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AN ENTIRE FUNCTION SHARING A POLYNOMIAL WITH LINEAR DIFFERENTIAL POLYNOMIALS

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Abstract. The uniqueness problems on entire functions sharing at least two values with their derivatives or linear differential polynomials have been studied and many results on this topic have been obtained. In this paper, we study an entire function $f(z)$ that shares a nonzero polynomial $a(z)$ with $f^{(1)}(z)$, together with its linear differential polynomials of the form: $L = L(f) = a_1(z)f^{(1)}(z)+a_2(z)f^{(2)}(z)+\cdots+a_n(z)f^{(n)}(z)$, where the coefficients $a_k(z)(k = 1, 2, \ldots, n)$ are rational functions and $a_n(z) \not\equiv 0$.

1. Introduction, definitions and results

In the paper, by meromorphic functions we shall always mean meromorphic functions in the complex plane $\mathbb{C}$. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [2]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a non-constant meromorphic function h, we denote by $T(r, h)$ any quantity satisfying $S(r, h) = o\{T(r, h)\}$ as $r \to \infty$ and $r \notin E$.

Let f and g be two nonconstant meromorphic functions and let a be a small function of f. We denote by $E(a; f)$ the set of a-points of f, where each point is counted according its multiplicity. We denote by $\overline{E}(a; f)$ the reduced form of $E(a; f)$. We say that f, g share a CM, provided that $E(a; f) = E(a; g)$, and we say that f and g share a IM, provided that $\overline{E}(a; f) = \overline{E}(a; g)$. In addition, we say that f and g share ∞ CM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM, and we say that f and g share ∞ IM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 IM.

In 1977, L. A. Rubel and C. C. Yang [8] first investigated the uniqueness of entire functions, which share certain values with their derivatives. The following is the result of Rubel and Yang [8].

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Theorem 1.1 ([8]). *Let f be a nonconstant entire function. If $E(a; f) = E(a; f^{(1)})$ and $E(b; f) = E(b; f^{(1)})$ for distinct finite complex numbers a and b, then $f \equiv f^{(1)}$.*

In 1979, E. Mues and N. Steinmetz [7] took up the case of IM shared values in the place of CM shared values and proved the following theorem.

Theorem 1.2 ([7]). *Let f be a nonconstant entire function. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and $\overline{E}(b; f) = \overline{E}(b; f^{(1)})$ for distinct finite complex numbers a and b, then $f \equiv f^{(1)}$.*

Afterwards in 1986 G. Jank, E. Mues and L. Volkman [3] considered the case of a single shared value by the first two derivatives of an entire function. They proved the following result:

Theorem 1.3 ([3]). *Let f be a nonconstant entire function and $a(\neq 0)$ be a finite number. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; f^{(2)})$, then $f \equiv f^{(1)}$.*

In [11] it was observed by the following example that in Theorem 1.3 the second derivative can not be straightway replaced by a higher order derivative.

Example 1.1. Let $(k \geq 3)$ be a positive integer and $w(\neq 1)$ be a root of the algebraic equation $w^{k-1} = 1$. We put $f = e^{wz} + w - 1$, then $E(w; f) = E(w; f^{(1)}) = E(w; f^{(k)})$ but $f \not\equiv f^{(1)}$.

In this context Zhong [11] extended Theorem 1.3 to higher order derivatives and proved the following result.

Theorem 1.4 ([11]). *Let f be a nonconstant entire function and $a(\neq 0)$ be a finite complex number. If f and $f^{(1)}$ share the value a CM and $\overline{E}(a; f) \subset \overline{E}(a; f^{(n)}) \cap \overline{E}(a; f^{(n+1)})$ for $n(\geq 1)$, then $f \equiv f^{(n)}$.*

For $A \subset \mathbb{C} \cup \{\infty\}$, we denote by $N_A(r, a; f)(\overline{N}_A(r, a; f))$ the counting function (reduced counting function) of those a-points of f which belong to A.

In 2011, I. Lahiri and G. K. Ghosh [4] improved Theorem 1.4 in the following manner.

