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UPPER BOUND ON THE THIRD HANKEL DETERMINANT FOR FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVE OPERATOR

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Abstract. Let S denote the class of analytic and univalent functions in the open unit disk $D = \{z : |z| < 1\}$ with the normalization conditions $f(0) = 0$ and $f'(0) = 1$. In the present article, an upper bound for third order Hankel determinant $H_3(1)$ is obtained for a certain subclass of univalent functions generated by Ruscheweyh derivative operator.

1. Introduction

Let $\mathbb{D}$ be the unit disk $\{z : |z| < 1\}$, $\mathcal{A}$ be the class of functions analytic in $\mathbb{D}$, satisfying the conditions

$$f(0) = 0 \text{ and } f'(0) = 1. \tag{1}$$

Then each function f in $\mathcal{A}$ has the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{2}$$

because of the conditions (1). Let S denote class of analytic and univalent functions in $\mathbb{D}$ with the normalization conditions (1).

The q^{th} Hankel determinant for $q \geq 1$ and $n \geq 0$ is stated by Noonan and Thomas [17] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & & \cdots \\ \vdots & & & \vdots \\ a_{n+q-1} & \cdots & & a_{n+2q-2} \end{vmatrix}. \tag{3}$$

This determinant has also been considered by several authors. For example, Noor [18] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions f given by (1) with bounded boundary rotations. Ehrenborg [5] studied the

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Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed in Layman's article [13]. A classical theorem of Fekete and Szegö [6] is considered second Hankel determinant $H_2(1) = a_3 - a_2^2$ for univalent functions. This functional corresponds to the Hankel determinant with $q = 2$ and $n = 1$. It is well known that the sharp inequality $\left|a_3 - a_2^2\right| \leq 1$ holds for $f \in S$ and given by (2). This result is given in the article [4]. Further, Fekete and Szegö [6] introduced the generalized functional, known as Fekete-Szegö functional, $\left|a_3 - \mu a_2^2\right|$ where μ is a real number. Hankel determinant in case of $q = 2$ and $n = 2$ is known as the second Hankel determinant, given by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

In particular, sharp bound on $|H_2(2)|$ is obtained by several authors for different subclasses of univalent functions (See [8–11, 16, 23, 25, 26]).

The third order Hankel determinant is constructed in the case of $q = 3$ and $n = 1$, given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

It is obvious that

$$H_3(1) = a_3\left(a_2 a_4 - a_3^2\right) - a_4\left(a_4 - a_2 a_3\right) + a_5\left(a_3 - a_2^2\right).$$

By applying the triangle inequality, we obtain

$$|H_3(1)| \leq |a_3|\left|a_2 a_4 - a_3^2\right| + |a_4|\left|a_4 - a_2 a_3\right| + |a_5|\left|a_3 - a_2^2\right|. \tag{4}$$

Recently, many authors have considered to investigate an upper bound for $|H_3(1)|$ of functions in different subclasses of univalent or p-valent functions (See [1–3, 19, 20, 24]).

Let $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$ be analytic functions in $\mathbb{D}$. The Hadamard product (convolution) of f and g, denoted by $f * g$, is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \ z \in \mathbb{D}. \tag{5}$$

Let $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The Ruscheweyh derivative [21] of the n^{th} order of f, denoted by $D^n f(z)$, is defined by

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(n+k)}{\Gamma(n+1)(k-1)!} a_k z^k. \tag{6}$$

The Ruscheweyh derivative gave an impulse for various generalization of well known classes of functions. By using the Ruscheweyh derivative, we can generalize the class of the starlike and convex functions, denoted by S^* and C, which are defined as

$$S^* = \left\{ f(z) \in S : \operatorname{Re}\left(\frac{zf'(z)}{f(z)} \right) > 0,\ z \in \mathbb{D} \right\} \tag{7}$$

and

$$C = \left\{ f(z) \in S : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} \right) > 0,\ z \in \mathbb{D} \right\}. \tag{8}$$

The class R_n was studied by Singh and Singh [22], which is given by the following definition

$$\operatorname{Re} \frac{z\left(D^n f(z)\right)'}{D^n f(z)} > 0,\ z \in \mathbb{D}. \tag{9}$$

We denote that $R_0 = S^*$ and $R_1 = C$.

