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GRADED PRIMITIVE AND INC-EXTENSIONS

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Abstract. It is well-known that quasi-Prüfer domains are characterized as those domains D, such that every extension of D inside its quotient field is a primitive extension and that primitive extensions are characterized in terms of INC-extensions.

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain graded by an arbitrary torsionless grading monoid Γ and $\star$ be a semistar operation on R. The main purpose of this paper is to give new characterizations of gr-$\star$-quasi-Prüfer domains in terms of graded primitive and INC-extensions. Applications include new characterizations of UMt-domains.

1. Introduction

Let D be a (commutative) integral domain with quotient field $qf(D)$. Recall that D is called a *quasi-Prüfer domain* if D has Prüfer integral closure [8], and as a t-operation analogue, D is called a *UMt-domain* if every upper to zero in the polynomial ring $D[X]$ is a maximal t-ideal [12]. Gilmer and Hoffmann characterized quasi-Prüfer domains as those domains D, such that the embedding $D \subseteq qf(D)$ is a primitive-extension [10, Theorem 2], and Dobbs [6] characterized primitive-extensions in terms of INC-domains.

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded (commutative) integral domain graded by an arbitrary grading torsionless monoid Γ. In [11] the authors studied quasi-Prüfer and UMt-domain properties of graded integral domains. For this reason they introduced the graded analogue of $\star$-quasi-Prüfer domains [4] called gr-$\star$-quasi-Prüfer domains. The graded integral domain R is called a *gr-$\star$-quasi-Prüfer domain* in case, if Q is a prime ideal in $R[X]$ and $Q \subseteq P[X]$, for some homogeneous quasi-$\star$-prime ideal P of R, then $Q = (Q \cap R)[X]$. When $\star = d$ the identity operation on R, then we call the gr-d-quasi-Prüfer domain a *gr-quasi-Prüfer domain*. It is shown that R is a gr-$\star$-quasi-Prüfer domain if and only if R_P is a quasi-Prüfer domain, for each homogeneous quasi-$\star$-prime ideal P of R [11, Proposition 2.2]. Also it is known that R is a UMt-domain if and only if R is a gr-t-quasi-Prüfer domain if and only if R_P is a quasi-Prüfer

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domain for each *homogeneous* prime t-ideal P of R [11, Theorem 3.2]. If $\star$ is a (semi)star operation on R, then R is a gr-$\star_f$-quasi-Prüfer domain if and only if R is a UMt-domain and $\widetilde{\star}$ and w coincide on nonzero homogeneous ideals of R [11, Theorem 3.9]. In particular R is a gr-quasi-Prüfer domain if and only if R is a UMt-domain and d and w coincide on nonzero homogeneous ideals of R. (Relevant definitions are reviewed in the sequel.)

The main purpose of this paper is to give new characterizations of gr-$\star$-quasi-Prüfer domains in terms of graded primitive extension and graded incomparable or INC-extension (see [3], [6] and [10]).

To facilitate the reading of the paper, we review some basic facts on semistar operations on (graded) integral domains. Let Γ be a nonzero torsionless grading monoid, that is, Γ is a torsionless commutative cancellative monoid (written additively), and $\langle \Gamma \rangle = \{a - b \mid a, b \in \Gamma\}$ be the quotient group of Γ; so $\langle \Gamma \rangle$ is a torsionfree abelian group. It is known that a cancellative monoid is torsionless if and only if it can be given a total order compatible with the monoid operation [14, page 123]. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a Γ-graded integral domain. That is, $\deg(x) = \alpha$ for each $0 \neq x \in R_\alpha$ and $\deg(0) = 0$, and thus each nonzero $f \in R$ can be written uniquely as $f = x_{\alpha_1} + \cdots + x_{\alpha_n}$ with $\deg(x_{\alpha_i}) = \alpha_i$ and $\alpha_1 < \cdots < \alpha_n$. A nonzero $x \in R_\alpha$ for all $\alpha \in \Gamma$ is said to be *homogeneous*, and so if $H = \bigcup_{\alpha \in \Gamma}(R_\alpha \setminus \{0\})$, then H is the saturated multiplicative set of nonzero homogeneous elements of R. Then $R_H = \bigoplus_{\alpha \in \langle \Gamma \rangle}(R_H)_\alpha$, called the *homogeneous quotient field of* R, is a $\langle \Gamma \rangle$-graded integral domain whose nonzero homogeneous elements are units. An integral ideal I of R is said to be homogeneous if $I = \bigoplus_{\alpha \in \Gamma}(I \cap R_\alpha)$. A fractional ideal I of R is *homogeneous* if sI is an integral homogeneous ideal of R for some $s \in H$ (thus $I \subseteq R_H$). An overring T of R, with $R \subseteq T \subseteq R_H$ will be called a *homogeneous overring* if $T = \bigoplus_{\alpha \in \langle \Gamma \rangle}(T \cap (R_H)_\alpha)$. Thus T is a ($\langle \Gamma \rangle$-)graded integral domain with $T_\alpha = T \cap (R_H)_\alpha$ for all $\alpha \in \langle \Gamma \rangle$. For more on graded integral domains and their divisibility properties (see [1], [14]).