Theorem 1.5 ([4]). *Let f be a nonconstant entire function and a be a nonzero finite number. Suppose that $A = \overline{E}(a; f) \setminus \overline{E}(a; f^{(1)})$ and $B = \overline{E}(a; f^{(1)}) \setminus \{\overline{E}(a; f^{(n)}) \cap \overline{E}(a; f^{(n+1)})\}$ for $n(\geq 1)$. If each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity and $N_A(r, a; f) + N_B(r, a; f^{(1)}) = S(r, f)$, then $f = \lambda e^z$ or $f = \lambda e^z + a$, where $\lambda(\neq 0)$ is a constant.*

In 1999 P. Li [5] extended Theorem 1.4 to linear differential polynomials and proved the following result.

Theorem 1.6 ([5]). *Let f be a nonconstant entire function and $L = L(f) = a_1 f^{(1)}(z) + a_2 f^{(2)}(z) + \cdots + a_n f^{(n)}(z)$ be a linear differential polynomial, where $a_1, a_2, \ldots, a_n(\neq 0)$ are complex numbers. Suppose that a is a nonzero finite value. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; L) \cap \overline{E}(a; L^{(1)})$, then $f \equiv f^{(1)} \equiv L$.*

In the paper we extend Theorem 1.6 by considering shared polynomial instead of shared value also by considering linear differential polynomials with rational coefficients instead of linear differential polynomials with constant coefficients.

For two subsets A and B of $\mathbb{C}$, we denote by $A\Delta B$ the set $(A-B)\cup(B-A)$, which is called the symmetric difference of the sets A and B.

We now state the main result of the paper.

Theorem 1.7. *Let f be a nonconstant entire function and $a=a(z)(\not\equiv 0)$ be a polynomial with* $\deg(a)\neq\deg(f)$. *Suppose that $A=\overline{E}(a;f)\Delta\overline{E}(a;f^{(1)})$, $B=\overline{E}(a;f^{(1)})\setminus\{\overline{E}(a;f^{(2)})\cap\overline{E}(a;L)\cap\overline{E}(a;L^{(1)})\}$, and $L=L(f)=a_1(z)f^{(1)}(z)+a_2(z)f^{(2)}(z)+\cdots+a_n(z)f^{(n)}(z)$ be a linear differential polynomials, where the coefficients $a_k(z)(k=1,2,\ldots,n)$ are rational functions and $a_n(z)\not\equiv 0$. Then $f\equiv f^{(1)}\equiv L$, provided the following hold*:

(i) $E_{1)}(a;f)\subset\overline{E}(a;f^{(1)})$,

(ii) $N_{A\cup B}(r,a;f)+N_A(r,a;f^{(1)})=S(r,f)$, *and*

(iii) *each common zero of $f-a$ and $f^{(1)}-a$ has the same multiplicity.*

Putting $A=B=\emptyset$ we obtain the following corollary.

Corollary 1.1. *Let f be a nonconstant entire function and $a=a(z)(\not\equiv 0)$ be a polynomial with* $\deg(a)\neq\deg(f)$. *If $E(a;f)=E(a;f^{(1)})$ and $\overline{E}(a;f)\subset\overline{E}(a;f^{(2)})\cap\overline{E}(a;L)\cap\overline{E}(a;L^{(1)})$, $n(\geq 1)$, where L is defined in Theorem 1.7, then $f\equiv f^{(1)}\equiv L$.*

2. Lemmas

In this section we need the following lemmas.

Lemma 2.1 ([1]; see also [9])**.** *Let f be a meromorphic function and k be a positive integer. Suppose that f is a solution of the following differential equation*: $a_0w^{(k)}+a_1w^{(k-1)}+\cdots+a_kw=0$, *where $a_0(\neq 0),a_1,a_2,\ldots,a_k$ are constants. Then $T(r,f)=O(r)$. Furthermore, if f is transcendental, then $r=O(T(r,f))$.*

Lemma 2.2 ([1])**.** *Let f be a meromorphic function and n be a positive integer. If there exist meromorphic functions $a_0(\not\equiv 0),a_1,a_2,\ldots,a_n$ such that*

$$a_0f^n+a_1f^{n-1}+\cdots+a_{n-1}f+a_n\equiv 0,$$

then

$$m(r,f)\leq nT(r,a_0)+\sum_{j=1}^{n}m(r,a_j)+(n-1)\log 2.$$

Lemma 2.3 ([6]; see also [10, p. 28])**.** *Let f be a nonconstant meromorphic function. If*

$$R(f)=\frac{a_0f^p+a_1f^{p-1}+\cdots+a_p}{b_0f^q+b_1f^{q-1}+\cdots+b_q}$$

is an irreducible rational function in f with the coefficients being small functions of f and $a_0b_0 \not\equiv 0$, then

