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SOLUTION TO $\overline{\partial}$-PROBLEM WITH SUPPORT CONDITIONS IN WEAKLY q-CONVEX DOMAINS

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Abstract. Let X be a complex manifold of dimension $n \geqslant 2$ and let $\Omega \Subset X$ be a weakly q-convex domain with smooth boundary. Assume that E is a holomorphic line bundle over X and $E^{\otimes m}$ is the m-times tensor product of E for positive integer m. If there exists a strongly plurisubharmonic function on a neighborhood of $b\Omega$, then we solve the $\overline{\partial}$-problem with support condition in Ω for forms of type (r,s), $s \geqslant q$ with values in $E^{\otimes m}$. Moreover, the solvability of the $\overline{\partial}_b$-problem on boundaries of weakly q-convex domains with smooth boundary in Kähler manifolds are given. Furthermore, we shall establish an extension theorem for the $\overline{\partial}_b$-closed forms.

1. Introduction

In [3], Derridj considered the $\overline{\partial}$-problem with exact support by using Carleman type estimates for smooth domains with plurisubharmonic defining functions. In [14], Shaw has obtained a solution to this problem in a pseudo-convex domain in $\mathbb{C}^n$ with C^1 smooth boundary. Cao-Shaw-Wang [2] have obtained a solution to this problem in a locally Stein domain of the complex projective space. On strongly q-convex (or concave) domains, this problem has been studied by Sambou in [13]. In [10], the author studied this problem on a weakly q-pseudoconvex domain with C^1-smooth boundary in $\mathbb{C}^n$ and extended this result to a Stein manifold in [11]. Also Saber in [12], studies this problem on a weakly pseudoconvex domain with smooth boundary for forms in $E^{\otimes m}$ under the positivity condition on E. The purpose of this paper is to extend this result to a weakly q-convex domain for forms of type (r,s), $s \geqslant q$ with values in $E^{\otimes m}$ and under a different condition. More precisely, we prove the following result:

Theorem 1. *Let X be a complex manifold of dimension $n \geq 2$ and let $\Omega \Subset X$ be a weakly q-convex domain with smooth boundary in X. Assume that E is a holomorphic line bundle over X and $E^{\otimes m}$ is the m-times tensor product of E for positive integer m. Suppose that there exists a strongly plurisubharmonic*

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function on a neighborhood of the boundary of Ω. *Then, for* $\alpha \in L^2_{r,s}(X, E^{\otimes m})$, *supp* $\alpha \subset \overline{\Omega}$, *with* $s \geqslant q$, *satisfying* $\overline{\partial}\alpha = 0$ *in the distribution sense in* X, *there exists* $u \in L^2_{r,s-1}(X, E^{\otimes m})$, *supp* $u \subset \overline{\Omega}$ *such that* $\overline{\partial}\, u = \alpha$ *in the distribution sense in* X.

Applications to the solvability of the $\overline{\partial}_b$-problem on boundaries of weakly q-convex domains with smooth boundary in Kähler manifolds are given. Furthermore, we shall establish an extension theorem for the $\overline{\partial}_b$-closed forms.

2. Notation and preliminaries

Let X be an n-dimensional complex manifold. Let Ω be an open subset of X and ρ be its defining function. Let E be a holomorphic line bundle over X and let E^* be its dual. Let $\{U_j\}_{j\in J}$ be an open covering of X such that $E|_{U_j}$ is trivial, namely $\pi^{-1}(U_j) = U_j \times \mathbb{C}$, and $(z_j^1, z_j^2, \dots, z_j^n)$ be local coordinates on U_j. Let $\{e_{jk}\}$ be a system of transition functions of E with respect to a covering $\{U_j\}_{j\in J}$. An (r,s) forms $\alpha = \{\alpha_j\}$ on X can be expressed as follows:

$$\alpha_j = \sum_{C_r, D_s}{}' \alpha_{jC_r\overline{D}_s}\, dz_j^{C_r} \wedge dz_j^{\overline{D}_s},$$

where $C_r = (c_1, \dots, c_r)$ and $D_s = (d_1, \dots, d_s)$ are multiindices and so on. The notation $\sum'$ means the summation over strictly increasing multiindices. Let

$$ds^2 = \sum_{\alpha,\beta=1}^{n} g_{j,\alpha\overline{\beta}}(z)\, dz_j^\alpha\, d\overline{z}_j^\beta$$

be a hermitian metric on X. We associate to ds^2, the $(1,1)$ differential form $\omega = \frac{\sqrt{-1}}{2} \sum_{\alpha,\beta=1}^{n} g_{j,\alpha\overline{\beta}}(z)\, dz_j^\alpha \wedge d\overline{z}_j^\beta$. If $d\omega = 0$, the metric ds^2 is called Kähler metric and ω is called the Kähler form associated to the metric ds^2. A complex manifold X is called Kähler manifold if we can define Kähler metric on it. Let $h = \{h_j\}$ be a hermitian metric of $E = \{e_{jk}\}$ with respect to the covering $\{U_j\}_{j\in J}$ satisfies $h_j = |e_{jk}|^2 h_k$ on $U_j \cap U_k$. For integers $r, s \geq 0$, $m \geqslant 1$, we define the following notations:

- $C^\infty_{r,s}(\Omega, E^{\otimes m})$: the complex vector space of $E^{\otimes m}$-valued differential forms of class C^∞ and of type (r,s) on Ω.
- $C^\infty_{r,s}(\overline{\Omega}, E^{\otimes m})$: the subspace of $C^\infty_{r,s}(\Omega, E^{\otimes m})$ whose elements can be extended smoothly up to $b\Omega$.
- $\mathcal{D}_{r,s}(\Omega, E^{\otimes m})$: the space of $E^{\otimes m}$-valued differential forms of type (r,s) with compact support in Ω.
- The operator $\star : C^\infty_{r,s}(X, E^{\otimes m}) \longrightarrow C^\infty_{n-s,n-r}(X, E^{\otimes m})$ is the Hodge star operator.
- The operator $\#_{E^{\otimes m}} : C^\infty_{r,s}(X, E^{\otimes m}) \longrightarrow C^\infty_{s,r}(X, E^{*\otimes m})$ is defined by $\#_{E^{\otimes m}}\, \alpha = h^m\, \overline{\alpha}$, which commutes with the Hodge star operator, and

the corresponding operator $\#_{E^{*\otimes m}} : C^\infty_{r,s}(X, E^{*\otimes m}) \longrightarrow C^\infty_{s,r}(X, E^{\otimes m})$ is defined by

$$\#_{E^{*\otimes m}}\alpha = \overline{(h^m)^*}\overline{\alpha} = \overline{{}^t(h^m)^{-1}}\overline{\alpha} = h^{-m}\overline{\alpha} = \#^{-1}_{E^{\otimes m}}\alpha.$$

Thus $\#_{E^{*\otimes m}}\alpha = \#^{-1}_{E^{\otimes m}}\alpha$.