Let D be an integral domain with quotient field K. Let $\overline{\mathcal{F}}(D)$ denote the set of all nonzero D-submodules of K, $\mathcal{F}(D)$ be the set of all nonzero fractional ideals of D, and $f(D)$ be the set of all nonzero finitely generated fractional ideals of D. Obviously, $f(D) \subseteq \mathcal{F}(D) \subseteq \overline{\mathcal{F}}(D)$. As in [15], a *semistar operation on* D is a map $\star : \overline{\mathcal{F}}(D) \to \overline{\mathcal{F}}(D)$, $E \mapsto E^\star$, such that, for all $0 \neq x \in K$, and for all $E, F \in \overline{\mathcal{F}}(D)$, the following properties hold: $(\star_1)$ $(xE)^\star = xE^\star$; $(\star_2)$: $E \subseteq F$ implies that $E^\star \subseteq F^\star$; $(\star_3)$ $E \subseteq E^\star$; and $(\star_4)$ $E^{\star\star} := (E^\star)^\star = E^\star$.

A semistar operation $\star$ is called a *(semi)star operation on* D, if $D^\star = D$. Let $\star$ be a semistar operation on D. For every $E \in \overline{\mathcal{F}}(D)$, put $E^{\star_f} := \bigcup F^\star$, where the union is taken over all $F \in f(D)$ with $F \subseteq E$. It is easy to see that $\star_f$ is a semistar operation on D. We say that a nonzero ideal I of D is a *quasi-$\star$-ideal* of D, if $I^\star \cap D = I$; a *quasi-$\star$-prime* (ideal of D), if I is a prime quasi-$\star$-ideal of D; and a *quasi-$\star$-maximal* (ideal of D), if I is maximal in the set of all proper quasi-$\star$-ideals of D. Each quasi-$\star$-maximal ideal is a prime ideal. It is shown in [7, Lemma 4.20] that if $D^\star \neq K$, then each proper quasi-$\star_f$-ideal of D is contained in a quasi-$\star_f$-maximal ideal of D. We denote by $\mathrm{QMax}^\star(D)$ (resp., $\mathrm{QSpec}^\star(D)$) the set of all quasi-$\star$-maximal ideals (resp., quasi-$\star$-prime ideals) of D.

Given a semistar operation $\star$ on D, it is possible to construct a semistar operation $\widetilde{\star}$, which is defined as follows, for each $E \in \overline{\mathcal{F}}(D)$, $E^{\widetilde{\star}} := \bigcap_{P \in \mathrm{QMax}^{\star_f}(D)} ED_P$.

The most widely studied (semi)star operations on D have been the identity d_D, v_D, $t_D := (v_D)_f$, and $w_D := \widetilde{v_D}$ operations, where $A^{v_D} := (A^{-1})^{-1}$, with $A^{-1} := (D : A) := \{x \in K \mid xA \subseteq D\}$. We usually use these operations without subscripts. If $\star$ is a (semi)star operation on D, then $d \leq \star \leq v$.