$$T(r, R(f)) = \max\{p, q\}T(r, f) + S(r, f).$$

Lemma 2.4. *Let f, $a_0, a_1, a_2, \ldots, a_p, b_0, b_1, b_2, \ldots, b_q$ be meromorphic functions. If*

$$R(f) = \frac{a_0 f^p + a_1 f^{p-1} + \cdots + a_p}{b_0 f^q + b_1 f^{q-1} + \cdots + b_q} \qquad (a_0b_0 \not\equiv 0),$$

then

$$T(r, R(f)) = O(T(r, f) + \sum_{i=0}^{p} T(r, a_i) + \sum_{j=0}^{q} T(r, b_j)).$$

Proof. The lemma follows from the first fundamental theorem and the properties of the characteristic function. □

Lemma 2.5 ([2, p. 68])**.** *Let f be a transcendental meromorphic function and $f^nP(z) = Q(z)$, where $P(z), Q(z)$ are differential polynomials generated by f and the degree of Q is at most n. Then $m(r, P) = S(r, f)$.*

Lemma 2.6 ([2, p. 69])**.** *Let f be a nonconstant meromorphic function and*

$$g(z) = f^n(z) + P_{n-1}(f),$$

where $P_{n-1}(f)$ is a differential polynomial generated by f and of degree at most $n - 1$.

If $N(r, \infty; f) + N(r, 0; g) = S(r, f)$, then $g(z) = h^n(z)$, where $h(z) = f(z) + \frac{a(z)}{n}$ and $h^{n-1}(z)a(z)$ is obtained by substituting $h(z)$ for $f(z)$, $h^{(1)}(z)$ for $f^{(1)}(z)$ etc. in the terms of degree $n - 1$ in $P_{n-1}(f)$.

Let us note the special case, where $P_{n-1}(f) = a_0(z)f^{n-1}+$ terms of degree $n - 2$ at most. Then $h^{n-1}(z)a(z) = a_0(z)h^{n-1}(z)$ and so $a(z) = a_0(z)$. Hence $g(z) = (f(z) + \frac{a_0(z)}{n})^n$.

Lemma 2.7 ([2, p. 47])**.** *Let f be a nonconstant meromorphic function and a_1,a_2,a_3 be three distinct meromorphic functions satisfying $T(r, a_\mu) = S(r, f)$ for $\mu = 1, 2, 3$. Then*

$$T(r, f) \leq \overline{N}(r, 0; f - a_1) + \overline{N}(r, 0; f - a_2) + \overline{N}(r, 0; f - a_3) + S(r, f).$$

3. Proof of the theorem

Proof of Theorem 1.7. **Step 1.** We verify that f cannot be a polynomial. If f is a polynomial, then $T(r, f) = O(\log r)$ and so $N_{A\cup B}(r, a; f) = S(r, f)$ implies that $A = B = \emptyset$. Therefore $\overline{E}(a; f)\Delta\overline{E}(a; f^{(1)}) = \{\overline{E}(a; f) - \overline{E}(a; f^{(1)})\} \cup \{\overline{E}(a; f^{(1)}) - \overline{E}(a; f)\} = \emptyset$ implies that $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$.

Let $\deg(f) = m$ and $\deg(a) = p$. If $m \geq p + 1$, then $\deg(f - a) = m$ and $\deg(f^{(1)} - a) \leq m - 1$. Since $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity, we arrive at a contradiction.

If $m \le p-1$, then $\deg(f-a)=\deg(f^{(1)}-a)=p$. Since $\overline{E}(a;f)=\overline{E}(a;f^{(1)})$ and each common zero of $f-a$ and $f^{(1)}-a$ has the same multiplicity, we have $f^{(1)}-a=k(f-a)$, where $(k\neq 0)$ is a constant. If $k\neq 1$, then $kf-f^{(1)}\equiv (k-1)a$, which is impossible as $\deg(k-1)a=p>m=\deg(kf-f^{(1)})$. If $k=1$, then $f^{(1)}\equiv f$ but f is a polynomial, which is a contradiction. Therefore f is a transcendental entire function.

