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RELATIVE MULTIFRACTAL SPECTRUM

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Abstract. We obtain a relation between generalized Hausdorff and packing multifractal premeasures and generalized Hausdorff and packing multifractal measures. As an application, we study a general formalism for the multifractal analysis of one probability measure with respect to an other.

1. Introduction

Multifractal theory was first introduced by Mandelbrot in [11, 12] as a description of measure arising in turbulance. Given a finite measure μ on $\mathbb{R}^n$, $n \geq 1$, we define the local dimension or the pointwise Hölder exponent of μ at x, when the limit exists, by

$$\alpha_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B_x(r))}{\log r},$$

where $B_x(r)$ denote the closed ball of center x and radius r.

The level set of the local dimension of μ contains crucial information on the geometrical properties of μ. The aim of multifractal analysis of a measure is to relate the Hausdorff and packing dimensions of these levels sets to the Legendre transform of some concave function [1, 2, 6, 13].

Cole introduced in [8] a general formalism for the multifractal analysis of one probability measure μ with respect to an other measure ν. More specifically, he calculated, for $\alpha \geq 0$, the size of the set

$$E(\alpha) = \Big\{ x \in \operatorname{supp}\mu \cap \operatorname{supp}\nu; \ \ \lim_{r\to 0} \frac{\log \mu(B_x(r))}{\log \nu(B_x(r))} = \alpha \Big\},$$

where $\operatorname{supp}\mu$ is the topologic support of μ. These sets were first introduced by Billingsley in [5] and studied in the setting of symbolic dynamics by Cajar in [7]. In several recent papers many authors have begun to discuss the idea of performing multifractal analysis with respect to an arbitrary reference measure [3, 9, 10, 14]. The special case when ν is the Lebesgue measure was studied by Olsen in [13] and he computes the Hausdorff and packing dimensions of

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$E(\alpha)$. Later, Ben Nasr, Bhouri and Heurteaux in [4] developed a necessary and sufficient condition for the validity of the multifractal formalism.

In this paper, we obtain a relation between generalized Hausdorff (resp. packing) multifractal premeasure $\overline{\mathcal{H}}^{q,t}_{\mu,\nu}$ (resp. $\overline{\mathcal{P}}^{q,t}_{\mu,\nu}$) and generalized Hausdorff (resp. packing) multifractal measure $\mathcal{H}^{q,t}_{\mu,\nu}$ (resp. $\mathcal{P}^{q,t}_{\mu,\nu}$). In particular, we give a sufficient condition about the validity of the multifractal formalism which extends the result of the sufficient condition in [8].

2. Preliminaries

2.1. Generalized packing and Hausdorff measures

Fix an integer $n \geq 1$ and denote by $\mathcal{P}(\mathbb{R}^n)$ the family of Borel probability measures on $\mathbb{R}^n$. We define, for $q \in \mathbb{R}$, the function $\varphi_q : [0,+\infty) \to [0,+\infty]$ by

$$\varphi_q(x) = \begin{cases} \left.\begin{array}{lll} \infty & \text{for} & x=0 \\ x^q & \text{for} & x>0 \end{array}\right\} & \text{for} \quad q<0, \\ 1 & \text{for} \quad q=0, \\ \left.\begin{array}{lll} 0 & \text{for} & x=0 \\ x^q & \text{for} & x>0 \end{array}\right\} & \text{for} \quad q>0. \end{cases}$$

Consider two measures μ and ν of $\mathcal{P}(\mathbb{R}^n)$ and two real numbers q and t. We suppose that $S_{\mu,\nu} = \operatorname{supp}\mu \cap \operatorname{supp}\nu \neq \emptyset$. For any subset E of $S_{\mu,\nu}$, we define

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu,\delta}(E) = \begin{cases} \sup \ \sum_i \varphi_q\big(\mu(B_{x_i}(r_i))\big)\varphi_t\big(\nu(B_{x_i}(r_i))\big) & \text{if} \quad E \neq \emptyset, \\ 0 & \text{if} \quad E = \emptyset, \end{cases}$$

where the supremum is taken over all centered δ-packing of E. We also define

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E) = \inf_{\delta>0} \overline{\mathcal{P}}^{q,t}_{\mu,\nu,\delta}(E) \quad \text{and} \quad \mathcal{P}^{q,t}_{\mu,\nu}(E) = \inf_{E\subset\bigcup_i E_i} \sum_i \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E_i).$$

$\mathcal{P}^{q,t}_{\mu,\nu}$ is called the generalized packing measure relatively to μ and ν. In a similar way we define

$$\overline{\mathcal{H}}^{q,t}_{\mu,\nu,\delta}(E) = \begin{cases} \inf \sum_i \varphi_q\big(\mu(B_{x_i}(r_i))\big)\varphi_t\big(\nu(B_{x_i}(r_i))\big) & \text{if} \quad E \neq \emptyset, \\ 0 & \text{if} \quad E = \emptyset, \end{cases}$$

where the infinimum is taken over all centered δ-covering of E. Also define

$$\overline{\mathcal{H}}^{q,t}_{\mu,\nu}(E) = \sup_{\delta>0} \overline{\mathcal{H}}^{q,t}_{\mu,\nu,\delta}(E) \quad \text{and} \quad \mathcal{H}^{q,t}_{\mu,\nu}(E) = \sup_{F\subseteq E} \overline{\mathcal{H}}^{q,t}_{\mu,\nu}(F).$$

$\mathcal{H}^{q,t}_{\mu,\nu}$ is called the generalized Hausdorff measure relatively to μ and ν.

The functions $\mathcal{H}^{q,t}_{\mu,\nu}$ and $\mathcal{P}^{q,t}_{\mu,\nu}$ are metric outer measures and are, thus, measures on the Borel family of subsets of $\mathbb{R}^n$. An important feature of the Hausdorff and packing measures is that $\mathcal{P}^{q,t}_{\mu,\nu} \leq \overline{\mathcal{P}}^{q,t}_{\mu,\nu}$ and there exists an integer $\xi \in \mathbb{N}$, such that $\mathcal{H}^{q,t}_{\mu,\nu} \leq \xi\mathcal{P}^{q,t}_{\mu,\nu}$. For more details about these measures, the reader can see [8].