Let $\star$ be a semistar operation on a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$. We say that $\star$ is *homogeneous preserving* if $\star$ sends homogeneous fractional ideals to homogeneous ones. It is known that d, t, and v are homogeneous preserving [1, Proposition 2.5], $\widetilde{\star}$ is homogeneous preserving [16, Proposition 2.3], and that if $\star$ is homogeneous preserving, then so is $\star_f$ [16, Lemma 2.4]. Denote by h-$\mathrm{QSpec}^\star(R)$ the homogeneous ideals of $\mathrm{QSpec}^\star(R)$ and let h-$\mathrm{QMax}^\star(R)$ denote the set of ideals of R which are maximal in the set of all proper homogeneous quasi-$\star$-ideals of R (if $\star$ is a (semi)star operation we denote these sets by h-$\mathrm{Spec}^\star(R)$ and h-$\mathrm{Max}^\star(R)$ respectively). It is shown that if $R^\star \subsetneq R_H$ and $\star = \star_f$ homogeneous preserving, then h-$\mathrm{QMax}^{\star_f}(R)(\subseteq h$-$\mathrm{QSpec}^\star(R))$ is nonempty, each proper homogeneous quasi-$\star_f$-ideal is contained in a homogeneous maximal quasi-$\star_f$-ideal [16, Lemma 2.1], and h-$\mathrm{QMax}^{\star_f}(R) = h$-$\mathrm{QMax}^{\widetilde{\star}}(R)$ [16, Proposition 2.5].

2. Graded primitive extension

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with quotient field $qf(R)$, H be the set of nonzero homogeneous elements of R, and $\star$ be a semistar operation on R such that $R^\star \subsetneq R_H$. In this section we give a characterization of gr-$\star$-quasi-Prüfer domains in terms of graded semistar primitive extensions.

For $a \in R$, denote by $C(a)$ the ideal of R generated by homogeneous components of a. The *homogeneous content ideal* for a polynomial $f = a_0 + a_1X + \cdots + a_nX^n \in R[X]$, is defined by $\mathcal{A}_f := \mathcal{A}_f^R := \sum_{i=0}^n C(a_i)$ [17]. It can be seen that if R has trivial grading, i.e., $\Gamma = \{0\}$, then $\mathcal{A}_f$ coincides with the usual content ideal of f (that is the ideal generated by coefficients of f). Assume that L is a fractional ideal of $R[X]$ such that $L \subseteq R_H[X]$, and set $\mathcal{A}_L := \sum_{f \in L} \mathcal{A}_f$. Now assume that $R \subseteq T$ is an extension of graded integral domains such that each homogeneous element of R is a homogeneous element of T. We say that an element $u \in T$ is *gr-$\star$-primitive over* R if u is a root of a nonzero polynomial $g \in R[X]$ with $(\mathcal{A}_g^R)^\star = R^\star$. The extension $R \subseteq T$ of graded integral domains is called a *gr-$\star$-primitive extension* if each homogeneous element of T is gr-$\star$-primitive over R. We call the extension $R \subseteq T$ a *gr-primitive extension* if $R \subseteq T$ is a gr-d_R-primitive extension. It is clear that if R and T have trivial grading, then gr-primitive extension coincides with the usual primitive extension of Gilmer and Hoffmann [10].

Recall that R is said to be a *graded-Prüfer domain* if each nonzero finitely generated homogeneous ideal of R is invertible [2]. We say that R is a *graded valuation domain* (gr-valuation domain) if either $u \in R$ or $u^{-1} \in R$ for every nonzero homogeneous $u \in R_H$.

Proposition 2.1 ([16, Theorem 4.4])**.** *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then the following statements are equivalent.*

1. *R is a graded-Prüfer domain.*
2. *R_P is a valuation domain for all $P \in h$-Spec(R) (resp., $P \in h$-Max(R)).*
3. *$R_{H\setminus P}$ is a gr-valuation domain for all $P \in h$-Spec(R) (resp., $P \in h$-Max(R)).*

Proposition 2.2. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be an integrally closed graded domain. Then R is a graded-Prüfer domain if and only if R_H is a gr-primitive extension of R.*

Proof. Assume that R is a graded-Prüfer domain and let $u = a/b \in R_H$ be a nonzero homogeneous element, where $a, b \in H$. Then there exists an integer $n > 1$ such that $a^{n-1}b \in (a^n, b^n)$ by [16, Theorem 4.1]. It follows that $a^{n-1}b = r_1a^n + r_2b^n$ for some $r_1, r_2 \in R$; dividing both sides of this equation by b^n yields $f(X) = r_1X^n - X^{n-1} + r_2 \in R[X]$ with $f(u) = 0$ and $\mathcal{A}_f = R$, so u is gr-primitive over R.