Step 2. We prove that

$$N_{(2}(r,a;f)=S(r,f). \tag{3.1}$$

From the hypothesis (iii) it can be easily seen that each common zero of $f-a$ and $f^{(1)}-a$ has the same multiplicity. Let z_0 be a zero of $f-a$ and $f^{(1)}-a$ with multiplicity $q(\ge 2)$. Then z_0 is a zero of $f^{(1)}-a^{(1)}$ with multiplicity $q-1$. Hence z_0 is a zero of $a-a^{(1)}=(f^{(1)}-a^{(1)})-(f^{(1)}-a)$ with multiplicity $q-1$. Since $q\le 2(q-1)$, we have $N_{(2}(r,a;f)\le 2N(r,0;a-a^{(1)})+N_A(r,a;f)=O(\log r)+S(r,f)$, but f is a transcendental entire function, so $N_{(2}(r,a;f)=S(r,f)$.

Step 3. We prove that

$$T(r,f)\le 2\overline{N}(r,0;f-a)+S(r,f). \tag{3.2}$$

By the first fundamental theorem we get

$$\begin{aligned}
T(r,f)&=T(r,f-a)+S(r,f)\\
&=T(r,\frac{1}{f-a})+S(r,f)\\
&=N(r,\frac{1}{f-a})+m(r,\frac{1}{f-a})+S(r,f)\\
&\le N(r,\frac{1}{f-a})+m(r,\frac{1}{f^{(1)}-a^{(1)}})+S(r,f)\\
&=N(r,\frac{1}{f-a})+T(r,f^{(1)})-N(r,\frac{1}{f^{(1)}-a^{(1)}})+S(r,f). \qquad (3.3)
\end{aligned}$$

Now by Lemma 2.7 we get

$$T(r,f^{(1)})\le \overline{N}(r,0;f^{(1)}-a)+\overline{N}(r,0;f^{(1)}-a^{(1)})+\overline{N}(r,\infty;f^{(1)}).$$

Then from (3.3) we get

$$\begin{aligned}
T(r,f)\le\; & N(r,0;f-a)+\overline{N}(r,0;f^{(1)}-a)+\overline{N}(r,0;f^{(1)}-a^{(1)})\\
&-N(r,0;f^{(1)}-a^{(1)})+S(r,f).
\end{aligned} \tag{3.4}$$

Let us denote by $N^p_{(k}(r,0;G)$ the counting function of zeros of G with multiplicities not less than k and a zero of multiplicity $q(\ge k)$ is counted $q-p$ times, where $p\le k$.

Now

$$\begin{aligned}
&N(r,0;f-a)+\overline{N}(r,0;f^{(1)}-a^{(1)})-N(r,0;f^{(1)}-a^{(1)})\\
=\;&\overline{N}(r,0;f-a)+N^1_{(2}(r,0;f-a)-N^1_{(2}(r,0;f^{(1)}-a^{(1)})
\end{aligned}$$

$$= \overline{N}(r,0;f-a) + \overline{N}_{(2}(r,0;f-a) + N^2_{(3}(r,0;f-a) - N^1_{(2}(r,0;f^{(1)}-a^{(1)})$$
$$\leq \overline{N}(r,0;f-a) + N^1_{(2}(r,0;f^{(1)}-a^{(1)}) - N^1_{(2}(r,0;f^{(1)}-a^{(1)}) + S(r,f)$$
$$= \overline{N}(r,0;f-a) + S(r,f).$$

Therefore from (3.4) we get

$$T(r,f) \leq \overline{N}(r,0;f-a) + \overline{N}(r,0;f^{(1)}-a) + S(r,f). \tag{3.5}$$

Since

$$\begin{aligned} \overline{N}(r,0;f^{(1)}-a) &\leq \overline{N}(r,0;f-a) + N_A(r,0;f^{(1)}-a) \\ &= \overline{N}(r,0;f-a) + S(r,f). \end{aligned} \tag{3.6}$$

Then from (3.5) and (3.6) we get (3.2).

Now we suppose that $\lambda = \frac{f^{(1)}-a}{f-a}$ and $F = f - a$. Then

$$F^{(1)} = \lambda F + a - a^{(1)} = \lambda_1 F + \mu_1, \tag{3.7}$$

where $\lambda_1 = \lambda$ and $\mu_1 = a - a^{(1)} = b$, say.