As with generalized Hausdorff and packing measures, we can define, for any subset E of $S_{\mu,\nu}$ and any real q,

$$\overline{\dim}^q_{\mu,\nu}(E) = \sup\Big\{t,\ \overline{\mathcal{H}}^{q,t}_{\mu,\nu}(E)=\infty\Big\} = \inf\Big\{t,\ \overline{\mathcal{H}}^{q,t}_{\mu,\nu}(E)=0\Big\},$$

$$\dim^q_{\mu,\nu}(E) = \sup\Big\{t,\ \mathcal{H}^{q,t}_{\mu,\nu}(E)=\infty\Big\} = \inf\Big\{t,\ \mathcal{H}^{q,t}_{\mu,\nu}(E)=0\Big\},$$

$$\mathrm{Dim}^q_{\mu,\nu}(E) = \sup\Big\{t,\ \mathcal{P}^{q,t}_{\mu,\nu}(E)=\infty\Big\} = \inf\Big\{t,\ \mathcal{P}^{q,t}_{\mu,\nu}(E)=0\Big\},$$

$$\Delta^q_{\mu,\nu}(E) = \sup\Big\{t,\ \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E)=\infty\Big\} = \inf\Big\{t,\ \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E)=0\Big\}.$$

Coming back to the definition, we can see obviously, for $t>0$, that

$$\overline{\mathcal{H}}^{0,t}_{\mu,\nu} = \overline{\mathcal{H}}^{t}_{\nu},\quad \mathcal{H}^{0,t}_{\mu,\nu} = \mathcal{H}^{t}_{\nu},\quad \mathcal{P}^{0,t}_{\mu,\nu} = \mathcal{P}^{t}_{\nu}\quad\text{and}\quad \overline{\mathcal{P}}^{0,t}_{\mu,\nu} = \overline{\mathcal{P}}^{t}_{\nu}.$$

Hence, we denote ν-pre-Hausdorff, ν-Hausdorff, ν-packing and ν-pre-packing dimension by $\overline{\dim}_\nu, \dim_\nu, \mathrm{Dim}_\nu$ and Δ_ν respectively, then, for $E\subset S_{\mu,\nu}$, we have

$$\overline{\dim}_\nu(E) = \overline{\dim}^0_{\mu,\nu}(E),\quad \dim_\nu(E) = \dim^0_{\mu,\nu}(E)$$

and

$$\mathrm{Dim}_\nu(E) = \mathrm{Dim}^0_{\mu,\nu}(E),\quad \Delta_\nu(E) = \dim^0_{\mu,\nu}(E).$$

We can see immediately that the dimensions defined above satisfy

$$\overline{\dim}^q_{\mu,\nu}(E) \le \dim^q_{\mu,\nu}(E) \le \mathrm{Dim}^q_{\mu,\nu}(E) \le \Delta^q_{\mu,\nu}(E).$$

Next, we define the multifractal functions

$$\Theta_{\mu,\nu}(q) = \overline{\dim}^q_{\mu,\nu}(S_{\mu,\nu}),$$

$$b_{\mu,\nu}(q) = \dim^q_{\mu,\nu}(S_{\mu,\nu}),$$

$$B_{\mu,\nu}(q) = \mathrm{Dim}^q_{\mu,\nu}(S_{\mu,\nu}),$$

$$\Lambda_{\mu,\nu}(q) = \Delta^q_{\mu,\nu}(S_{\mu,\nu}).$$

For $\mu\in\mathcal{P}(\mathbb{R}^n)$ and $a>0$, write

$$P_a(\mu) = \limsup_{r\searrow 0}\ \sup_{x\in\operatorname{supp}\mu}\frac{\mu(B_x(ar))}{\mu(B_x(r))}\quad\text{and}\quad d_\mu(a) = \liminf_{r\to 0}\ \inf_{x\in\operatorname{supp}\mu}\frac{\mu(B_x(ar))}{\mu(B_x(r))}.$$

We recall that in [13], it was proved that

$$\Big(P_a(\mu)<\infty\ \text{ for some }\ a>1\Big)\quad\text{if and only if}\quad\Big(P_a(\mu)<\infty\ \text{ for all }\ a>1\Big).$$

Also, define the family $\mathcal{P}_D(\mathbb{R}^n)$ of doubling probability measures on $\mathbb{R}^n$, by

$$\mathcal{P}_D(\mathbb{R}^n) = \big\{\mu\in\mathcal{P}(\mathbb{R}^n)\,|\,P_a(\mu)<\infty\quad\text{for some }\ a>1\big\}.$$

Obviously, the set $\mathcal{P}_D(\mathbb{R}^n)$ is independent of a and we have (see [15]) that

$$\mu\in\mathcal{P}_D(\mathbb{R}^n)\ \text{ if and only if }\quad d_\mu(1^-) = \lim_{a\to 1^-} d_\mu(a) > 0.$$

Finally, if $f:\mathbb{R}\to\mathbb{R}$ is a real-valued function, let $f^*:\mathbb{R}\to[-\infty,+\infty]$ denote the following Legendre transform of

$$f^*(x)=\inf_{x\in\mathbb{R}}\big(xy+f(y)\big).$$

2.2. Relative multifractal analysis

Let us define, for μ and $\nu\in\mathcal{P}(\mathbb{R}^n)$,

$$\underline{a}_{\mu,\nu}=\sup_{q>0}-\frac{b_{\mu,\nu}(q)}{q};\quad \overline{a}_{\mu,\nu}=\inf_{q<0}-\frac{b_{\mu,\nu}(q)}{q}.$$

Recall the level set $E(\alpha)$ introduced in the introduction. Cole in [8] proved the upper bound of generalizes Hausdorff and packing dimension of this set. More precisely he get the following result.