Conversely, suppose that R_H is a gr-primitive extension of R. Let M be a homogeneous maximal ideal of R and u be a nonzero homogeneous element of R_H. Then there exists a polynomial f in $R[X]$ such that $f(u) = 0$ and $\mathcal{A}_f = R$. Since M is homogeneous, one has $f \notin M[X]$. It follows from [19, Lemma in Page 19], that u or u^{-1} is in R_M. Thus u or u^{-1} is in $R_{H\setminus M}$. Consequently $R_{H\setminus M}$ is a gr-valuation domain and hence R is a graded-Prüfer domain by Proposition 2.1. ☐

Let $N := \{f \in R[X] \mid f \neq 0 \text{ and } \mathcal{A}_f = R\}$; then N is a multiplicatively closed subset of $R[X]$, and set $\mathrm{NA}(R) := R[X]_N$. It is known that $N = R[X]\setminus\bigcup\{P[X] \mid P \in h\text{-}\mathrm{Max}(R)\}$ and $\mathrm{Max}(\mathrm{NA}(R)) = \{P\,\mathrm{NA}(R) \mid P \in h\text{-}\mathrm{Max}(R)\}$ [17, Proposition 2.3].

The integral closure of R is denoted by $\bar{R}$. Then $\bar{R}$ is a homogeneous overring of R (cf. [13, Theorem 2.10]).

Remark 2.3. It is shown in the proof of part (2) $\Rightarrow$ (7) of [11, Theorem 2.9] that $\bar{R}[X]_N = \mathrm{NA}(\bar{R})$, where $N = R[X]\setminus\bigcup\{M[X] \mid M \in h\text{-}\mathrm{Max}(R)\}$.

Lemma 2.4. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. If R_H is a gr-primitive extension of $\bar{R}$, then R_H is a gr-primitive extension of R.*

Proof. Let $u \in R_H$, $Q' = \{f \in \bar{R}[X] \mid f(u) = 0\}$, and set $Q = Q' \cap R[X]$. If $N = R[X]\setminus\bigcup\{M[X] \mid M \in h\text{-}\mathrm{Max}(R)\}$ and $N' = \bar{R}[X]\setminus\bigcup\{M'[X] \mid M' \in h\text{-}\mathrm{Max}(\bar{R})\}$, then $\mathrm{NA}(R) = R[X]_N$ and $\mathrm{NA}(\bar{R}) = \bar{R}[X]_{N'}$. The hypothesis that

R_H is a gr-primitive extension of $\bar{R}$ implies that $Q' \cap N' \neq \emptyset$. It suffices to show that $Q \cap N \neq \emptyset$. We first observe that $Q\,\mathrm{NA}(R) = Q'\bar{R}[X]_N \cap \mathrm{NA}(R)$. That the right side contains the left side is clear, and if $f/n = d/m \in Q'\bar{R}[X]_N \cap \mathrm{NA}(R)$, where $f \in Q'$, $d \in R[X]$, and $n, m \in N$, then $fm = dn \in Q' \cap R[X] = Q$, so that $f/n = fm/nm \in Q\,\mathrm{NA}(R)$. Thus $Q'\bar{R}[X]_N \cap \mathrm{NA}(R) \subseteq Q\,\mathrm{NA}(R)$. It follows from Remark 2.3, that $\mathrm{NA}(\bar{R}) = \bar{R}[X]_N$; hence

$$\begin{aligned} Q\,\mathrm{NA}(R) &= Q'\bar{R}[X]_N \cap \mathrm{NA}(R) = Q'\,\mathrm{NA}(\bar{R}) \cap \mathrm{NA}(R) \\ &= \mathrm{NA}(\bar{R}) \cap \mathrm{NA}(R) = \mathrm{NA}(R), \end{aligned}$$

which means that $Q \cap N \neq \emptyset$. □

Gilmer and Hoffmann characterized Prüfer domains as those integrally closed domains D, such that every extension of D inside its quotient field is a primitive extension [10, Theorem 2].