Differentiating (3.7) and using (3.7) repeatedly we get

$$F^{(k)} = \lambda_k F + \mu_k, \tag{3.8}$$

where $\lambda_{k+1} = \lambda_k^{(1)} + \lambda_1\lambda_k$ and $\mu_{k+1} = \mu_k^{(1)} + \mu_1\lambda_k$ for $k = 1, 2, \ldots$.

Step 4. Now we shall prove that $T(r,\lambda) = S(r,f)$. If λ is constant, then obviously $T(r,\lambda) = S(r,f)$. So we suppose that λ is nonconstant. From the hypotheses we get

$$\begin{aligned} N(r,0;\lambda) + N(r,\infty;\lambda) &\leq N_A(r,0;f-a) + N_A(r,0;f^{(1)}-a) \\ &= S(r,f). \end{aligned} \tag{3.9}$$

Put $k = 1$ in $\lambda_{k+1} = \lambda_k^{(1)} + \lambda_1\lambda_k$ we get $\lambda_2 = \lambda^2 + d_1\lambda$, where $d_1 = \frac{\lambda^{(1)}}{\lambda}$. Again putting $k = 2$ in $\lambda_{k+1} = \lambda_k^{(1)} + \lambda_1\lambda_k$ we get $\lambda_3 = \lambda_2^{(1)} + \lambda_1\lambda_2$, so $\lambda_3 = \lambda^3 + 3d_1\lambda^2 + d_2\lambda$, where $d_2 = d_1^2 + d_1^{(1)}$. Similarly $\lambda_4 = \lambda_3^{(1)} + \lambda_1\lambda_3 = \lambda^4 + 6d_1\lambda^3 + (6d_1^2 + 3d_1^{(1)} + d_2)\lambda^2 + (d_2^{(1)} + d_1d_2)\lambda$. Therefore, in general, we get for $k \geq 2$

$$\lambda_k = \lambda^k + \sum_{j=1}^{k-1} \alpha_j \lambda^j, \tag{3.10}$$

where $T(r,\alpha_j) = O(\overline{N}(r,0;\lambda) + \overline{N}(r,\infty;\lambda)) + S(r,\lambda) = S(r,f)$ for $j = 1, 2, \ldots, k-1$.

Again put $k = 1$ in $\mu_{k+1} = \mu_k^{(1)} + \mu_1\lambda_k$ we get $\mu_2 = \mu_1^{(1)} + \mu_1\lambda_1 = b\lambda + b^{(1)}$. Also putting $k = 2$ in $\mu_{k+1} = \mu_k^{(1)} + \mu_1\lambda_k$ we obtain by (3.10), $\mu_3 = \mu_2^{(1)} + \mu_1\lambda_2 = b\lambda^{(1)} + b^{(1)}\lambda + b^{(2)} + b(\lambda^2 + d_1\lambda) = b\lambda^2 + (b^{(1)} + 2bd_1)\lambda + b^{(2)}$. Similarly $\mu_4 = b\lambda^3 + (2bd_1 + b^{(1)} + b\alpha_2)\lambda^2 + (b^{(2)} + 2b^{(1)}d_1 + bd_1 + \alpha_1^{(1)} + bd_1^2 + \alpha_1 d^{(1)} + b\alpha_1)\lambda + b^{(3)}$.

Therefore, in general, for $k \geq 2$

$$\mu_k = \sum_{j=1}^{k-1} \beta_j \lambda^j + b^{(k-1)}, \tag{3.11}$$

where $T(r, \beta_j) = O(\overline{N}(r, 0; \lambda) + \overline{N}(r, \infty; \lambda)) + S(r, \lambda) = S(r, f)$ for $j = 1, 2, \ldots,$ $k-1$ and $\beta_{k-1} = b$.

First suppose that either $n \geq 2$ or $n = 1$ and $a_1 \neq 1$. Let

$$\Psi = \frac{(a - L(a))(f^{(1)} - a^{(1)}) - (a - a^{(1)})(L - L(a))}{f - a}. \tag{3.12}$$