Fekete-Szegö problem was discussed for a certain subclass which includes R_n in the paper [12]. On the other hand, the second Hankel determinant problem of functions in R_n is investigated in [25].

Motivated from the results in [2], [3], [19], and [20], we obtain an upper bound for $|H_3(1)|$ for the functions belonging to the class R_n.

2. Preliminary results

The following lemmas are required to prove our main results. Let P be the family of all functions p analytic in $\mathbb{D}$ for which $\operatorname{Re}(p(z)) > 0$ and

$$p(z) = 1 + c_1 z + c_2 z + \cdots . \tag{10}$$

Lemma 2.1 (Duren [4])**.** *If $p \in P$, then $|c_k| \le 2$ for each $k \in \mathbb{N}$. The inequality is sharp for each k.*

Lemma 2.2 (Grenander & Szegö, [7])**.** *Let $p \in P$. Then*

$$2c_2 = c_1^2 + \left(4 - c_1^2\right) x, \tag{11}$$

$$4c_3 = c_1^3 + 2c_1\left(4 - c_1^2\right) x - c_1\left(4 - c_1^2\right) x^2 + 2c_1\left(4 - c_1^2\right)\left(1 - |x|^2\right) z \tag{12}$$

for some x and z satisfying $|x| \le 1$, $|z| \le 1$.

We obtain following lemma as a special case of the parameter of class of functions defined in [12].

Lemma 2.3 ([12])**.** *Let $f \in R_n$. Then for $\mu \in \mathbb{R}$, we have*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{2}{(n+1)(n+2)}\left[3 - 2\mu\frac{n+2}{n+1}\right], & \mu \le \frac{n+1}{n+2}, \\ \frac{2}{(n+1)(n+2)}, & \frac{n+1}{n+2} \le \mu \le \frac{3(n+1)}{n+2}, \\ \frac{2}{(n+1)(n+2)}\left[2\mu\frac{n+2}{n+1} - 3\right], & \mu \ge \frac{3(n+1)}{n+2}. \end{cases} \tag{13}$$

For each μ there exists a function in R_n such that equality holds.

Lemma 2.4 ([25])**.** *Let the function f given by* (2) *be in the class in R_n. Then*

$$\left|a_2a_4 - a_3^2\right| \le \begin{cases} 1, & n = 0, \\ \frac{1}{8}, & n = 1, \\ \frac{12(n-1)}{(n+1)^2(n+2)^2(n+3)}, & n > 1. \end{cases} \tag{14}$$

3. Main results

Before we get our main results, we need to obtain upper bounds for coefficients of functions in the class R_n.

Theorem 3.1. *Let $f(z) \in R_n$. Then all $k \in \mathbb{N}$, we have the following sharp inequalities*

$$|a_k| \le \frac{k!}{(n+1)(n+2)\cdots(n+k-1)}. \tag{15}$$

Proof. Let the function $f \in R_n$. Define a function

$$F(z) = D^n f(z) = z + \sum_{k=2}^{\infty} A_k z^k, \tag{16}$$

where

$$A_k = \frac{\Gamma(n+k)}{\Gamma(n+1)(k-1)!} a_k, \ \ k \ge 2, \ \ A_1 = 1. \tag{17}$$

Then, there exists an analytic function $p(z) \in P$ in the unit disk $\mathbb{D}$ with $p(0) = 1$ and $\mathrm{Re}(p(z)) > 0$ such that

$$\frac{zF'(z)}{F(z)} = p(z). \tag{18}$$

Hence, we have from (10)

$$z + \sum_{k=2}^{\infty} kA_k z^k = \left\{ z + \sum_{k=2}^{\infty} A_k z^k \right\} \times \left\{ 1 + \sum_{k=1}^{\infty} c_k z^k \right\}. \tag{19}$$

We need to use the principle of the mathematical induction to get desired result.