- $\mathcal{B}_{r,s}(\overline{\Omega}, E^{\otimes m}) = \{\alpha \in C^\infty_{r,s}(\overline{\Omega}, E^{\otimes m}) : \star\#_{E^{\otimes m}}\,\alpha|_{b\Omega} = 0\}$.
- dV is the volume element with respect to ds^2.
- $\overline{\partial} : C^\infty_{r,s-1}(\Omega, E^{\otimes m}) \longrightarrow C^\infty_{r,s}(\Omega, E^{\otimes m})$ is the Cauchy-Riemann operator and ϑ_m its formal adjoint.
- dom$(\overline{\partial},\, E^{\otimes m})$, range$(\overline{\partial}, E^{\otimes m})$ and ker$(\overline{\partial}, E^{\otimes m})$ is the domain, the range and the kernel of $\overline{\partial}$, respectively.
- $H^{r,s}(X, E^{\otimes m}) = \frac{C^\infty_{r,s}(X,\, E^{\otimes m})\,\cap\,\ker(\overline{\partial}, E^{\otimes m})}{\overline{\partial}(C^\infty_{r,s-1}(X,\, E^{\otimes m}))}$.
- $C^\infty_{r,s}(b\Omega, E^{\otimes m}) = C^\infty_{r,s}(\overline{\Omega}, E^{\otimes m})/\mathcal{D}_{r,s}(\Omega, E^{\otimes m})$.
- We put

$$\pi_{r,s} : C^\infty_{r,s}(\overline{\Omega}, E^{\otimes m}) \longrightarrow C^\infty_{r,s}(b\Omega, E^{\otimes m}),$$

$$\sigma_{r,s} : \oplus_{p,q} C^\infty_{(p,q)}(\overline{\Omega}, E^{\otimes m}) \longrightarrow C^\infty_{r,s}(b\Omega, E^{\otimes m})$$

the natural projections. For simplicity we put

$$\pi_{r,s}(u) = u|_{b\Omega}.$$

- The $\overline{\partial}_b$-operator

$$\overline{\partial}_b : C^\infty_{r,s}(b\Omega, E^{\otimes m}) \longrightarrow C^\infty_{r,s+1}(b\Omega, E^{\otimes m})$$

is defined by

$$\overline{\partial}_b = \sigma_{r,s+1} \circ d \circ (\pi_{r,s})^{-1}.$$

Differentiable functions f on $b\Omega$ satisfying $\overline{\partial}_b\, f = 0$ are called $\mathcal{CR}$ functions on $b\Omega$. It is clear that f is $\mathcal{CR}$ if there exists a differentiable function F on $\overline{\Omega}$ satisfying $F|_{b\Omega} = f$ and $\overline{\partial}F = 0$. Then the space $C^\infty_{r,s}(b\Omega, E)$ and the operator

$$\overline{\partial}_b : C^\infty_{r,s}(b\Omega, E^{\otimes m}) \longrightarrow C^\infty_{r,s+1}(b\Omega, E^{\otimes m})$$

are defined similarly as above.

For $\alpha,\, u \in C^\infty_{r,s}(X, E^{\otimes m})$, the local inner product $(\alpha,\, u)_m$ is defined by

$$(\alpha,\, u)_m\, dV = \alpha_j \wedge \star\, h^m \overline{u}_j = \alpha_j \wedge \star\, \#_{E^{\otimes m}}\, u_j.$$

For $\alpha,\, u \in C^\infty_{r,s}(X, E^{\otimes m})$, the global inner product $\langle \alpha,\, u\rangle_{m,\Omega}$ and the norm $\|\alpha\|_{m,\Omega}$ are defined by

$$\langle \alpha,\, u\rangle_{m,\Omega} = \int_\Omega \alpha \wedge \star\, \#_{E^{\otimes m}}\, u,$$
$$\|\alpha\|^2_{m,\Omega} = \langle \alpha, \alpha\rangle_{m,\Omega}.$$

For $\alpha \in C^\infty_{r,s}(\Omega, E^{\otimes m})$ and $\eta \in \mathcal{D}_{r,s-1}(\Omega, E^{\otimes m})$, the formal adjoint operator ϑ_m of the operator $\overline{\partial} : C^\infty_{r,s-1}(\Omega, E^{\otimes m}) \longrightarrow C^\infty_{r,s}(\Omega, E^{\otimes m})$ is defined by:

$$\begin{aligned} \langle \vartheta_m \alpha, \eta\rangle_{m,\Omega} &= \langle \alpha, \overline{\partial}\eta\rangle_{m,\Omega}, \\ \vartheta_m &= -\#_{E^{\otimes m}} \star \overline{\partial} \star \#_{E^{\otimes m}}. \end{aligned} \tag{2.1}$$

Other notations are the following:

- $L^2_{r,s}(\Omega, E^{\otimes m})$ is the Hilbert space obtained by completing $C^\infty_{r,s}(\overline{\Omega}, E^{\otimes m})$ under the norm $\|\alpha\|^2_{m,\Omega}$.
- $\overline{\partial} : L^2_{r,s-1}(\Omega, E^{\otimes m}) \longrightarrow L^2_{r,s}(\Omega, E^{\otimes m})$ is the maximal closed extension of the the Cauchy-Riemann operator $\overline{\partial}$ and $\overline{\partial}^*_m$ its Hilbert space adjoint.
- $\square^m = \square^m_{r,s} = \overline{\partial}\,\overline{\partial}^*_m + \overline{\partial}^*_m\overline{\partial} : \mathrm{dom}(\square_{r,s}, E^{\otimes m}) \longrightarrow L^2_{r,s}(\Omega, E^{\otimes m})$ is the Laplace-Beltrami operator $\square^m$ for $E^{\otimes m}$-valued forms, where

$$\begin{aligned} \mathrm{dom}(\square^m_{r,s}, E^{\otimes m}) = \{u \in L^2_{r,s}(\Omega, E^{\otimes m}) : u \in \mathrm{dom}(\overline{\partial}, E^{\otimes m}) \cap \mathrm{dom}(\overline{\partial}^*_m, E^{\otimes m}); \\ \overline{\partial}u \in \mathrm{dom}(\overline{\partial}^*_m, E^{\otimes m}) \text{ and } \overline{\partial}^*_m u \in \mathrm{dom}(\overline{\partial}, E^{\otimes m})\}. \end{aligned}$$