By the lemma of logarithmic derivative, we have $m(r, \Psi) = S(r, f)$. It is not difficult to see that the poles of Ψ arise from the poles of $a_k(z)(k = 1, 2, \ldots, n)$, poles of a, and the multiple zero of $f - a$ but any multiple zero of $f - a$ is a zero of $a - a^{(1)}$. Now from (3.12) by (3.1) and by hypotheses we get $N(r, \Psi) \leq N_{(2}(r, a; f) + N_{A \cup B}(r, a; f) + (n+1)N(r, \infty; a) + N(r, \infty; a_k) = S(r, f) + O(\log r)$, and so $T(r, \Psi) = m(r, \Psi) + O(\log r) + S(r, f) = S(r, f)$, because f is a transcendental entire function. Using (3.8), (3.10) and (3.11) we get

$$\begin{aligned} L(F) &= a_1(z)F^{(1)} + \sum_{k=2}^{n} a_k(z)F^{(k)} \\ &= a_1(z)(\lambda F + b) + \sum_{k=2}^{n} a_k(z)(\lambda^k + \sum_{j=1}^{k-1} \alpha_j \lambda^j)F \\ &\quad + \sum_{k=2}^{n} a_k(z)(\sum_{j=1}^{k-1} \beta_j \lambda^j + b^{(k-1)}). \end{aligned}$$

Therefore from (3.12) we get

$$\begin{aligned} &\{\Psi + a_1(z)b\lambda + \sum_{k=2}^{n} a_k(z)b(\lambda^k + \sum_{j=1}^{k-1} \alpha_j \lambda^j) - \lambda(a - L(a))\}F \\ &+ b\{ba_1(z) + \sum_{k=2}^{n} a_k(z)(\sum_{j=1}^{k-1} \beta_j \lambda^j + b^{(k-1)}) - (a - L(a))\} = 0. \end{aligned} \tag{3.13}$$

If $\Psi + a_1(z)b\lambda + \sum_{k=2}^{n} a_k(z)b(\lambda^k + \sum_{j=1}^{k-1} \alpha_j \lambda^j) - \lambda(a - L(a)) \equiv 0$, then by Lemma 2.2 we get $m(r, \lambda) = S(r, f)$. Therefore by (3.9) we have $T(r, \lambda) = S(r, f)$.

Next suppose that $\Psi+a_1(z)b\lambda+\sum\limits_{k=2}^{n}a_k(z)b(\lambda^k+\sum\limits_{j=1}^{k-1}\alpha_j\lambda^j)-\lambda(a-L(a))\not\equiv 0$. Then from (3.13) we get

$$F=-\frac{b\{ba_1(z)+\sum\limits_{k=2}^{n}a_k(z)(\sum\limits_{j=1}^{k-1}\beta_j\lambda^j+b^{(k-1)})-(a-L(a))\}}{\Psi+a_1(z)b\lambda+\sum\limits_{k=2}^{n}a_k(z)b(\lambda^k+\sum\limits_{j=1}^{k-1}\alpha_j\lambda^j)-\lambda(a-L(a))}. \tag{3.14}$$

Then from (3.14) we get by Lemma 2.4, $T(r,F)=O(T(r,\lambda))+S(r,f)$ but $T(r,f)=T(r,F+a)\le T(r,F)+S(r,f)$ also $T(r,F)\le T(r,f)+S(r,f)$, i.e., $T(r,f)=T(r,F)+S(r,f)=O(T(r,\lambda))+S(r,f)$ this implies that $S(r,f)$ is replaceable by $S(r,\lambda)$.

Also from (3.14) we see that F is a rational function in λ, which can be made irreducible. We put

$$F=\frac{P_l(\lambda)}{Q_{l+1}(\lambda)}, \tag{3.15}$$

where $P_l(\lambda)$ and $Q_{l+1}(\lambda)$ are relatively prime polynomials in λ of respective degrees l and $l+1$. Also the coefficients of the both the polynomials are rational functions. Without loss of generality we assume that $Q_{l+1}(\lambda)$ is a monic polynomial. We further note that the counting function of the common zeros of $P_l(\lambda)$ and $Q_{l+1}(\lambda)$, if any, is $S(r,\lambda)$, because $P_l(\lambda)$ and $Q_{l+1}(\lambda)$ are relatively prime and the coefficients are rational functions.

Since $N(r,\infty;F)=O(\log r)=S(r,f)=S(r,\lambda)$, we see from (3.15) that $N(r,0;Q_{l+1}(\lambda))=S(r,\lambda)$. Also by (3.9) we know that $N(r,\infty;\lambda)=S(r,f)=S(r,\lambda)$. So by Lemma 2.6 we get

$$Q_{l+1}(\lambda)=(\lambda+\frac{a_0(z)}{l+1})^{l+1}, \tag{3.16}$$

where $a_0(z)$ is the coefficient of λ^l in $Q_{l+1}(\lambda)$.