Theorem 1. *Let $\mu,\nu\in\mathcal{P}(\mathbb{R}^n)$ and $\alpha\geq 0$.*

(1) *If $\alpha\in(\underline{a}_{\mu,\nu},\overline{a}_{\mu,\nu})$, then*

$$\dim_\nu(E(\alpha))\leq b^*_{\mu,\nu}(\alpha)\quad\textit{and}\quad \mathrm{Dim}_\nu(E(\alpha))\leq B^*_{\mu,\nu}(\alpha).$$

(2) *If $\alpha\in\mathbb{R}^*_+\backslash[\underline{a}_{\mu,\nu},\overline{a}_{\mu,\nu}]$, then $\dim_\nu(E(\alpha))=\mathrm{Dim}_\nu(E(\alpha))=0$.*

3. Relations of multifractals measures

Let μ,ν in $\mathcal{P}(\mathbb{R}^n)$ and q,t in $\mathbb{R}$. Without loss of generality, we suppose that $S_{\mu,\nu}\neq\emptyset$. In general case, we only know that, for all set E

$$\overline{\mathcal{H}}^{q,t}_{\mu,\nu}(E)\leq\mathcal{H}^{q,t}_{\mu,\nu}(E)\quad\text{and}\quad\mathcal{P}^{q,t}_{\mu,\nu}(E)\leq\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E).$$

In this section, we are interested in the others inequalities. This result will be used to obtain a relative multifractal formalism which will be discussed in the next section.

Theorem 2. *Let $\mu,\nu\in\mathcal{P}_D(\mathbb{R}^n)$. Then, for all $E\subset\mathbb{R}^n$, for all $q,t\in\mathbb{R}$, there exists a constant $c>0$ which depends on q and t such that*

$$c\mathcal{H}^{q,t}_{\mu,\nu}(E)\leq\overline{\mathcal{H}}^{q,t}_{\mu,\nu}(E)\leq\mathcal{H}^{q,t}_{\mu,\nu}(E).$$

Proof. Let $\delta>0$, $F\subset E$ and $\Omega=\{B(x_i,r_i)\}_i$ is a centered δ-covering of E. We set

$$\Omega'=\{B_{x_i}(r_i);\ \ B_{x_i}(r_i)\in\Omega\ \text{ and }\ B_{x_i}(r_i)\cap F\neq\emptyset\}.$$

For all $B_{x_i}(r_i)\in\Omega'$, let $y_i\in B_{x_i}(r_i)\cap F$. Then, $B_{x_i}(r_i)\subset B_{y_i}(2r_i)$ and $\Lambda=\big\{B_{y_i}(2r_i)\big\}$ is a 2δ-covering of F.

(1) If $q\leq 0$ and $t\leq 0$, then

$$\begin{aligned}\sum_{B_{x_i}(r_i)\in\Omega}\mu\big(B_{x_i}(r_i)\big)^q\nu\big(B_{x_i}(r_i)\big)^t&\geq\sum_{B_{x_i}(r_i)\in\Omega'}\mu\big(B_{x_i}(r_i)\big)^q\nu\big(B_{x_i}(r_i)\big)^t\\&\geq\sum_{B_{y_i}(2r_i)\in\Lambda}\mu\big(B_{y_i}(2r_i)\big)^q\nu\big(B_{y_i}(2r_i)\big)^t\end{aligned}$$

when we have used the fact that

$$\mu\big(B_{x_i}(r_i)\big)^q \geq \mu\big(B_{y_i}(2r_i)\big)^q \quad \text{and} \quad \nu\big(B_{x_i}(r_i)\big)^t \geq \nu\big(B_{y_i}(2r_i)\big)^t.$$

Hence

$$\overline{\mathcal{H}}^{q,t}_{\mu,\nu,\delta}(E) \geq \overline{\mathcal{H}}^{q,t}_{\mu,\nu,2\delta}(F).$$

Letting $\delta \to 0$ we get

$$\overline{\mathcal{H}}^{q,t}_{\mu,\nu}(E) \geq \overline{\mathcal{H}}^{q,t}_{\mu,\nu}(F),$$

and we conclude since F is arbitrary.

(2) If $q > 0$, $t > 0$ and $\mu, \nu \in \mathcal{P}_D(\mathbb{R}^n)$, then

$$\begin{aligned} &\sum_{B_{x_i}(r_i)\in\Omega} \mu\big(B_{x_i}(r_i)\big)^q \nu\big(B_{x_i}(r_i)\big)^t \\ \geq &\sum_{B_{x_i}(r_i)\in\Omega'} \mu\big(B_{x_i}(r_i)\big)^q \nu\big(B_{x_i}(r_i)\big)^t \\ \geq\ & c_1^{-2q} c_2^{-2t} \sum_{B_{x_i}(r_i)\in\Omega'} \mu\big(B_{x_i}(4r_i)\big)^q \nu\big(B_{x_i}(4r_i)\big)^t \\ \geq\ & c_1^{-2q} c_2^{-2t} \sum_{B_{y_i}(2r_i)\in\Lambda} \mu\big(B_{y_i}(2r_i)\big)^q \nu\big(B_{y_i}(2r_i)\big)^t \end{aligned}$$

when we have used the fact that

$$c_1^{2q}\mu\big(B_{x_i}(r_i)\big)^q \geq \mu\big(B_{x_i}(4r_i)\big)^q \quad \text{and} \quad c_2^{2t}\nu\big(B_{x_i}(r_i)\big)^t \geq \nu\big(B_{x_i}(4r_i)\big)^t.$$

Hence

$$\overline{\mathcal{H}}^{q,t}_{\mu,\nu,\delta}(E) \geq c_1^{-2q} c_2^{-2t} \overline{\mathcal{H}}^{q,t}_{\mu,\nu,2\delta}(F).$$

Letting $\delta \to 0$ we get

$$\overline{\mathcal{H}}^{q,t}_{\mu,\nu}(E) \geq c_1^{-2q} c_2^{-2t} \overline{\mathcal{H}}^{q,t}_{\mu,\nu}(F),$$

and we conclude since F is arbitrary.