- $\mathcal{H}^m_{r,s}(E^{\otimes m}) = \{u \in \mathrm{dom}(\square^m_{r,s}, E^{\otimes m}) : \overline{\partial}u = \overline{\partial}^*_m u = 0\}$ is a closed subspace of $\mathrm{dom}(\square^m_{r,s}, E^{\otimes m})$ since $\square^m_{r,s}$ is a closed operator.
- The $\overline{\partial}$-Neumann operator $N^m = N^m_{r,s} : L^2_{r,s}(\Omega, E^{\otimes m}) \longrightarrow L^2_{r,s}(\Omega, E^{\otimes m})$ is defined as the inverse of the restriction of $\square^m_{r,s}$ to $(\mathcal{H}^m_{r,s}(E^{\otimes m}))^\perp$, i.e.,

$$N^m_{r,s}u = \begin{cases} 0 & \text{if } u \in \mathcal{H}^m_{r,s}(E^{\otimes m}), \\ v & \text{if } u = \square^m_{r,s}v, \text{ and } v \perp \mathcal{H}^m_{r,s}(E^{\otimes m}). \end{cases}$$

  In other words, $N^m_{r,s}u$ is the unique solution v to the equations $\square^m_{r,s}v = u - \Pi^m_{r,s}\,u$ and $\Pi^m_{r,s}v = 0$, where $\Pi^m_{r,s} : L^2_{r,s}(\Omega, E^{\otimes m}) \longrightarrow \mathcal{H}^m_{r,s}(E^{\otimes m})$ is the orthogonal projection from the space $L^2_{r,s}(\Omega, E^{\otimes m})$ onto the space $\mathcal{H}^m_{r,s}(E^{\otimes m})$.

The following proposition is due to Hörmander [7] Propositions 1.2.3 and 1.2.4.

Proposition 1. *$\mathcal{B}_{r,s}(\overline{\Omega}, E^{\otimes m})$ is dense in dom$(\overline{\partial}^*_m, E^{\otimes m})$ (resp. dom$(\overline{\partial}, E^{\otimes m})$ $\cap$ dom$(\overline{\partial}^*_m, E^{\otimes m})$) with respect to the graph norm $(\|\alpha\|^2_m + \|\overline{\partial}^*_m\alpha\|^2_m)^{1/2}$ (resp. $(\|\alpha\|^2_m + \|\overline{\partial}\alpha\|^2_m + \|\overline{\partial}^*_m\alpha\|^2_m)^{1/2}$).*

The curvature form associated to the metric h is defined by $\Theta = \{\Theta_j\}$,

$$\Theta_j = \sqrt{-1}\,\overline{\partial}\partial \log h_j = \sqrt{-1}\sum_{\alpha,\beta=1}^{n} \Theta_{j\alpha\overline{\beta}}\, dz^\alpha_j \wedge d\overline{z}^\beta_j,$$

where $\Theta_{j\alpha\overline{\beta}} = -\frac{\partial^2 \log h_j}{\partial z^\alpha_j \partial \overline{z}^\beta_j}$ is the coefficients of the curvature form Θ associated to the metric h.

Definition 1. A holomorphic line bundle $\pi : E \longrightarrow X$ is said to be positive on a subset Ω of X if there exist a coordinate cover $\{U_j\}_{j\in J}$ of X such that $\pi^{-1}(U_j)$ are trivial and a hermitian metric $h = \{h_j\}$ along the fibres of E such that $-\log h_j$ is strictly plurisubharmonic on $U_j \cap \Omega$ for any $j \in J$.

By a complex tensor calculus for Kähler manifolds with boundary, one obtain the following theorem (see [15]).

Proposition 2. *Let X be a Kähler manifold of dimension n and let $\Omega \Subset X$ be an open subset of X. Assume that E is a holomorphic line bundle over X and $E^{\otimes m}$ is the m-times tensor product of E for positive integer m. Let U^* be a neighborhood of $b\Omega$ and let $\overline{\nabla}$ be the covariant differentiation associated to ds^2. If $m \geqslant 1$, we have*

$$\begin{aligned}
&\|\overline{\partial}\alpha\|_m^2 + \|\overline{\partial}_m^*\alpha\|_m^2 \\
&= \|\overline{\nabla}\alpha\|_m^2 + \int_{b\Omega} h_j^m |grad\,\rho|^{-1} \sum_{\beta,\gamma=1}^{n} \frac{\partial^2\rho}{\partial z^\beta \partial z^{\overline{\gamma}}}\ \alpha^{\beta}_{jC_r\overline{B}_{s-1}} \overline{\alpha_j^{\overline{C}_r\,\gamma B_{s-1}}}\, dS \qquad (2.2)\\
&\quad + \int_m h_j^m \sum_{\beta,\gamma=1}^{n} s\left(\delta_\tau^\sigma [m\Theta_{\overline{\alpha}}^{\overline{\beta}} + R_{\overline{\alpha}}^{\overline{\beta}}] - R_{\tau\overline{\alpha}}^{\sigma\overline{\beta}}\right) \times \alpha^{\beta}_{jC_r\overline{B}_{s-1}} \overline{\alpha_j^{\overline{C}_r\,\gamma B_{s-1}}}\, dV
\end{aligned}$$

for $\alpha \in \mathcal{B}_{r,s}(\overline{\Omega}, E^{\otimes m})$, such that supp $\alpha \Subset U^*$, $r \geqslant 0$, and $s \geqslant 1$, where

$$\|\overline{\nabla}\alpha\|_m^2 = \int_\Omega \sum_{\alpha,\beta=1}^{n} g_j^{\overline{\beta}\alpha} \overline{\nabla}_\beta \alpha_{jC_r\overline{D}_s} \overline{\overline{\nabla}_\alpha \alpha_j^{\overline{C}_r\, D_s}}\, dV,$$

$R^{\alpha}_{\beta\overline{\nu}\,\gamma} = -\dfrac{\partial}{\partial \overline{z}_j^{\nu}}\left(\sum g_j^{\overline{\sigma}\,\alpha} \dfrac{\partial}{\partial z_j^{\gamma}} g_{j\beta\overline{\sigma}}\right)$ *is the Riemann curvature tensor,*

$R_{\alpha\overline{\nu}} = -\dfrac{\partial^2}{\partial z_j^\alpha \partial \overline{z}_j^\nu}(\log\det g_{j\alpha\overline{\beta}})$ *is the Ricci curvature tensor,*

$\Theta_{\alpha\overline{\nu}} = -\dfrac{\partial^2}{\partial z_j^\alpha \partial \overline{z}_j^\nu}(\log h)$ *is the curvature tensor of E and,*

δ_τ^σ *denotes the Kronecker's delta.*

For a given boundary point $z_0 \in b\Omega$, we consider a boundary complex frame which means an orthonormal basis $dz^1, \ldots, dz^n = \partial\rho$ of $(1,0)$-forms with C^∞ coefficients on a small neighborhood U of z_0. We denote by $\left(\frac{\partial^2\rho(z)}{\partial z_j \overline{\partial} z_j}\right)$, $1 \leqslant i, j \leqslant n-1$, the matrix of the Levi form $\partial\overline{\partial}\rho(z)$ in the complex tangential direction at z with respect to the basis $dz^1, \ldots, dz^n$. Let $\lambda_1(z) \leqslant \cdots \leqslant \lambda_{n-1}(z)$ be the eigenvalues of $\left(\frac{\partial^2\rho(z)}{\partial z_j \overline{\partial} z_j}\right)$.