Theorem 2.5. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then R is a gr-quasi-Prüfer domain if and only if R_H is a gr-primitive extension of R.*

Proof. Suppose that R_H is a gr-primitive extension of R. Then R_H is a gr-primitive extension of $\bar{R}$. Therefore by Proposition 2.2, $\bar{R}$ is a graded-Prüfer domain and hence R is a gr-quasi-Prüfer domain by [11, Corollary 2.10]. Conversely, if R is a gr-quasi-Prüfer domain, then $\bar{R}$ is a graded-Prüfer domain ([11, Corollary 2.10]). Then by Proposition 2.2, R_H is a gr-primitive extension of $\bar{R}$ and hence, by Lemma 2.4, R_H is a gr-primitive extension of R. □

The following is the main result of this section.

Theorem 2.6. *Let $\star$ be a homogeneous preserving semistar operation on a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ such that $R^\star \subsetneq R_H$. Then the following statements are equivalent*:

1. *$R \subseteq R_H$ is a gr-$\star_f$-primitive extension.*
2. *$R_{H\setminus P} \subseteq R_H$ is a gr-primitive extension for each $P \in h$-$\mathrm{QSpec}^{\star_f}(R)$.*
3. *R_P is a quasi-Prüfer domain, for each $P \in h$-$\mathrm{QSpec}^{\star_f}(R)$.*
4. *$R_P \subseteq qf(R)$ is a primitive extension, for each $P \in h$-$\mathrm{QSpec}^{\star_f}(R)$.*
5. *R is a gr-$\star_f$-quasi-Prüfer domain.*

Proof. (1) ⇒ (2) Let $P \in h$-$\mathrm{QSpec}^{\star_f}(R)$ and let u be a nonzero homogeneous element of R_H. Then by assumption there is a polynomial $0 \neq g \in R[X]$ such that $\mathcal{A}_g^{\star_f} = R^\star$ and $g(u) = 0$. Clearly, $g \in R_{H\setminus P}[X]$ and $\mathcal{A}_g^R \nsubseteq P$. So

$$\mathcal{A}_g^{R_{H\setminus P}} = \mathcal{A}_g^R R_{H\setminus P} = R_{H\setminus P},$$

and u is primitive over $R_{H\setminus P}$.

(2) ⇒ (1) Let u be a nonzero homogeneous element of R_H and let I be the nonzero ideal of $R[X]$ generated by polynomials $f \in R[X]$ such that $f(u) = 0$. We show that $\mathcal{A}_I^{\star_f} = R^\star$. Since $R_{H\setminus P} \subseteq R_H$ is a gr-primitive extension for each $P \in h$-$\mathrm{QMax}^{\star_f}(R)$, there is a nonzero polynomial $g \in R_{H\setminus P}[X]$, such

that $g(u) = 0$ and $\mathcal{A}_g^{R_{H\setminus P}} = R_{H\setminus P}$. Let $0 \neq s \in H\setminus P$ with $sg \in R[X]$. Then $\mathcal{A}_{sg}^R \nsubseteq P$ (otherwise, $R_{H\setminus P} = sR_{H\setminus P} = s\mathcal{A}_g^{R_{H\setminus P}} = \mathcal{A}_{sg}^{R_{H\setminus P}} = \mathcal{A}_{sg}^R R_{H\setminus P} \subseteq PR_{H\setminus P}$, a contradiction). Clearly, $sg \in I$ and so $\mathcal{A}_I \nsubseteq P$ for each $P \in h$-$\mathrm{QMax}^{\star_f}(R)$, therefore $\mathcal{A}_I^{\star_f} = R^\star$. Hence we can find $f \in I$ such that $\mathcal{A}_f^{\star_f} = R^\star$ and $f(u) = 0$. So u is gr-$\star_f$-primitive over R.