If $a_0(z)\not\equiv 0$, then by Lemma 2.7 we obtain

$$\begin{aligned}T(r,\lambda)&\le\overline{N}(r,0;\lambda)+\overline{N}(r,\infty;\lambda)+\overline{N}(r,-\frac{a_0(z)}{l+1};\lambda)+S(r,\lambda)\\&=\overline{N}(r,0;Q_{l+1}(\lambda))+S(r,\lambda)\\&=S(r,\lambda),\end{aligned}$$

a contradiction. Therefore $a_0(z)\equiv 0$ and we get from (3.15) and (3.16)

$$F=\frac{P_l(\lambda)}{\lambda^{l+1}}. \tag{3.17}$$

Differentiating (3.17) we obtain $F^{(1)}=d_1\frac{\lambda P_l^{(1)}(\lambda)-(l+1)P_l(\lambda)}{\lambda^{l+1}}$, where $d_1=\frac{\lambda^{(1)}}{\lambda}$ and $T(r,d_1)=O(\overline{N}(r,0;\lambda)+\overline{N}(r,\infty;\lambda))+m(r,d_1)=S(r,f)+S(r,\lambda)=$

$S(r,\lambda)$. So by Lemma 2.3 we have

$$T(r,F^{(1)}) = (l+1-p)T(r,\lambda) + S(r,\lambda) \tag{3.18}$$

for some integer p, $0 \le p \le l$.

Again since $F^{(1)} = \lambda F + b$, where $b = a - a^{(1)} \not\equiv 0$, we get from (3.17) $F^{(1)} = \frac{P_l(\lambda)}{\lambda^l} + b$ and so by Lemma 2.3 we have

$$T(r,F^{(1)}) = (l-p)T(r,\lambda) + S(r,\lambda), \tag{3.19}$$

where p is same as in (3.18). Now from (3.18) and (3.19) we get $T(r,\lambda) = S(r,\lambda)$, a contradiction.

Next we suppose that $n = 1$ and $a_1(z) \equiv 1$. Then we consider

$$\Phi = \frac{(a - L^{(1)}(a))(a - L(a)) - (a - L(a))(L^{(1)} - L^{(1)}(a))}{f - a}.$$

Since in this case $L = f^{(1)}$, we get

$$\begin{aligned}\Phi &= \frac{(a - a^{(2)})(f^{(1)} - a^{(1)}) - (a - a^{(1)})(f^{(2)} - a^{(2)})}{f - a}\\ &= \frac{(a - a^{(2)})F^{(1)} - bF^{(2)}}{F}. \qquad (3.20)\end{aligned}$$

By the hypothesis we get $T(r,\Phi) = S(r,f)$. Using (3.8), (3.10), (3.11) and (3.20) we get

$$\{\Phi + b\lambda^2 + (\alpha_1 b - a + a^{(2)})\lambda\}F + b(a^{(2)} - a + \beta_1\lambda + b^{(1)}) \equiv 0. \tag{3.21}$$

Following the similar argument of the preceding case and using (3.21) we can show that $m(r,\lambda) = S(r,f)$. So by (3.9) we have $T(r,\lambda) = S(r,f)$.

Step 5. Since $T(r,\lambda) = S(r,f)$, we see that $T(r,\lambda_k) + T(r,\mu_k) = S(r,f)$ for $k = 1, 2, \ldots$, where λ_k and μ_k are defined in (3.8). Now

$$\begin{aligned}L &= \sum_{k=1}^{n} a_k f^{(k)} = \sum_{k=1}^{n} a_k F^{(k)} + L(a)\\ &= (\sum_{k=1}^{n} a_k\lambda_k)F + \sum_{k=1}^{n} a_k\mu_k + L(a) = \xi F + \eta, \text{ say.} \qquad (3.22)\end{aligned}$$

But since $a_k(z)(k = 1, 2, \ldots, n)$ are rational functions and f is a transcendental entire function so $T(r,\xi) + T(r,\eta) = O(\log r) = S(r,f)$. Differentiating (3.22) we get

$$L^{(1)} = \xi^{(1)}F + \xi F^{(1)} + \eta^{(1)}. \tag{3.23}$$

Let z_1 be a zero of $F = f - a$ such that $z_1 \notin A \cup B$. Then from (3.22) and (3.23) we get $a(z_1) - \eta(z_1) = 0$ and $\xi(z_1)(a(z_1) - a^{(1)}(z_1)) + \eta^{(1)}(z_1) - a(z_1) = 0$.