Definition 2 (cf. [6])**.** Let Ω be a smooth domain in $\mathbb{C}^n$ and ρ be its defining function, Ω is weakly q-convex ($q \geqslant 1$) if at every point $z \in b\Omega$ we have

$$\sum_{|K|}{}' \sum_{j,\,k} \rho_{jk}\, u_{jK}\, \overline{u_{kK}} \geqslant 0 \quad \text{for every } (0,q)\text{-form } u = \sum_{|J|=q} u_J\, \overline{\omega}^J \text{ such that}$$

$$\sum_{j=1}^{n} L_j(\rho)\, u_{jK} = 0 \ \text{ for all } \ |K| = q-1.$$

Lemma 1 (cf. [6])**.** *Let Ω be a smooth domain in $\mathbb{C}^n$ and ρ be its defining function. The following two conditions are equivalent*:

(1) Ω *is weakly q-convex.*

(2) *For any $z \in b\Omega$ the sum of any q eigenvalues $\rho_{i_1}, \dots, \rho_{i_q}$, with distinct subscripts, of the Levi-form at z satisfies $\sum_{j=1}^{q} \rho_{i_j} \geqslant 0$.*

Definition 3. $\alpha \in L^2_{r,s}(\Omega, E^{\otimes m})$ is supported in $\overline{\Omega}$ (supp $\alpha \subset \overline{\Omega}$) or α vanishes to infinite order at the boundary of Ω if α vanishes on $b\Omega$.

To prove the basic estimate (3.6), the following lemma which is Theorem 1.1.3 of [7] is needed.

Lemma 2. *Let $H_j (j = l, 2, 3)$ be three Hilbert spaces and $T : H_1 \longrightarrow H_2$ and $S : H_2 \longrightarrow H_3$ be closed linear operators with dense domains such that $ST = 0$. Assume that for any sequence $\{f_\nu\}$ such that $f_\nu \in H_2 \cap dom\, S \cap dom\, T$, $\|\alpha_\nu\|^2_{H_2} \leqslant 1$ and $\lim\limits_{\nu \longrightarrow \infty} \|S\alpha_\nu\|^2_{H_3} = 0$, $\lim\limits_{\nu \longrightarrow \infty} \|T\alpha_\nu\|^2_{H_1} = 0$, one can choose a strongly convergent subsequence of $\{f_\nu\}$. Then rangeange(T) is closed and $\mathcal{H}(S)/range(T)$ is a finite dimensional vector space.*

3. Proof of Theorem 1

Let X be an n-dimensional complex manifold and let $\Omega \Subset X$ be a weakly q-convex domain with smooth boundary $b\Omega$. Let $E \longrightarrow X$ be a holomorphic line bundle which is positive on a neighborhood V of $b\Omega$. Let $h = \{h_j\}$ be the metric of E on X which gives the positivity of E on V with respect to a suitable covering $\{U_j\}_{j\in J}$ of X. Then the curvature form $\sum_{\alpha,\beta=1}^{n} \left(-\frac{\partial^2 \log h_j}{\partial z_j^\alpha \partial \overline{z}_j^\beta}\right) dz^\alpha \wedge d\overline{z}^\beta$ of a provides a Kähler metric $d\sigma^2 = \sum_{\alpha,\beta=1}^{n} \left(-\frac{\partial^2 \log h_j}{\partial z_j^\alpha \partial \overline{z}_j^\beta}\right) dz^\alpha\, d\overline{z}^\beta$ on V. We may assume that the defining function ρ of $b\Omega$ is constructed from the geodesic distance with respect to the metric $d\sigma^2$ and we obtain the following lemma.

Lemma 3. *There exist neighborhoods V and V' of $b\Omega$, a coordinate covering $\{U_j\}_{j\in J}$ of X, a fibre metric $h = \{h_j\}$ of E on X and a hermitian metric $ds^2 = \sum_{\alpha,\beta=1}^{n} g_{j\alpha\overline{\beta}}(z)\, dz_j^\alpha\, d\overline{z}_j^\beta$ on X such that*

1) *$V \Subset V'$ and $\overline{V}'$ is contained in a smooth product neighborhood of $b\Omega$,*

2) *$\pi^{-1}(\overline{U}_j)$ is trivial for any $j \in J$ and $U_j \Subset V$ if $U_j \cap b\Omega \neq \varnothing$,*

3) *E is positive on V' with respect to h,*

4) *the restriction of* ds^2 *onto* V' *coincides with the Kähler metric* $d\sigma^2$.

Under the situation of Lemma 3, one obtain the following estimate (see Appendix II in [16]).

Proposition 3. *There exist a positive constant* C *not depending on* m *and a positive integer* m_0 *such that for any* $m \geqslant m_0$, $r \geqslant 0$, $s \geqslant q$, *we have*

$$\|\overline{\nabla}\alpha\|^2_{m,\Omega\setminus K}+(m-m_0)\|\alpha\|^2_{m,\Omega\setminus K} \leqslant C(\|\overline{\partial}\alpha\|^2_{m,\Omega}+\|\overline{\partial}^*_m\alpha\|^2_{m,\Omega}+\|\alpha\|^2_{m,K}), \tag{3.1}$$

where K *is the compact subset of* Ω *defined by* $K = \Omega\setminus(\Omega \cap V)$ *and* $\overline{\nabla}$ *is the covariant differentiation of type* $(0, 1)$ *associated to the metric* ds^2.