(2) ⇔ (3) For each $P \in h$-$\mathrm{QSpec}^{\star_f}(R)$, $R_{H\setminus P} \subseteq R_H$ is a gr-primitive extension if and only if $R_{H\setminus P}$ is a gr-quasi-Prüfer domain by Theorem 2.5, if and only if R_P is a quasi-Prüfer domain by [11, Theorem 2.9].

(5) ⇔ (3) Follows from [11, Theorem 2.9].

(3) ⇔ (4) Follows from [4, Theorem 1.1]. □

Corollary 2.7. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then the following statements are equivalent*:

(1) *$R \subseteq R_H$ is a gr-primitive extension.*
(2) *$R_{H\setminus P} \subseteq R_H$ is a gr-primitive extension for each $P \in h$-$\mathrm{Spec}(R)$.*
(3) *$R_P \subseteq qf(R)$ is a primitive extension, for each $P \in h$-$\mathrm{Spec}(R)$.*
(4) *R is a gr-quasi-Prüfer domain.*

Proof. Set $\star = d$ in Theorem 2.6. □

Corollary 2.8. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then the following statements are equivalent*:

(1) *$R \subseteq R_H$ is a gr-t-primitive extension.*
(2) *$R_{H\setminus P} \subseteq R_H$ is a gr-primitive extension for each $P \in h$-$\mathrm{Spec}^t(R)$.*
(3) *$R_P \subseteq qf(R)$ is a primitive extension, for each $P \in h$-$\mathrm{Spec}^t(R)$.*
(4) *R is a UMt-domain.*

Proof. Follows from Theorem 2.6 by setting $\star = v$ and [11, Theorem 3.2]. □

3. Graded INC-extension

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with quotient field $qf(R)$, H be the set of nonzero homogeneous elements of R, and $\star$ be a semistar operation on R such that $R^\star \subsetneq R_H$. In this section we give a characterization of gr-$\star$-quasi-Prüfer domains in terms of graded semistar INC-extensions.

Assume that $R \subseteq T$ is an extension of graded integral domains. We say that T is a *gr-$\star$-INC-extension* of R if whenever Q_1 and Q_2 are nonzero homogeneous prime ideals of T such that $Q_1 \cap R = Q_2 \cap R$ and $(Q_1 \cap R)^\star \subsetneq R^\star$ then Q_1 and Q_2 are incomparable. We also say that R is a *gr-$\star$-INC-domain* if each homogeneous overring of R is a gr-$\star$-INC-extension of R. We call R a *gr-INC-domain* if it is a gr-d_R-INC-domain. It is clear that if R has trivial grading, then gr-INC extension coincides with the usual INC extension of Dobbs [6].

For an ideal I of $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ let I_h denote the ideal of R generated by the set of homogeneous elements of R in I.

Lemma 3.1. *Let $R \subseteq T \subseteq S \subseteq R_H$ be such that T and S are homogeneous overrings of a graded integral domain $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$.*

(1) *If $R \subseteq T$ and $T \subseteq S$ are gr-INC-extensions, then $R \subseteq S$ is a gr-INC-extension.*

(2) *If $R \subseteq S$ is a gr-INC-extension and $T \subseteq S$ is an integral extension, then $R \subseteq T$ is a gr-INC-extension.*

(3) *If $R \subseteq S$ is a gr-INC-extension, then $T \subseteq S$ is a gr-INC-extension.*

Proof. (1) Assume that Q_1 and Q_2 are homogeneous prime ideals of S such that $Q_1 \subsetneq Q_2$. Since $T \subseteq S$ is a gr-INC-extension, then $Q_1 \cap T \subsetneq Q_2 \cap T$. Note that $Q_1 \cap T$ and $Q_2 \cap T$ are homogeneous prime ideals of T. Since $R \subseteq T$ is a gr-INC-extension, then $Q_1 \cap T \cap R \subsetneq Q_2 \cap T \cap R$, i.e., $Q_1 \cap R \subsetneq Q_2 \cap R$.