(3) If $q > 0$, $t \leq 0$ and $\mu \in \mathcal{P}_D(\mathbb{R}^n)$, then the proof is similar and we get $c = c_1^{-2q}$.

(4) If $q \leq 0$, $t > 0$ and $\nu \in \mathcal{P}_D(\mathbb{R}^n)$, then the proof is similar and we get $c = c_2^{-2t}$. □

Similarly, we will give a relation between generalized packing multifractal premeasure $\overline{\mathcal{P}}^{q,t}_{\mu,\nu}$ and generalized packing multifractal measure $\mathcal{P}^{q,t}_{\mu,\nu}$. First we start with the following result.

Proposition 1. *Let $\overline{E}$ be the closure of $E \subset S_{\mu,\nu}$. Then*

1. *for $q \leq 0$ and $t \leq 0$, we have $\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E) = \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(\overline{E})$,*
2. *for $q \geq 0$ and $t \geq 0$, we have $d_\mu(1^-)^q d_\nu(1^-)^t \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(\overline{E}) \leq \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E) \leq \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(\overline{E})$,*
3. *for $q \leq 0$ and $t \geq 0$, we have $d_\nu(1^-)^t \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(\overline{E}) \leq \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E) \leq \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(\overline{E})$,*

(4) *for* $q \geq 0$ *and* $t \leq 0$*, we have* $d_\mu(1^-)^q \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(\overline{E}) \leq \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E) \leq \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(\overline{E})$.

Proof. Obviously, for all $E \subset S_{\mu,\nu}$ and $q, t \in \mathbb{R}$, we have $\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E) \leq \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(\overline{E})$. Fix $\delta > 0$ and $\eta \in (0,1)$. Let $\{B_{x_i}(r_i)\}_i$ be a centred δ-packing of $\overline{E}$. Then, there exists $\{B_{y_i}((1-\eta)r_i)\}_i$ a centred δ-packing of E such that

$$B_{y_i}((1-\eta)r_i) \subset B_{x_i}(r_i) \subset B_{y_i}((1+\eta)r_i). \tag{3.1}$$

From the definition of $\overline{\mathcal{P}}^{q,t}_{\mu,\nu,\delta}$, we have

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu,\delta}(E) \geq \sum_i \mu\Big(B_{y_i}((1-\eta)r_i)\Big)^q \nu\Big(B_{y_i}((1-\eta)r_i)\Big)^t.$$

(1) If $q \leq 0$ and $t \leq 0$ we have

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu,\delta}(E) \geq \sum_i \mu\Big(B_{x_i}(r_i)\Big)^q \nu\Big(B_{x_i}(r_i)\Big)^t,$$

which yields $\overline{\mathcal{P}}^{q,t}_{\mu,\nu,\delta}(E) \geq \overline{\mathcal{P}}^{q,t}_{\mu,\nu,\delta}(\overline{E})$ and so $\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E) \geq \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(\overline{E})$.

(2) If $q \geq 0$ and $t \geq 0$ we have

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu,\delta}(E) \geq \sum_i \mu\Big(B_{y_i}((1-\eta))r_i)\Big)^q \nu\Big(B_{y_i}((1-\eta)r_i)\Big)^t.$$

Notice that, from (3.1), for each i, we have

$$\begin{aligned}\mu\big(B_{y_i}((1-\eta)r_i)\big) &= \frac{\mu\big(B_{y_i}((1-\eta)r_i)\big)}{\mu\big(B_{y_i}((1+\eta)r_i)\big)}\mu\big(B_{y_i}((1+\eta)r_i)\big)\\ &\geq \Big(\inf_{0<r\leq\delta}\ \inf_{y\in\operatorname{supp}\mu}\ \frac{\mu\big(B_{y_i}((1-\eta)r_i)\big)}{\mu\big(B_{y_i}((1+\eta)r_i)\big)}\Big)\mu\big(B_{x_i}(r_i)\big).\end{aligned}$$

Similarly, we have

$$\nu\big(B_{y_i}((1-\eta)r_i)\big) \geq \Big(\inf_{0<r\leq\delta}\ \inf_{y\in\operatorname{supp}\nu}\ \frac{\nu\big(B_{y_i}((1-\eta)r_i)\big)}{\nu\big(B_{y_i}((1+\eta)r_i)\big)}\Big)\nu\big(B_{x_i}(r_i)\big).$$

Which yields, by letting $\delta \to 0$ and $\eta \to 0$,

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu,\delta}(E) \geq d_\mu(1^-)^q d_\nu(1^-)^t \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(\overline{E}).$$

The other cases are similar. ☐

Corollary 1. *Let* $\mu, \nu \in \mathcal{P}_D(\mathbb{R}^n)$ *and* $q \in \mathbb{R}$*. Then, for all subset* $E \subset \mathbb{R}^n$ *we have*

$$\overline{\dim}^q_{\mu,\nu}(E) = \dim^q_{\mu,\nu}(E).$$

In particular, we get $\Theta_{\mu,\nu}(q) = b_{\mu,\nu}(q)$*.*

Theorem 3. *Let* E *be a compact subset of* $S_{\mu,\nu}$ *such that* $\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E) < \infty$*.*

(1) *For* $q \leq 0$, *we have*

$$\mathcal{P}^{q,t}_{\mu,\nu}(E) \geq \begin{cases} \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E) & \text{if} \quad t \leq 0, \\ d_\nu(1^-)^{2t}\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E) & \text{if} \quad t > 0 \quad \text{and} \quad \nu \in \mathcal{P}_D(E). \end{cases}$$

(2) *For* $q > 0$, *we have*

$$\mathcal{P}^{q,t}_{\mu,\nu}(E) \geq \begin{cases} d_\nu(1^-)^{2t}\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E) & \text{if} \quad t \leq 0 \quad \text{and} \quad \mu \in \mathcal{P}_D(E), \\ d_\mu(1^-)^{2q}d_\nu(1^-)^{2t}\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E) & \text{if} \quad t > 0 \quad \text{and} \quad \mu, \nu \in \mathcal{P}_D(E). \end{cases}$$

Proof. For $\epsilon > 0$ and F is a compact subset of E, let F_ϵ be the open ϵ-neighborhood of F. Obviously we have

$$a := \inf_{\epsilon > 0} \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(F_\epsilon \cap E) < \infty.$$

Let us announced this two lemmas, the first one can be found in [15] and the second will be proved in the end of this section.