Proof. In the situation of Lemma 3, assume that χ is a C^∞-function on X such that $\operatorname{supp}\chi \Subset V'$ and $\chi = 1$ on $\overline{V}$. Then one can apply the formula (2.2) to $\chi\alpha$. Since the third term of the right-hand side of (2.2) is non-negative by the weakly q-convexity of $b\Omega$ for $s \geqslant q$, one obtain

$$\begin{aligned} &\|\overline{\nabla}(\chi\alpha)\|^2_m + \int_m h^m \sum_{\beta,\gamma=1}^{n} s\left(\delta^\sigma_\tau[m\Theta^{\overline{\beta}}_{\overline{\alpha}} + R^{\overline{\beta}}_{\overline{\alpha}}] - rR^{\sigma\overline{\beta}}_{\tau\overline{\alpha}}\right) \\ &\times (\chi\alpha)^\beta_{j,C_p\overline{B}_{s-1}} \overline{(\chi\alpha)^{\overline{C}_p\,\gamma B_{s-1}}_j}\, dV \leqslant \|\overline{\partial}(\chi\alpha)\|^2_m + \|\overline{\partial}^*_m(\chi\alpha)\|^2_m. \end{aligned} \tag{3.2}$$

Since the integrand of the first term of the left-hand side of (3.2) is nonnegative on V', one obtain

$$\|\overline{\nabla}\alpha\|^2_{m,\Omega\setminus K} \leqslant \|\overline{\nabla}(\chi\alpha)\|^2_m, \tag{3.3}$$

where $K = \Omega\setminus(\Omega \cap V)$. From the construction of ds^2, the matrix $(g_{j\alpha\overline{\beta}})$ coincides with the one $(\Theta_{\alpha\overline{\beta}})$ at each point of V'. Hence

$$\Theta^{\overline{\beta}}_{\overline{\alpha}} = \sum_{\gamma=1}^{n} g^{\overline{\beta}\gamma}_j \Theta_{\gamma\overline{\alpha}} = \delta^\beta_\alpha.$$

Also, at each point of $\operatorname{supp}\chi$, there exists a positive constant C not depending on m such that the hermitian form

$$\sum_{\beta,\gamma=1}^{n} s\left(\delta^\sigma_\tau R^{\overline{\beta}}_{\overline{\alpha}} - rR^{\sigma\overline{\beta}}_{\tau\overline{\alpha}}\right)(\chi\alpha)_{j,\sigma C_{r-1}\overline{\beta}\,\overline{D}_{s-1}} \overline{(\chi\alpha)^{\overline{\tau}\overline{C}_{r-1}\,\alpha D_{s-1}}_j}$$

is greater than

$$-C\sum(\chi\alpha)_{j,C_r\overline{D}_s}\overline{(\chi\alpha)^{\overline{C}_r\,D_s}_j}.$$

Setting $m_0 = [C] + 1$ for every $m \geqslant m_0$, one obtain

$$\begin{aligned} (m-m_0)\|\alpha\|^2_{m,\Omega\setminus K} &\leqslant (m-m_0)\|\chi\alpha\|^2_m \\ &\leqslant \int_m h^m \sum_{\beta,\gamma=1}^{n} s\left(\delta^\sigma_\tau[m\Theta^{\overline{\beta}}_{\overline{\alpha}} + R^{\overline{\beta}}_{\overline{\alpha}}] - rR^{\sigma\overline{\beta}}_{\tau\overline{\alpha}}\right) \\ &\quad\times (\chi\alpha)^\beta_{jC_r\overline{B}_{s-1}} \overline{(\chi\alpha)^{\overline{C}_r\,\gamma B_{s-1}}_j} dV. \end{aligned} \tag{3.4}$$

Moreover we have

$$
\begin{aligned}
&\|\overline{\partial}(\chi\alpha)\|_m^2 + \|\overline{\partial}_m^*(\chi\alpha)\|_{m,\Omega}^2 \\
&\leqslant 2(\|\overline{\partial}\chi\wedge\alpha\|_m^2 + \|\overline{\partial}\chi\wedge\star\alpha\|_m^2 + \|\chi\overline{\partial}\alpha\|_m^2 + \|\chi\overline{\partial}_m^*\alpha\|_m^2) \qquad (3.5)\\
&\leqslant C(\|\overline{\partial}\alpha\|_m^2 + \|\overline{\partial}_m^*\alpha\|_m^2 + \|\alpha\|_{m,\Omega\setminus K}^2)
\end{aligned}
$$

for a positive constant $C \geqslant 4\cdot\max\{l, c_0\cdot\sup|\mathrm{grad}\,\chi|_{ds^2}(x)\}$ and $m \geqslant 1$ where c_0 is a positive constant depending only on the dimension of X. From (3.3), (3.4) and (3.5) into (3.2), we obtain the desired estimate. □

Proposition 4. *There exists a positive constant m_* such that for any $m \geqslant m_*$, the harmonic space $\mathcal{H}_{r,s}^m(E^{\otimes m})$ has finite dimension and there exists a positive constant C_m depending on m such that*

$$
\|\alpha\|_m^2 \leqslant C_m(\|\overline{\partial}\alpha\|_m^2 + \|\overline{\partial}_m^*\alpha\|_m^2) \qquad (3.6)
$$

for $\alpha \in dom\,(\overline{\partial}, E^{\otimes m}) \cap dom\,(\overline{\partial}_m^*, E^{\otimes m})$ with $s \geqslant q$.

Proof. Let m_0, C and K be the same as in Proposition 3, then we determine a positive integer m_* as $m_* = m_0 + 1$. As in Proposition 3, let χ be a real-valued C^∞-function on X such that supp $\chi \Subset X$ and $\chi = 1$ on K. If $m \geqslant m_*$ and $\alpha \in \mathcal{B}_{r,s}(\overline{\Omega}, E^{\otimes m})$, then from (3.1), we obtain the following estimate:

$$
\|\alpha\|_m^2 \leqslant C_m(\|\overline{\partial}\alpha\|_m^2 + \|\overline{\partial}_m^*\alpha\|_m^2 + \|\chi\alpha\|_m^2),
$$

where C_m is a positive constant depending on m.

Take any sequence $\{\alpha_\nu\}$ such that $\alpha_\nu \in dom\,\overline{\partial} \cap dom\,\overline{\partial}_m^*$, $\|\alpha_\nu\|^2 \leqslant 1$ and $\lim_{\nu\longrightarrow\infty}\|\overline{\partial}\alpha_\nu\|_m^2 = 0$, $\lim_{\nu\longrightarrow\infty}\|\overline{\partial}_m^*\alpha_\nu\|_m^2 = 0$. Then, from Lemma 2 there exists a subsequence $\{\alpha_{\nu_k}\}$ of $\{\alpha_\nu\}$ which converges strongly on Ω. In fact ds^2 is complete, $\mathcal{D}_{r,s}(\Omega, E^{\otimes m})$ is dense in $dom\,\overline{\partial} \cap dom\,\overline{\partial}_m^*$ with respect to the norm $\|\alpha\|_m^2 + \|\overline{\partial}\alpha\|_m^2 + \|\overline{\partial}_m^*\alpha\|_m^2$ ([17], Theorem 1.1). Hence we may assume $\chi\alpha_\nu \in \mathcal{D}_{r,s}(\Omega, E^{\otimes m})$. Thus

$$
\|\overline{\partial}(\chi\alpha_\nu)\|_m^2 + \|\overline{\partial}_m^*(\chi\alpha_\nu)\|_m^2 + \|(\chi\alpha_\nu)\|_m^2 = \langle\Box^m(\chi\alpha_\nu), \chi\alpha_\nu\rangle_m + \langle\chi\alpha_\nu, \chi\alpha_\nu\rangle_m
$$