For $n = 1$, $A_1 = 1$. Assume that $|A_l| \le l$, $l = 2, 3, \ldots, k-1$. After that, we have to show that $|A_k| \le k$. According to (19), we obtain the following relation

$$(k-1)A_k = c_{k-1}A_1 + c_{k-2}A_2 + \cdots + c_1 A_{k-1}. \tag{20}$$

Applying the triangle inequality with Lemma 1, we obtain

$$(k-1)\left|A_k\right| \le 2 \sum_{l=1}^{k-1} \left|A_l\right|.$$

According to our assumption, we get the following desired result

$$|a_k| \le \frac{k!}{(n+1)(n+2)\cdots(n+k-1)}.$$

This completes the proof. □

We prove the following theorem by using the classical method of Libera and Zlotkiewicz [14], [15].

Theorem 3.2. *Let the function given by* (1.2) *be in the class* R_n. *Then we have the following sharp inequalities*:

$$|a_2a_3 - a_4| \le \begin{cases} 2, & n = 0, \\ \frac{4}{9\sqrt{3}}, & n = 1, \\ \frac{4(5n+3)}{3(n+1)^2(n+2)(n+3)}\sqrt{\frac{2(5n+3)}{3(n+3)}}, & n > 1. \end{cases} \tag{21}$$

Proof. By using the series expansion of $F(z)$ and $p(z)$ in (19), equating coefficients in (20) yields

$$\begin{aligned} a_2 &= \frac{1}{n+1}c_1, \\ a_3 &= \frac{1}{(n+1)(n+2)}\left\{c_2 + c_1^2\right\}, \\ a_4 &= \frac{1}{(n+1)(n+2)(n+3)}\left\{2c_3 + 3c_1c_2 + c_1^3\right\}. \end{aligned} \tag{22}$$

Hence, we get from (22)

$$a_2a_3 - a_4 = A(n)\left\{c_1c_2 + c_1^3 - B(n)\left(2c_3 + 3c_1c_2 + c_1^3\right)\right\}, \tag{23}$$

where

$$A(n) = \frac{1}{(n+1)^2(n+2)}, \tag{24}$$

and

$$B(n) = \left(\frac{n+1}{n+3}\right), \quad n = 0, 1, 2, \dots. \tag{25}$$

Using (11) and (12) in (23) ,we get

$$\begin{aligned} |a_2a_3 - a_4| = A(n)\Bigg| & 3\left(\frac{1}{2} - B(n)\right)c_1^3 + B(n)\frac{c_1\left(4 - c_1^2\right)x^2}{2} \\ & + \frac{1}{2}\left(1 - 5B(n)\right)c_1\left(4 - c_1^2\right)x - B(n)c_1\left(4 - c_1^2\right)\left(1 - |x|^2\right)z\Bigg|. \end{aligned}$$

Since the function $p(e^{i\theta}z)$, $(\theta \in \mathbb{R})$ is also in the class P, we assume that without loss of generality that $c_1 \ge 0$. For convenience of notation, we take $c_1 = c$, $c \in [0, 2]$. Applying the triangle inequality with the assumptions $c_1 = c \in [0, 2]$, $|x| = \rho$ and $|z| \le 1$, it is obtained that

$$|a_2a_3 - a_4| \le A(n)\left\{3\left|\frac{1}{2} - B(n)\right|c^3 + B(n)\frac{c\left(4 - c^2\right)\rho^2}{2}\right.$$

$$+\frac{1}{2}\left(5B(n)-1\right)c\left(4-c^2\right)\rho+B(n)c\left(4-c^2\right)\left(1-\rho^2\right)\Big\} \tag{26}$$
$$= A(n)G(c,\rho).$$