Lemma 1. *For* $w > 0$, *there exist* $\epsilon, \delta \in \mathbb{R}^*_+, p \in \mathbb{N}$ *and* $\big\{B_{x_i}(r_i)\big\}_{i=1}^p$ *a* δ-*packing of* $F_\epsilon \cap E$ *such that*

$$a - w \leq \sum_{i=1}^{p} \mu\big(B_{x_i}(r_i)\big)^q \nu\big(B_{x_i}(r_i)\big)^t \leq \overline{\mathcal{P}}^{q,t}_{\mu,\nu,\delta}(F_\epsilon \cap E) \leq a + 2w. \tag{3.2}$$

Moreover there exist, for all $i \in \{1, \ldots, p\}$, $y_i \in F$ *and* $r_i', r_i'' \geq 0$ *such that*

$$r_i' + r_i'' = r_i \quad \text{and} \quad \Big\{B_{y_i}(r_i'),\ r_i' > 0\Big\} \text{ is a } \delta\text{-packing of } F.$$

In addition, there exists a constant $c(q,t) \in \mathbb{R}_+$,

$$\sum_{i, r_i'' > 0} \mu\Big(B_{x_i}(r_i'')\Big)^q \nu\Big(B_{x_i}(r_i'')\Big)^t \leq c(q,t)w. \tag{3.3}$$

Lemma 2.

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(F) \geq d_\mu(1^-)^q a, \quad (t < 0), \tag{3.4}$$

and

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(F) \geq d_\mu(1^-)^q d_\nu(1^-)^t a, \quad (t > 0,\ \ \nu \in \mathcal{P}_D(E)). \tag{3.5}$$

Now we will give the proof of Theorem 3 for $q > 0$, $t > 0$ and $\mu, \nu \in \mathcal{P}_D(E)$. The others cases are similar.

Let $(F_i)_i$ be any sequence of subsets of E such that $E \subset \cup_i F_i$. Let w be given arbitrarily. By inequality (3.5), for each i there exists an open set θ_i with $\overline{F}_i \subset \theta_i$ such that

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(\overline{F}_i) \geq d_\mu(1^{-1})^q d_\nu(1^{-1})^t \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(\theta_i \cap E) - \frac{w}{2^i}. \tag{3.6}$$

Since E is compact and $E \subset \cup_i \theta_i$, there exists an integer N such that $E \subset \cup_{i=1}^N \theta_i$. From Proposition 1 and the inequality (3.6) it follows that

$$\begin{aligned} d_\mu(1^{-1})^{2q} d_\nu(1^{-1})^{2t} \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E) &\leq d_\mu(1^{-1})^{2q} d_\nu(1^{-1})^{2t} \sum_{i=1}^N \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(\theta_i \cap E) \\ &\leq d_\mu(1^{-1})^{q} d_\nu(1^{-1})^{t} \Big[\sum_{i=1}^N \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(\overline{F_i}) + w \Big] \\ &\leq \sum_{i=1}^N \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(F_i) + d_\mu(1^{-1})^{q} d_\nu(1^{-1})^{t} w. \end{aligned}$$

Letting $w \to 0$ we get

$$d_\mu(1^{-1})^{2q} d_\nu(1^{-1})^{2t} \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E) \leq \inf_{E \subset \cup_i F_i} \sum_{i=1}^N \overline{\mathcal{P}}^{q,t}_{\mu,\nu}(F_i) = \mathcal{P}^{q,t}_{\mu,\nu}(E).$$

□

Corollary 2. *Let $\mu, \nu \in \mathcal{P}_D(\mathbb{R}^n)$ and $q \in \mathbb{R}$. Then, for all compact $E \subset S_{\mu,\nu}$ we have*

$$\operatorname{Dim}^q_{\mu,\nu}(E) = \Delta^q_{\mu,\nu}(E).$$

In particular, if $S_{\mu,\nu}$ is compact, we get $B_{\mu,\nu}(q) = \Lambda_{\mu,\nu}(q)$.

Proof of Lemma 2. Recall the notation and the definition in Lemma 1. Under the assumption of Theorem 3: $\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(E) < \infty$ we get, for $q \leq 0$, that $t \geq 0$.

(1) Case $q \leq 0$ and $\nu \in \mathcal{P}_D(E)$.

$$\begin{aligned} \overline{\mathcal{P}}^{q,t}_{\mu,\nu,\delta}(F) &\geq \sum_{i: r_i' > 0} \mu\big(B_{y_i}(r_i')\big)^q \nu\big(B_{y_i}(r_i')\big)^t \\ &\geq \sum_{i: r_i'' < \frac{r_i}{2}} \mu\big(B_{y_i}(r_i')\big)^q \nu\big(B_{y_i}(r_i')\big)^t \\ &\geq \sum_{i: r_i'' < \frac{r_i}{2}} \mu\big(B_{x_i}(r_i)\big)^q \nu\big(B_{x_i}(r_i)\big)^t \Big(\frac{\nu(B_{y_i}(r_i'))}{\nu(B_{x_i}(r_i))} \Big)^t \\ &\qquad \text{since} \quad q \leq 0 \quad \text{and} \quad r_i' < r_i \\ &\geq \sum_{i: r_i'' < \frac{r_i}{2}} \mu\big(B_{x_i}(r_i)\big)^q \nu\big(B_{x_i}(r_i)\big)^t \Big(\frac{\nu(B_{y_i}(r_i/2))}{\nu(B_{x_i}(\frac{3r_i}{2}))} \Big)^t \\ &\geq \Big(\inf_{0<r\leq\delta} \inf_{x \in E} \frac{\nu(B_x(\frac{r}{3}))}{\nu(B_x(r))} \Big)^t \sum_{i: r_i'' < \frac{r_i}{2}} \mu\big(B_{x_i}(r_i)\big)^q \nu\big(B_{x_i}(r_i)\big)^t. \end{aligned}$$

