  $|\varphi| < 1$  a.e. on  $\partial\mathbb{D}$ , then  $M_u C_\varphi$  is strongly (and so weakly) asymptotically Toeplitz with asymptotic symbol zero.*

*Proof.* For  $f \in H^2$ ,

$$\begin{aligned} \|S^{*n}(M_u C_\varphi)S^n f\|^2 &\leq \|u\varphi^n f \circ \varphi\|^2 \\ &= \int_{\partial\mathbb{D}} |u(e^{i\theta})\varphi^n(e^{i\theta})f(\varphi(e^{i\theta}))|^2 dm(\theta) \\ &= \int_{\{|\varphi|<1\}} |u(e^{i\theta})\varphi^n(e^{i\theta})f(\varphi(e^{i\theta}))|^2 dm(\theta). \end{aligned}$$

By the Lebesgue Dominated Convergence Theorem,  $\|S^{*n}(M_u C_\varphi)S^n f\| \rightarrow 0$ . Thus  $M_u C_\varphi$  is strongly asymptotically Toeplitz with asymptotic symbol zero.  $\square$

We could obtain the converse of the theorem above under the hypothesis.

**Theorem 2.5.** *Let  $u$  be a non-zero bounded analytic function on  $\mathbb{D}$  and  $\varphi$  a non-constant analytic self-map of  $\mathbb{D}$  with  $\varphi(z) \not\equiv z$ . Suppose that  $\varphi(0) = 0$ . If  $M_u C_\varphi$  is strongly asymptotically Toeplitz with asymptotic symbol zero, then  $|\varphi| < 1$  a.e. on  $\partial\mathbb{D}$ .*

*Proof.* As  $\varphi(0) = 0$ ,  $\varphi(z) = z\psi(z)$  where  $\psi$  is analytic on  $\mathbb{D}$ . Then

$$S^{*n}(M_u C_\varphi)S^n 1 = S^{*n}(u\varphi^n) = u\psi^n.$$

Suppose  $m(\Gamma(\varphi)) > 0$ . We have

$$\begin{aligned} \|S^{*n}(M_u C_\varphi)S^n 1\|^2 &= \int_{\partial\mathbb{D}} |u(e^{i\theta})\psi^n(e^{i\theta})|^2 dm(\theta) \\ &\geq \int_{\Gamma(\varphi)} |u(e^{i\theta})|^2 dm(\theta). \end{aligned}$$

Thus, since  $\|S^{*n}(M_u C_\varphi)S^n 1\| \rightarrow 0$  as  $n \rightarrow \infty$ ,  $u = 0$  on  $\Gamma(\varphi)$ . By the analyticity of  $u$ ,  $u \equiv 0$ . This is a contradiction.  $\square$

Finally we obtain the criterion for  $M_u C_\varphi$  to be weakly asymptotically Toeplitz.

**Theorem 2.6.** *Let  $u$  be a non-zero bounded analytic function on  $\mathbb{D}$  and  $\varphi$  a non-constant analytic self-map of  $\mathbb{D}$ . If  $M_u C_\varphi$  is weakly asymptotically Toeplitz with asymptotic symbol zero, then  $\varphi$  is not a nontrivial rotation. Furthermore, if  $\varphi$  is not a rotation with  $\varphi(0) = 0$ ,  $M_u C_\varphi$  is weakly asymptotically Toeplitz with asymptotic symbol zero.*

The proof is done by the same way as in [14]. In this case the behavior of the weight  $u$  does not cause the weakly asymptotic toeplitzness.

### 3. Adjoint asymptotic toeplitzness

In this section we consider the adjoint of  $M_u C_\varphi$ . But it is easily checked that the toeplitzness, uniformly asymptotic toeplitzness and weakly asymptotic toeplitzness of  $(M_u C_\varphi)^*$  are ones of  $M_u C_\varphi$ .

We could show the following by the same method as in [14].

**Theorem 3.1.** *Let  $u$  be a non-zero bounded analytic function on  $\mathbb{D}$  and  $\varphi$  a non-constant analytic self-map of  $\mathbb{D}$ . Suppose that  $\varphi(0) = 0$  and  $\varphi$  is not a rotation. Then  $(M_u C_\varphi)^*$  is strongly asymptotically Toeplitz.*

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