(2) Assume that P_1 and P_2 are homogeneous prime ideals of T such that $P_1 \subsetneq P_2$. Since $T \subseteq S$ is an integral extension, there are prime ideals $Q_1 \subsetneq Q_2$ of S such that $P_i = Q_i \cap T$ for $i = 1, 2$ ([9, Corollary 11.6]). Since P_i is homogeneous,

$$P_i = (P_i)_h = (Q_i \cap T)_h = (Q_i)_h \cap T$$

for $i = 1, 2$ ([11, Lemma 2.7]). Hence we may assume that Q_i is homogeneous for $i = 1, 2$. Since $R \subseteq S$ is a gr-INC-extension, then $Q_1 \cap R \subsetneq Q_2 \cap R$ and so

$$P_1 \cap R = Q_1 \cap T \cap R = Q_1 \cap R \subsetneq Q_2 \cap R = Q_2 \cap T \cap R = P_2 \cap R.$$

(3) Assume that Q_1 and Q_2 are homogeneous prime ideals of S such that $Q_1 \subsetneq Q_2$. Since $R \subseteq S$ is a gr-INC-extension, we have $Q_1 \cap R \subsetneq Q_2 \cap R$. Now if $Q_1 \cap T = Q_2 \cap T$, then $Q_1 \cap R = Q_2 \cap R$ a contradiction. □

Recall that Ayache and Jaballah introduced the residually algebraic extension of integral domains and characterized quasi-Prüfer domains as those domains D, such that every extension of D inside its quotient field is a residually algebraic extension [3, Corollary 2.8].

The extension $R \subseteq T$ of graded integral domains is called a *gr-residually algebraic extension*, if for each homogeneous prime ideal Q of T, T/Q is algebraic over $R/(Q \cap R)$, equivalently $qf(R/(Q \cap R)) \hookrightarrow qf(T/Q)$ is an algebraic extension. We say that R is a *gr-residually algebraic domain* if each homogeneous overring of R is a gr-residually algebraic extension of R. Recall that if $I = \bigoplus_{\alpha\in\Gamma} I_\alpha$ is a homogeneous ideal of a graded ring $T = \bigoplus_{\alpha\in\Gamma} T_\alpha$, then T/I is a Γ-graded ring such that for each $\alpha \in \Gamma$, $(T/I)_\alpha = (T_\alpha + I)/I \cong T_\alpha/(T_\alpha \cap I) = T_\alpha/I_\alpha$.

The following lemma is the graded version of [3, Theorem 2.3].

Lemma 3.2. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then R is a gr-residually algebraic domain if and only if it is a gr-INC-domain.*

Proof. Assume that R is a gr-residually algebraic domain which is not a gr-INC-domain. So there exists a homogeneous overring T of R such that $R \subseteq T$ is not a gr-INC-extension. Let Q_1 and Q_2 be homogeneous prime ideals of T such that $P := Q_1 \cap R = Q_2 \cap R$ and $Q_1 \subsetneq Q_2$. Moreover, $R/P \subseteq T/Q_1$ is an algebraic extension of domains, although Q_2/Q_1 is a nonzero ideal of T/Q_1 that intersects R/P in 0, the desired contradiction.

Conversely, assume that R is a gr-INC-domain which is not a gr-residually algebraic domain. So there exists a homogeneous overring T of R such that $R \subseteq T$ is not a residually algebraic extension. Let Q be a homogeneous prime ideal of T such that $R/(Q \cap R) \hookrightarrow T/Q$ is not an algebraic extension. Therefore there exists $u \in T$ such that $u + Q$ is a transcendental element of T/Q over R/P, where $P := Q \cap R$. Let $u + Q = \sum_{i=1}^{n}(u_i + Q)$ be the decomposition of $u + Q$ to homogeneous components. Hence there is a $u_j + Q$ which is a homogeneous transcendental element of T/Q (in fact if for each i, $u_i + Q$ is algebraic over R/P, then $u + Q$ is algebraic over R/P). Note that u_j is a homogeneous element of T and hence $R[u_j](= R_0[H \cup \{u_j\}])$ is a homogeneous overring of R. Set $x = u_j + Q$ and $T_1 = R[u_j] + Q$. Then T_1 is a homogeneous overring of R and $T_1/Q \cong (R/P)[x]$. Let $\wp$ be the prime ideal of $(R/P)[x]$ generated by x. Then $\wp$ contracts to the zero ideal in R/P, and if Q_1 is a homogeneous prime of T_1 minimal over $u_jT_1 + Q$, then $Q \subsetneq Q_1$ and $\wp = Q_1/Q$. It is easy to check that $Q_1 \cap R = Q \cap R = P$. This contradicts the graded incomparability of the extension $R \subseteq T_1$. □