In addition, for $q \leq 0$, we have, for δ small enough,

$$\sum_{i: r_i'' \geq \frac{r_i}{2}} \mu\big(B_{x_i}(r_i)\big)^q \nu\big(B_{x_i}(r_i)\big)^t$$

$$\leq \sum_{i:r_i''\geq\frac{r_i}{2}} \mu\big(B_{x_i}(r_i'')\big)^q \nu\big(B_{x_i}(r_i'')\big)^t \Big(\frac{\nu(B_{x_i}(r_i)}{\nu(B_{x_i}(\frac{r_i}{2})}\Big)^t$$
$$\leq \sum_{i:r_i''\geq\frac{r_i}{2}} \mu\big(B_{x_i}(r_i'')\big)^q \nu\big(B_{x_i}(r_i'')\big)^t \Big(\sup_{0\leq r\leq\delta}\sup_{x\in E}\frac{\nu(B_x(2r)}{\nu(B_x(r)}\Big)^t$$
$$\leq c_1^t \sum_{i:r_i''\geq\frac{r_i}{2}} \mu\big(B_{x_i}(r_i'')\big)^q \nu\big(B_{x_i}(r_i'')\big)^t,$$

where C_1 is the constant in the doubling condition. Then, from the fact that

$$\sum_{i:r_i''<\frac{r_i}{2}} \mu\big(B_{x_i}(r_i)\big)^q \nu\big(B_{x_i}(r_i)\big)^t$$
$$\geq \sum_{i=1}^{p} \mu\big(B_{x_i}(r_i)\big)^q \nu\big(B_{x_i}(r_i)\big)^t - \sum_{i:r_i\geq\frac{r_i}{2}} \mu\big(B_{x_i}(r_i)\big)^q \nu\big(B_{x_i}(r_i)\big)^t,$$

if $\delta \to 0$ we have

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(F) \geq d_\nu(1^{-1})^t(a - w - c_1^t c(q,t)w).$$

Letting $w \to 0$ we get $\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(F) \geq d_\nu(1^{-1})^t a$.

(2) Case $q > 0$, $t \leq 0$ and $\mu \in \mathcal{P}_D(E)$. This case is similar to the preview case.

(3) Case $q > 0$, $t > 0$ and $\mu, \nu \in \mathcal{P}_D(E)$.

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu,\delta}(F)$$
$$\geq \sum_{i:r_i'>0} \mu\big(B_{y_i}(r_i')\big)^q \nu\big(B_{y_i}(r_i')\big)^t$$
$$\geq \sum_{i:r_i''<\frac{r_i}{2}} \mu\big(B_{y_i}(r_i')\big)^q \nu\big(B_{y_i}(r_i')\big)^t$$
$$\geq \sum_{i:r_i''<\frac{r_i}{2}} \mu\big(B_{x_i}(r_i)\big)^q \nu\big(B_{x_i}(r_i)\big)^t \Big(\frac{\mu(B_{y_i}(r_i')}{\mu(B_{x_i}(r_i)}\Big)^q \Big(\frac{\nu(B_{y_i}(r_i')}{\nu(B_{x_i}(r_i)}\Big)^t$$
$$\geq \sum_{i:r_i''<\frac{r_i}{2}} \mu\big(B_{x_i}(r_i)\big)^q \nu\big(B_{x_i}(r_i)\big)^t \Big(\frac{\mu(B_{y_i}(r_i/2)}{\mu(B_{y_i}(3r_i/2)}\Big)^q \Big(\frac{\nu(B_{y_i}(r_i/2)}{\nu(B_{y_i}(3r_i/2)}\Big)^t$$
$$\geq \Big(\inf_{0<r\leq\delta}\inf_{x\in E}\frac{\mu(B_x(\frac{r}{3})}{\mu(B_x(r)}\Big)^q \Big(\inf_{0<r\leq\delta}\inf_{x\in E}\frac{\nu(B_x(\frac{r}{3})}{\nu(B_x(r)}\Big)^t \sum_{i:r_i''<\frac{r_i}{2}} \mu\big(B_{x_i}(r_i)\big)^q \nu\big(B_{x_i}(r_i)\big)^t.$$

Finally, since

$$\sum_{i:r_i''<\frac{r_i}{2}} \mu\big(B_{x_i}(r_i)\big)^q \nu\big(B_{x_i}(r_i)\big)^t$$

$$= \sum_{i=1}^{p} \mu\big(B_{x_i}(r_i)\big)^q \nu\big(B_{x_i}(r_i)\big)^t - \sum_{i:r_i'' \geq \frac{r_i}{2}} \mu\big(B_{x_i}(r_i)\big)^q \nu\big(B_{x_i}(r_i)\big)^t$$

and, for δ small enough,

$$\sum_{i:r_i'' \geq \frac{r_i}{2}} \mu\big(B_{x_i}(r_i)\big)^q \nu\big(B_{x_i}(r_i)\big)^t \leq c_1^t c_2^q \sum_{i:r_i'' \geq \frac{r_i}{2}} \mu\big(B_{x_i}(r_i'')\big)^q \nu\big(B_{x_i}(r_i'')\big)^t,$$

we get, by Lemma 1, if $\delta \to 0$

$$\overline{\mathcal{P}}^{q,t}_{\mu,\nu}(F) \geq d_\mu(1^{-1})^q d_\nu(1^{-1})^t(a - w - c_1^t c_2^q c(q,t)w).$$