If $a(z) - \eta(z) \not\equiv 0$, we get

$$\begin{aligned}\overline{N}(r,0;f-a) &\le N_{A\cup B}(r,0;f-a) + N(r,0;a-\eta) + S(r,f)\\ &= S(r,f),\end{aligned}$$

which contradicts (3.2).

Therefore

$$a(z) \equiv \eta(z). \tag{3.24}$$

Again if $\xi(z)(a(z) - a^{(1)}(z)) + \eta^{(1)}(z) - a(z) \not\equiv 0$, we get

$$\begin{aligned}\overline{N}(r, 0; f - a) \le{} & N_{A\cup B}(r, 0; f - a) \\ & + N(r, 0; \xi(z)(a(z) - a^{(1)}(z)) + \eta^{(1)}(z) - a(z)) + S(r, f) \\ ={} & S(r, f),\end{aligned}$$

which contradicts (3.2).

Therefore

$$\xi(z)(a(z) - a^{(1)}(z)) + \eta^{(1)}(z) - a(z) \equiv 0. \tag{3.25}$$

Since a is a polynomial so $a(z) \not\equiv a^{(1)}(z)$, from (3.24) and (3.25) we get $\xi(z) \equiv 1$. Hence from (3.22) and (3.24) we get $L \equiv F + a \equiv f$.

Step 6. Set

$$\tau = \frac{(a - a^{(1)})(f^{(2)} - a^{(2)}) - (a - a^{(2)})(f^{(1)} - a^{(1)})}{f - a}. \tag{3.26}$$

By the lemma of logarithmic derivative, we have $m(r, \tau) = S(r, f)$ and also $N(r, \tau) \le N_{(2}(r, a; f) + N_{A\cup B}(r, a; f) + N(r, \infty; a) = O(\log r)$. Since f is a transcendental entire function so, $T(r, \tau) = S(r, f)$. From (3.7) we get

$$F^{(1)} = b + \lambda F. \tag{3.27}$$

Differentiating (3.27) and using (3.27) we get

$$F^{(2)} = b^{(1)} + \lambda^{(1)} F + \lambda F^{(1)} = b^{(1)} + \lambda b + (\lambda^{(1)} + \lambda^2)F. \tag{3.28}$$

Now we rewrite (3.26) in the following form

$$\tau = \frac{bF^{(2)} - (b + b^{(1)})F^{(1)}}{F}. \tag{3.29}$$

By (3.29) we have

$$bF^{(2)} - (b + b^{(1)})F^{(1)} - \tau F = 0. \tag{3.30}$$

Then by (3.27), (3.28) and (3.30) we have

$$\{(\lambda^{(1)} + \lambda^2)b - (b + b^{(1)})\lambda - \tau\}F = b^2(1 - \lambda). \tag{3.31}$$

If $(\lambda^{(1)} + \lambda^2)b - (b + b^{(1)})\lambda - \tau \not\equiv 0$, then from (3.31) we get

$$F = -\frac{b^2\lambda - b^2}{b\lambda^2 + bd_1\lambda - (b + b^{(1)})\lambda - \tau}, \tag{3.32}$$

where $d_1 = \frac{\lambda^{(1)}}{\lambda}$ and $T(r, d_1) = O(\overline{N}(r, 0; \lambda) + \overline{N}(r, \infty; \lambda)) + m(r, d_1) = S(r, f) + S(r, \lambda) = S(r, \lambda)$. From (3.32) we see that F is a rational function in λ, which can be made irreducible. Following the similar argument of Step 4 and using (3.32) we get $T(r, \lambda) = S(r, \lambda)$, which is a contradiction. Therefore

$(\lambda^{(1)} + \lambda^2)b - (b + b^{(1)})\lambda - \tau \equiv 0$. Now since $(\lambda^{(1)} + \lambda^2)b - (b + b^{(1)})\lambda - \tau \equiv 0$ and $b \not\equiv 0$ then by (3.31) we deduce that $\lambda \equiv 1$, but $\lambda = \frac{f^{(1)}-a}{f-a}$, hence we get $f \equiv f^{(1)}$. This completes the proof of the theorem. $\square$

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