is bounded by the assumption. From coerciveness of elliptic differential operator $\Box^m$ on $\mathcal{D}_{r,s}(\Omega, E^{\otimes m})$ (cf. [4], (2.2.1) Theorem) and Rellich's lemma (cf. [4], Appendix (A.1.6) Proposition), it follows that $\{\alpha_\nu\}$ has a subsequence $\{\alpha_{\nu_k}\}$ which is strongly convergent on compact subset K of Ω. By (3.1), we conclude that $\{\alpha_{\nu_k}\}$ converges strongly on Ω. Thus, by Hörmander [7] Theorem 1.1.2 and Theorem 1.1.3, there exists a positive constant C_m such that

$$
\|\alpha\|_m^2 \leqslant C_m(\|\overline{\partial}\alpha\|_m^2 + \|\overline{\partial}_m^*\alpha\|_m^2) \qquad (3.7)
$$

for $\alpha \in dom\,(\overline{\partial}, E^{\otimes m}) \cap dom\,(\overline{\partial}_m^*, E^{\otimes m})$ with $\alpha \perp \mathcal{H}_{r,s}^m(E^{\otimes m})$, while each element α in $\mathcal{H}_{r,s}^m(E^{\otimes m})$ is a solution of the operator $\Box^m$. Namely α is a harmonic form with valued in $E^{\otimes m}$. Now, from (3.1), α vanishes identically on $\Omega\setminus K$. Since any connected component of Ω is not contained in K, by the above unique continuation property, α vanishes on each connected component and so

α vanishes identically on Ω. Hence $\mathcal{H}^m_{r,s}(E^{\otimes m})$ is the null space. Combining this with (3.7), the proof is completed. □

Remark 1. If there exists a strongly plurisubharmonic function ϕ on a neighborhood V of $b\Omega$, then any line bundle E is positive on a relatively compact neighborhood of $b\Omega$. In fact let h be a metric of E over X and extend ϕ to a C^∞-function Φ on X without changing the original near $b\Omega$ in a suitable manner. Then there exists a positive integer m^* such that $h_m = he^{-m\Phi}$ gives the positivity of E on a relatively compact neighborhood $V'(\Subset V)$ of $b\Omega$ for every $m \geqslant m^*$.

Remark 2. There are pseudoconvex domains with smooth boundary $b\Omega$ not possessing such a strongly plurisubharmonic function on any neighborhood of $b\Omega$ but possessing a line bundle which is positive on a neighborhood of $b\Omega$ (cf. [5]).

Theorem 2. *Let X be a complex manifold of dimension $n \geq 2$ and let $\Omega \Subset X$ be a weakly q-convex domain with smooth boundary in X. Assume that E is a holomorphic line bundle over X and $E^{\otimes m}$ is the m-times tensor product of E for positive integer m. Suppose that there exists a strongly plurisubharmonic function on a neighborhood of $b\Omega$. Then there exists a positive integer m^* such that, for $m \geqslant m^*$, $r \geqslant 0$, $s \geqslant q$, there exists a bounded linear operator $N^m : L^2_{r,s}(\Omega, E^{\otimes m}) \longrightarrow L^2_{r,s}(\Omega, E^{\otimes m})$ such that*

(i) *$range(N^m, E^{\otimes m}) \subset dom(\Box^m, E^{\otimes m})$,*
$N^m\Box^m = I - \Pi^m$ on $dom(\Box^m, E^{\otimes m})$,

(ii) *for $\alpha \in L^2_{r,s}(\Omega, E^{\otimes m})$, we have*

$$\alpha = \overline{\partial}\,\overline{\partial}^*_m N^m\alpha \oplus \overline{\partial}^*_m\overline{\partial}N^m\alpha \oplus \Pi^m\alpha,$$

(iii) *$N^m\overline{\partial} = \overline{\partial}N^m$ on $dom(\overline{\partial}, E^{\otimes m})$ and*

(iv) *$N^m\overline{\partial}^*_m = \overline{\partial}^*_m N^m$ on $dom(\overline{\partial}^*_m, E^{\otimes m})$,*

(v) *N^m, $\overline{\partial}N^m$, $\overline{\partial}^*_m N^m$ are bounded operators on $L^2_{r,s}(\Omega, E^{\otimes m})$.*

Proof. From (3.6), we obtain

$$\|\alpha\|_m \leqslant C_m\|\Box^m\alpha\|_m \tag{3.8}$$

for $\alpha \in dom\,(\overline{\partial}, E^{\otimes m}) \cap dom\,(\overline{\partial}^*_m, E^{\otimes m})$ $\mathrm{dom}\,\overline{\partial}^*_m$ with $s \geqslant q$. Since $\Box^m$ is a linear closed densely defined operator, then, from [7]; Theorem 1.1.1, range$(\Box^m, E^{\otimes m})$ is closed. Thus, from (1.1.1) in [7]] and the fact that $\Box^m$ is self adjoint, we have the Hodge decomposition

$$L^2_{r,s}(\Omega, E^{\otimes m}) = \overline{\partial}\,\overline{\partial}^*_m \mathrm{dom}\,\Box^m \oplus \overline{\partial}^*_m\overline{\partial}\,\mathrm{dom}\,\Box^m.$$

Since $\Box^m : \mathrm{dom}(\Box^m, E^{\otimes m}) \longrightarrow \mathrm{range}(\Box^m, E^{\otimes m}) = L^2_{r,s}(\Omega, E^{\otimes m})$ is one to one on dom $(\Box^m, E^{\otimes m})$ from (3.98), there exists a unique bounded inverse operator

$$N^m : L^2_{r,s}(\Omega, E^{\otimes m}) \longrightarrow \mathrm{dom}(\Box^m, E^{\otimes m})$$

such that $N^m\Box^m\alpha = \alpha$ on $\mathrm{dom}\,(\Box^m, E^{\otimes m})$. Also, from the definition of N^m, we obtain $\Box^m N^m = I$ on $L^2_{r,s}(\Omega, E^{\otimes m})$. Thus (i) and (ii) are satisfied. To show that $\overline{\partial}^*_m N^m = N^m\overline{\partial}^*_m$ on $\mathrm{dom}(\overline{\partial}^*_m, E^{\otimes m})$, by using (ii), we have $\overline{\partial}^*_m\alpha = \overline{\partial}^*_m\overline{\partial}\,\overline{\partial}^*_m N^m\alpha$ for $\alpha \in \mathrm{dom}\,(\overline{\partial}^*_m, E^{\otimes m})$. Thus

$$N^m\overline{\partial}^*_m\alpha = N^m\overline{\partial}^*_m\,\overline{\partial}\,\overline{\partial}^*_m N^m\alpha = N^m(\overline{\partial}^*_m\,\overline{\partial} + \overline{\partial}\,\overline{\partial}^*_m)\,\overline{\partial}^*_m N^m\alpha = \overline{\partial}^*_m N^m\alpha.$$