We now maximize the function $G(c,\rho)$ on the closed square $[0,2]\times[0,1]$.

$$\begin{aligned}\frac{\partial G(c,\rho)}{\partial\rho} &= -B(n)c\left(4-c^2\right)\rho+\frac{5B(n)-1}{2}c\left(4-c^2\right)\\ &\geq \frac{n}{n+3}c\left(4-c^2\right)>0.\end{aligned} \tag{27}$$

Hence, $G(c,\rho)$ can not have a maximum in the interior of the closed square $[0,2]\times[0,1]$. Moreover for a fixed $c\in[0,2]$, we have

$$\max_{0\leq\rho\leq1} G(c,\rho)=G(c,1)=F(c). \tag{28}$$

One can obtain that

$$|a_2a_3-a_4|\leq A(n)F(c), \tag{29}$$

where

$$F(c)=\frac{3\left|1-n\right|}{2(n+3)}c^3+\frac{6B(n)-1}{2}c\left(4-c^2\right). \tag{30}$$

Since

$$F'(c)=\begin{cases} 2, & n=0,\\ 4-3c^2, & n=1,\\ -3c^2+2\left(6B(n)-1\right), & n>1,\end{cases} \tag{31}$$

we have to consider following three cases:

Case 1. For $n=0$, $F'(c)>0$. Hence $F(c)\leq F(2)$. We get the following result

$$|a_2a_3-a_4|\leq A(0)F(2)=2. \tag{32}$$

This one coincides with the result for starlike functions in the article [2, Theorem 3.3]. This inequality is sharp and the equality is obtained for the Koebe function $k(z)=\frac{z}{(1-z)^2}$ and its rotations.

Case 2. Let $n=1$. After required calculations, it is obtained that $F(c)$ has a local maximum at $c=\frac{2}{\sqrt{3}}$. Since $F(0)=F(2)=0$, it is easy to see that $F(c)\leq F\left(\frac{2}{\sqrt{3}}\right)$. Hence, we have the following sharp estimates which is stated in [2]

$$|a_2a_3-a_4|\leq A(1)F\left(\frac{2}{\sqrt{3}}\right)=\frac{4}{9\sqrt{3}}. \tag{33}$$

Case 3. Let $n>1$. Then, $F'(c)=0$ for $c^*=\sqrt{\frac{2(5n+3)}{3(n+3)}}$. It is obvious that $F''(c)<0$. Then, $F(c)$ has a local maximum at $c=c^*$. Since, $F(0)=0$,

$F(2) = \frac{12(n-1)}{n+3}$ and

$$\frac{F(c^*)}{F(2)} = \frac{5n+3}{9(n-1)}\sqrt{\frac{2(5n+3)}{3(n+3)}} > 1 \text{ for all } n > 1,$$

we obtain

$$|a_2a_3 - a_4| \le A(n)F(c^*) = \frac{4(5n+3)}{3(n+1)^2(n+2)(n+3)}\sqrt{\frac{2(5n+3)}{3(n+3)}}.$$

This completes the proof of theorem. □

By using above results in (4) together with the known inequalities given by Lemma 2.3 and Lemma 2.4, after necessarily calculations we obtain the following corollary.

Corollary 3.1. *Let $f(z) \in R_n$. Then*

$$|H_3(1)| \le \begin{cases} 16, & n = 0, \\ \frac{11}{24} + \frac{4}{9\sqrt{3}} = 0,714\ldots, & n = 1, \\ \frac{4!}{(n+1)^3(n+2)^3(n+3)}\left\{\frac{13n^2+39n+8}{n+4} + \frac{4(5n+3)(n+2)}{3(n+3)}\sqrt{\frac{2(5n+3)}{3(n+3)}}\right\}, & n > 1. \end{cases} \tag{34}$$

Remark 3.1. By choosing $n = 0$ and $n = 1$ in (34), we obtain sharp upper bound for third hankel determinant of starlike and convex functions, respectively. These results also agree with those considered by Babalola [2].

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