Letting $w \to 0$ we get the result. □

4. Relative multifractal spectrum

Let μ, ν in $\mathcal{P}_D(\mathbb{R}^n)$ such that $S_{\mu,\nu}$ is a compact set. We will start by computing the ν-Hausdorff and ν-packing dimensions of the set $E(\alpha)$ and then, Corollary 3, give the validity of multifractal analysis:

$$\Theta_{\mu,\nu} = b_{\mu,\nu} = B_{\mu,\nu} = \Lambda_{\mu,\nu}.$$

Theorem 4. *Suppose that $b_{\mu,\nu}$ is differentiable at q and set $\alpha(q) = -b'_{\mu,\nu}(q)$, then, provided that $\Theta^*_{\mu,\nu}(\alpha(q)) \geq 0$ and $\mathcal{H}^{q,\Theta_{\mu,\nu}(q)}_{\mu,\nu}(E(\alpha(q))) > 0$, we have*

$$\dim_\nu E(\alpha(q)) = \Theta^*_{\mu,\nu}(\alpha(q)) = b^*_{\mu,\nu}(\alpha(q)).$$

Proof. Since μ, ν in $\mathcal{P}_D(\mathbb{R}^n)$, then, from Corollary 1, we have $\Theta_{\mu,\nu} = b_{\mu,\nu}$. In particular our assumption implies that $\mathcal{H}^{q,b_{\mu,\nu}(q)}_{\mu,\nu}(E(\alpha(q))) > 0$ and we deduce the result from Theorem 2.10 in [8]. □

Remark 1. For $q \in \mathbb{R}$, we have $\Theta_{\mu,\nu}(q) \leq b_{\mu,\nu}(q)$. Then $\mathcal{H}^{q,\Theta_{\mu,\nu}(q)}_{\mu,\nu}(E(\alpha(q))) > 0$ does not implies that $\mathcal{H}^{q,b_{\mu,\nu}(q)}_{\mu,\nu}(E(\alpha(q))) > 0$. Hence, if μ, ν in $\mathcal{P}_D(\mathbb{R}^n)$, the preview theorem improves Cole's result established in [8] (Theorem 2.10).

Theorem 5. *Let $q \in \mathbb{R}$ such that $\overline{\mathcal{P}}^{q,B_{\mu,\nu}(q)}_{\mu,\nu}(S_{\mu,\nu}) < \infty$. Suppose that $B_{\mu,\nu}$ is differentiable at q and set $\alpha(q) = -B'_{\mu,\nu}(q)$, then, provided that $B^*_{\mu,\nu}(\alpha(q)) \geq 0$ and $\mathcal{P}^{q,B_{\mu,\nu}(q)}_{\mu,\nu}(E(\alpha(q))) > 0$, we have*

$$\operatorname{Dim}_\nu E(\alpha(q)) = B^*_{\mu,\nu}(\alpha(q)) = \Lambda^*_{\mu,\nu}(\alpha(q)).$$

Proof. It follow from Corollary 2, that $B_{\mu,\nu} = \Lambda_{\mu,\nu}$ and we deduce the result from Theorem 2.11 in [8]. □

Corollary 3. *Suppose that $\Lambda_{\mu,\nu}$ is differentiable at q and set $\alpha(q) = -\Lambda'_{\mu,\nu}(q)$, then, provided that $\Theta^*_{\mu,\nu}(\alpha(q)) \geq 0$ and $\mathcal{H}^{q,\Lambda_{\mu,\nu}(q)}_{\mu,\nu}(S_{\mu,\nu}) > 0$, we have*

$$\begin{aligned}\dim_\nu E(\alpha(q)) &= \operatorname{Dim}_\nu E(\alpha(q)) = \Theta^*_{\mu,\nu}(\alpha(q)) \\ &= b^*_{\mu,\nu}(\alpha(q)) = B^*_{\mu,\nu}(\alpha(q)) = \Lambda^*_{\mu,\nu}(\alpha(q)).\end{aligned}$$

Proof. From the definition of generalized Hausdorff multifractal premeasure, the assumption $\mathcal{H}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(S_{\mu,\nu}) > 0$ implies that $\Lambda_{\mu,\nu}(q) \leq b_{\mu,\nu}(q)$ so we have the equality. In addition, since μ, ν in $\mathcal{P}_D(\mathbb{R}^n)$, we get $\Theta_{\mu,\nu} = b_{\mu,\nu}$. Finally, we only have to prove, according to Theorem 4, that $\mathcal{H}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(E(\alpha(q))) > 0$ or $\mathcal{H}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}\big(S_{\mu,\nu}\backslash E(\alpha(q))\big) = 0$. Since $\mu, \nu \in \mathcal{P}_D(\mathbb{R}^n)$ then, according to Theorem 4, we only have to prove that

$$\overline{\mathcal{H}}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}\big(S_{\mu,\nu}\backslash E(\alpha(q))\big) = 0.$$

For $\alpha \in \mathbb{R}_+^*$, let us introduce the sets

$$\overline{F}_\alpha = \Big\{x \in S_{\mu,\nu},\ \limsup_{r\to 0} \frac{\log(\mu(Bx(r))}{\log(\nu(B_x(r))} > \alpha\Big\}$$

and

$$\underline{F}_\alpha = \Big\{x \in S_{\mu,\nu},\ \liminf_{r\to 0} \frac{\log(\mu(Bx(r))}{\log(\nu(B_x(r))} < \alpha\Big\}.$$

We only have to prove that

$$\overline{\mathcal{H}}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(\overline{F}_\alpha) = 0\ ,\ \forall \alpha > \alpha(q), \tag{4.1}$$

$$\overline{\mathcal{H}}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(\underline{F}_\alpha) = 0\ ,\ \ \forall \alpha < \alpha(q). \tag{4.2}$$