A similar argument shows that $\overline{\partial}N^m = N^m\overline{\partial}$ on $\mathrm{dom}\,\overline{\partial}$. By using (iii) and the condition on α, $\overline{\partial}\alpha = 0$, we have $\overline{\partial}N^m\alpha = N^m\overline{\partial}\alpha = 0$. Then, by using (ii), we obtain $\alpha = \overline{\partial}\,\overline{\partial}^*_m N^m\alpha$. Thus the form $u = \overline{\partial}^*_m N^m\alpha$ satisfies the equation $\overline{\partial}u = \alpha$. Since $\mathrm{Rang}(N^m, E^{\otimes m}) \subset \mathrm{dom}(\Box^m, E^{\otimes m})$, then by applying (3.6) to $N^m\alpha$ instead of α, we obtain

$$\|N^m\alpha\|_m \leq C_m\|\alpha\|_m,$$
$$\|\overline{\partial}N^m\alpha\|_m + \|\overline{\partial}^*_m N^m\alpha\|_m \leq 2\sqrt{C_m}\,\|\alpha\|_m.$$

Thus the proof follows. □

Theorem 3. *Under the same assumption of Theorem 2, for $\alpha \in L^2_{r,s}(X, E^{\otimes m})$, $\mathrm{supp}\,\alpha \subset \overline{\Omega}$, with $s \geqslant q$, satisfying $\overline{\partial}\alpha = 0$ in the distribution sense in X, there exists $u \in L^2_{r,s-1}(X, E^{\otimes m})$, $\mathrm{supp}\,u \subset \overline{\Omega}$ such that $\overline{\partial}\,u = \alpha$ in the distribution sense in X.*

Proof. Let $\alpha \in L^2_{r,s}(X, E^{\otimes m})$, $\mathrm{supp}\,\alpha \subset \overline{\Omega}$, then $\alpha \in L^2_{r,s}(\Omega, E^{\otimes m})$. Following Theorem 2, $N^m_{n-r,n-s}$ exists for $n - s \geqslant q$. Thus, one can define $u \in L^2_{r,s-1}(\Omega, E^{\otimes m})$ by

$$u = -\star\,\#_{E^{\otimes m}}\,\overline{\partial}\,N^m_{n-r,n-s}\#_{E^{\otimes m}}\,\star\,\alpha. \tag{3.9}$$

Extend u to X by defining $u = 0$ in $X \setminus \overline{\Omega}$. To prove that u satisfies $\overline{\partial}\,u = \alpha$ in the distribution sense in X, we first prove that $\overline{\partial}\,u = \alpha$ in the distribution sense in Ω.

For $\eta \in \mathrm{dom}(\overline{\partial}, E^{*\otimes m})$, we have

$$\langle\overline{\partial}\eta, \#_{E^{\otimes m}}\star\alpha\rangle_{m,\Omega} = (-1)^{r+s}\langle\alpha, \#_{E^{*\otimes m}}\star\overline{\partial}\,\eta\rangle_{m,\Omega}.$$

From the density of the space $\mathcal{B}_{r,s}(\overline{\Omega}, E^{\otimes m})$ in $\mathrm{dom}\,(\overline{\partial}, E^{\otimes m}) \cap \mathrm{dom}\,(\overline{\partial}^*, E^{\otimes m})$ in the graph norm (cf. Proposition 1) and since $\vartheta^m = \overline{\partial}^*_m$ on $\mathcal{B}_{r,s}(\overline{\Omega}, E^{\otimes m})$, when ϑ^m acts in the distribution sense, we have from (2.1) that

$$\langle\overline{\partial}\eta, \#_{E^{\otimes m}}\star\alpha\rangle_{m,\Omega} = \langle\alpha, \overline{\partial}^*_m\,\#_{E^{*\otimes m}}\star\eta\rangle_{m,\Omega}.$$

Since $\mathrm{supp}\,\alpha \subset \overline{\Omega}$, then we obtain

$$\langle\overline{\partial}\eta, \#_{E^{\otimes m}}\star\alpha\rangle_{m,\Omega} = \langle\alpha, \overline{\partial}^*_m\,\#_{E^{*\otimes m}}\star\eta\rangle_{m,\Omega} = \langle\overline{\partial}\alpha, \#_{E^{*\otimes m}}\star\eta\rangle_{m,X} = 0.$$

It follows that

$$\overline{\partial}^*_m(\#_{E^{\otimes m}}\star\alpha) = 0 \quad \text{on } \Omega.$$

Using Theorem 2(iv), we have

$$\overline{\partial}^*_m N^m_{n-r,n-s}(\#_{E^{\otimes m}} \star \alpha) = N^m_{n-r,n-s-1}\overline{\partial}^*_m(\#_{E^{\otimes m}} \star \alpha) = 0. \tag{3.10}$$

Thus, in the distribution sense in Ω and from (2.1), (3.9) and (3.10), we obtain

$$\begin{aligned}\overline{\partial}\, u &= -\,\overline{\partial} \star \#_{E^{*\otimes m}}\, \overline{\partial} N^m_{n-r,n-s}\, \#_{E^{\otimes m}} \star \alpha \\ &= (-1)^{r+s} \star \#_{E^{*\otimes m}} \overline{\partial}^*_m \overline{\partial} N^m_{n-r,n-s} \#_{E^{\otimes m}} \star \alpha \\ &= (-1)^{r+s} \star \#_{E^{*\otimes m}} (\overline{\partial}^*_m \overline{\partial} + \overline{\partial}\,\overline{\partial}^*_m)\, N^m_{n-r,n-s} \#_{E^{\otimes m}} \star \alpha \\ &= (-1)^{r+s} \star \#_{E^{*\otimes m}} \#_{E^{\otimes m}} \star \alpha \\ &= \alpha.\end{aligned} \tag{3.11}$$

Because $u = 0$ in $X \setminus \Omega$, then for $\eta \in \mathrm{dom}(\overline{\partial}^*_m, E^{\otimes m}) \subset L^2_{r,s}(X, E^{\otimes m})$, one obtain

$$\langle\, u, \overline{\partial}^*_m \eta\rangle_{m,X} = \langle\, u, \overline{\partial}^*_m \eta\rangle_{m,\Omega} = \langle \#_{E^{\otimes m}} \star \overline{\partial}^*_m \eta, \#_{E^{\otimes m}} \star\, u\rangle_{m,\Omega}.$$

Since

$$\#_{E^{\otimes m}} \star\, u = (-1)^{r+s+1}\, \overline{\partial} N^m_{n-r,n-s} \#_{E^{\otimes m}} \star \alpha \in \mathrm{dom}(\overline{\partial}^*_m, E^{*\otimes m}).$$

Thus, from (2.1), we obtain

$$\begin{aligned}\langle\, u, \overline{\partial}^*_m \eta\rangle_{m,X} &= (-1)^{r+s} \langle \overline{\partial} \#_{E^{\otimes m}} \star \eta, \#_{E^{\otimes m}} \star\, u\rangle_{m,\Omega} \\ &= \langle \#_{E^{\otimes m}} \star \eta, \#_{E^{\otimes m}} \star \overline{\partial}\, u\rangle_{m,\Omega} = \langle \overline{\partial}\, u, \eta\rangle_{m,\Omega}.\end{aligned}$$

Thus, from (3.11),

$$\langle\, u, \overline{\partial}^*_m \eta\rangle_{m,X} = \langle \alpha, \eta\rangle_{m,\Omega} = \langle \alpha, \eta\rangle_{m,X}.$$

Thus $\overline{\partial}\, u = \alpha$ in the distribution sense in X. □

4. Solvability of the $\overline{\partial}_b$-problem

In this section, applications to the solvability of the $\overline{\partial}_b$-problem are given.

Theorem 4. *Let X be a Kähler manifold of dimension $n \geq 2$ and let $\Omega \Subset X$ be a weakly q-convex domain with smooth boundary in X. Let E be a holomorphic line bundle over X and $E^{\otimes m}$ be the m-times tensor product of E for positive integer m. Suppose that there exists a strongly plurisubharmonic function on a neighborhood of $b\Omega$. Then, for $f \in C^\infty_{r,s}(b\Omega, E^{\otimes m})$, $q \leqslant s \leqslant n-2$, satisfying $\overline{\partial}_b f = 0$, there exists $F \in C^\infty_{r,s}(\overline{D}, E^{\otimes m})$ such that $F|_{b\Omega} = f$ and $\overline{\partial} F = 0$.*

Proof. The proof follows as in Theorem 4.1 in Saber [12]. □

Theorem 5. *Under the same assumption of Theorem 4, if $f \in C^\infty_{r,s}(b\Omega, E^{\otimes m})$, $1 \leq s \leq n-2$, with $\overline{\partial}_b f = 0$, there exists $u \in C^\infty_{r,s-1}(b\Omega, E^{\otimes m})$ such that $\overline{\partial}_b u = f$.*

Proof. Let $f \in C^{\infty}_{r,s}(b\Omega, E^{\otimes m})$, $1 \leq s \leq n-2$, with $\overline{\partial}_b f = 0$. Then from Theorem 4, there exists $F \in C^{\infty}_{r,s}(\overline{\Omega}, E^{\otimes m})$ such that $F|_{b\Omega} = f$ and $\overline{\partial} F = 0$. Following Theorem 3, there exists $U \in C^{\infty}_{r,s-1}(\overline{\Omega}, E^{\otimes m})$ satisfying $\overline{\partial} U = f$ in Ω. Then $u = U|_{b\Omega}$ satisfies $\overline{\partial}_b u = f$. □

Corollary 6. *Let X be a Kähler manifold of dimension $n \geq 2$ and let $D \Subset X$ be a weakly q-concave domain with smooth boundary in X. Let E be a holomorphic line bundle over X and $E^{\otimes m}$ be the m-times tensor product of E for positive integer m. Suppose that there exists a strongly plurisubharmonic function on a neighborhood of $b\Omega$. If $H^{r,s}(X, E^{\otimes m}) = 0$, then, for $f \in C^{\infty}_{r,s}(\overline{D}, E^{\otimes m})$, $\overline{\partial} f = 0$, $q \leqslant s \leqslant n-2$, there exists $u \in C^{\infty}_{r,s-1}(\overline{D}, E^{\otimes m})$ such that $\overline{\partial} u = f$.*

Proof. The proof follows as in Corollary 4.3 in Saber [12]. □

The necessary and sufficient condition on $f \in W^{\frac{1}{2}}_{r,s}(b\Omega, E^{\otimes m})$ to have a $\bar{\partial}$-closed extension F on Ω is summarized as follows.

Theorem 7. *Let Ω, E and X be the same as in Theorem 4.1. For $f \in W^{\frac{1}{2}}_{r,s}(b\Omega, E^{\otimes m})$, $0 \leq r \leq n$, $q \leq s \leq n-2$. We assume that $\bar{\partial}_b f = 0$. Then there exists $F \in L^2_{r,s-1}(\Omega, E^{\otimes m})$ such that $F = f$ on $b\Omega$ and $\bar{\partial} F = 0$ in Ω.*

Proof. The proof follows as in Theorem 4.4 in Saber [12]. □

5. Extension from the boundary

Let X be a connected complex manifold of dimension $n \geq 2$, and let $\Omega \subset X$ be any domain with C^{∞}-smooth boundary. Let E be a holomorphic vector bundle over X. In this section we prove the following results:

Lemma 4. *For any $\alpha \in C^{\infty}_{r,s}(b\Omega, E)$ satisfying $\overline{\partial}_b f = 0$, there exists $\widetilde{\alpha} \in C^{\infty}_{r,s}(\overline{\Omega}, E)$ such that $\widetilde{\alpha}|_{b\Omega} = \alpha$ and that $\overline{\partial}\widetilde{\alpha}$ vanishes to the infinite order on $b\Omega$.*

Proof. The proof follows as in Lemma 4 in Oshawa [9]. □

By virtue of a theory of Kodaira-Andreotti-Vesentini (cf. Kodaira [8], Andreotti and Vesentini [1]), we can show that a sufficient condition for the C^k-extendability can be stated as follows.

Lemma 5. *Let X be a connected Kähler manifold of dimension n and let $\Omega \Subset X$ be a weakly q-convex domain with C^{∞}-smooth boundary. Let E be a holomorphic vector bundle over X. Suppose that Ω admits a C^{∞} defining function ρ such that*

$$\partial\overline{\partial}(-\log(-\rho)) \geq c\left(\partial(-\log(-\rho))\overline{\partial}(-\log(-\rho)) + \omega\right).$$

holds on Ω for some positive constant c. Then, for any $\psi \in C^{\infty}_{r,s}(b\Omega, E) \cap \ker(\overline{\partial}_b, E)$ with $s < n-1$, and for any nonnegative integer k, there exists a $\overline{\partial}$-closed E-valued (r,s)-form Ψ_k of class C^k on $\overline{\Omega}$ satisfying $\Psi_k|_{b\Omega} = \psi$.

Proof. The proof follows as in Theorem 5 in Oshawa [9]. □

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