In deed,

$$\begin{aligned} 0 &\leq \overline{\mathcal{H}}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(S_{\mu,\nu}\backslash E(\alpha(q)) \\ &\leq \overline{\mathcal{H}}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(\underline{F}_{\alpha(q)}) + \overline{\mathcal{H}}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(\overline{F}_{\alpha(q)}) \\ &\leq \overline{\mathcal{H}}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(\bigcup_{\alpha<\alpha(q)} \underline{F}_\alpha) + \overline{\mathcal{H}}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(\bigcup_{\alpha>\alpha(q)} \overline{F}_\alpha) \\ &\leq \sum_\alpha \overline{\mathcal{H}}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(\underline{F}_\alpha) + \overline{\mathcal{H}}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(\overline{F}_\alpha) = 0. \end{aligned}$$

Let us come back to prove the inequality (4.1) (the proof for (4.2) is similar). If $x \in \overline{F}_\alpha$, let $\delta > 0$ we can find $0 < r_x < \delta$ such that

$$\mu(B_x(r_x)) < \nu(B_x(r_x))^\alpha. \tag{4.3}$$

The family $(B_x(r_x))_{x\in\overline{F}_\alpha}$ is then a centered δ-covering of $\overline{F}_\alpha$. Using Besicovitch's Covering Theorem, we can construct ξ finite or countable sub-families

$$(B_{x_{1j}}(r_{1j}))_j, \ldots, (B_{x_{\xi j}}(r_{\xi j}))_j$$

such that each $\overline{F}_\alpha \subseteq \displaystyle\bigcup_{i=1}^{\xi}\bigcup_j B_{x_{ij}}(r_{ij})$ and $(B_{x_{ij}}(r_{ij}))_j$ is a δ-packing of $\overline{F}_\alpha$. From the inequality (4.3), we get, for $t > 0$,

$$\mu(B_{x_{ij}}(r_{ij}))^q \nu(B_{x_{ij}}(r_{ij}))^{\Lambda_{\mu,\nu}(q)} \leq \mu(B_{x_{ij}}(r_{ij}))^{q-t} \nu(B_{x_{ij}}(r_{ij}))^{\Lambda_{\mu,\nu}(q)+\alpha t}$$

and then

$$\overline{\mathcal{H}}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(\overline{F}_\alpha) \leq \xi \overline{\mathcal{P}}_{\mu,\nu}^{q-t,\Lambda_{\mu,\nu}(q)+\alpha t}(\overline{F}_\alpha).$$

Since $\alpha > -\Lambda'_{\mu,\nu}(q)$, we may choose $t > 0$ such that $\Lambda(q-t) > \Lambda(q) + \alpha t$ thereby

$$\overline{\mathcal{P}}_{\mu,\nu}^{q-t,\Lambda_{\mu,\nu}(q)+\alpha t}(S_{\mu,\nu}) = 0.$$

□

Computing the Hausdorff and packing dimension of the set $E(\alpha)$, respectively $\dim E(\alpha)$ and $\operatorname{Dim} E(\alpha)$, is difficult in general, but we can estimate from bellow Hausdorff and packing dimension of this level set. Indeed, we can decompose the set $E(\alpha)$ according to the ν-local dimension of theirs points and then calculate the size of the subset of $E(\alpha)$ whose points have ν-local dimension β. This idea can be found in [8,14]. We set, for $\alpha, \beta \geq 0$,

$$E(\alpha,\beta) = \Big\{ x \in S_{\mu\nu} \mid \lim_{r\to 0} \frac{\log \mu(B_x(r))}{\log \nu(B_x(r))} = \alpha;\ \lim_{r\to 0} \frac{\log \nu(B_x(r))}{\log r} = \beta \Big\}.$$

Theorem 6. *Let $q \in \mathbb{R}$ such that $b_{\mu,\nu}$ is differentiable at q. Set $\alpha(q) = -b'_{\mu,\nu}(q)$ and*

$$I = \Big\{ \beta \geq 0 \mid \mathcal{H}_{\mu,\nu}^{q,\Theta_{\mu,\nu}(q)}(E(\alpha(q),\beta)) > 0 \Big\}.$$

Suppose that $\Theta^*_{\mu,\nu}(\alpha(q)) \geq 0$ then

$$\dim E(\alpha(q)) \geq \sup_{\beta \in I} \beta \cdot \Theta^*_{\mu,\nu}(\alpha(q)).$$

Proof. It's clear that $E(\alpha(q),\beta) \subset E(\alpha(q))$. Then it's enough to prove that $\dim E(\alpha(q),\beta) = \beta \cdot \Theta^*_{\mu,\nu}(\alpha(q))$. From Corollary 2, we have $\Theta_{\mu,\nu} = b_{\mu,\nu}$. In particular our assumption implies that $\mathcal{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)}(E(\alpha(q))) > 0$ and we deduce the result from Theorem 2.14 in [8]. □

Theorem 7. *Let $q \in R$ such that $B_{\mu,\nu}$ is differentiable at q. Set $\alpha(q) = -B'_{\mu,\nu}(q)$ and*

$$J = \Big\{ \beta \geq 0 \mid \overline{\mathcal{P}}_{\mu,\nu}^{q,B_{\mu,\nu}(q)}(E(\alpha(q),\beta)) > 0 \Big\}.$$

Suppose that $B^*_{\mu,\nu}(\alpha(q)) \geq 0$ then

$$\operatorname{Dim} E(\alpha(q)) \geq \sup_{\beta \in J} \beta \cdot B^*_{\mu,\nu}(\alpha(q)).$$

Proof. By Theorem 3, the assumption $\overline{\mathcal{P}}_{\mu,\nu}^{q,B_{\mu,\nu}(q)}(E(\alpha(q),\beta)) > 0$ implies that

$$\mathcal{P}_{\mu,\nu}^{q,B_{\mu,\nu}(q)}(E(\alpha(q),\beta)) > 0$$

and we deduce the result from Theorem 2.15 in [8]. □

Remark 2. Theorems 6 and 7 improve Theorems 2.14 and 2.15 established in [8], if μ, ν in $\mathcal{P}_D(\mathbb{R}^n)$.

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