Lemma 3.3 (cf. [5, Lemma 10])**.** *Let $\{P_\lambda\}$ be a chain of homogeneous prime ideals of a graded domain $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$. Then there is a gr-valuation overring of R with a chain of homogeneous prime ideals that contract to $\{P_\lambda\}$.*

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain and T be a homogeneous overring of R. We say that T is an *h-flat* overring of R if for each homogeneous prime ideal Q of T, one has $R_{Q\cap R} = T_Q$.

Lemma 3.4. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then R is a graded-Prüfer domain if and only if each homogeneous overring of R is h-flat.*

Proof. Suppose that R is a graded-Prüfer domain and T is a homogeneous overring of R and let $Q \in h\text{-Spec}(T)$. Therefore $R_{Q\cap R} \subseteq T_Q$ is an inclusion of valuation domains. By [9, Theorem 17.6], $QT_Q \subseteq (Q \cap R)R_{Q\cap R} \subseteq QT_Q$, so $R_{Q\cap R} = T_Q$.

Conversely, assume that each homogeneous overring of R is h-flat. Let P be a homogeneous prime ideal of R. By Lemma 3.3, there exists a gr-valuation overring (V, M) of R such that $P = M \cap R$. Hence $V_M = R_{M\cap R} = R_P$ is a valuation domain. Thus R is a graded-Prüfer domain by Proposition 2.1. □

Corollary 3.5. *If R is a graded-Prüfer domain, then it is a gr-INC-domain.*

Proof. Let T be a homogeneous overring of R and Q_1 and Q_2 be homogeneous prime ideals of T such that $Q_1 \subsetneq Q_2$. By Lemma 3.4, the map $\phi : h\text{-Spec}(T) \to h\text{-Spec}(R)$ which sends Q to $Q \cap R$ is injective. Therefore $Q_1 \cap R \subsetneq Q_2 \cap R$. □

Lemma 3.6. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then R is a gr-residually algebraic domain if and only if $\bar{R}$ is a gr-residually algebraic domain.*

Proof. Since one implication is trivial, we will concentrate on the other. Let T be a homogeneous overring of R. Since $\bar{R} \subseteq \bar{T} \subseteq R_H$, then $\bar{R} \subseteq \bar{T}$ is a gr-residually algebraic extension by assumption. Moreover, $R \subseteq \bar{R}$ is a gr-residually algebraic extension, and thus $R \subseteq \bar{T}$ is also a gr-residually algebraic extension by Lemma 3.1(1), and Lemma 3.2. Hence $R \subseteq T$ is a gr-residually algebraic extension by Lemma 3.1(2), and Lemma 3.2. □

Proposition 3.7. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be an integrally closed graded domain. If R is a gr-INC-domain, then $R \subseteq R_H$ is a gr-primitive extension.*

Proof. Assume that $R \subseteq R_H$ is not a gr-primitive extension. So R is not a graded-Prüfer domain by Proposition 2.2. Hence there exists $P \in$ h-Max(R) such that R_P is not a valuation domain by Proposition 2.1. Thus there is a homogeneous element $u \in R_H$ such that $u \notin R_P$ and $u^{-1} \notin R_P$ by [16, Lemma 4.3]. Consequently by [18, Theorem 7], $PR[u]$ is a (non-maximal) prime ideal of $R[u]$, $P = PR[u] \cap R$ and $R[u]/PR[u] \cong (R/P)[X]$. Since u is a homogeneous element of R_H, $R[u]$ is a homogeneous overring of R. Thus $R \subseteq R[u]$ is a gr-residually algebraic extension by Lemma 3.2. Hence

$$R/P \hookrightarrow R[u]/PR[u] \cong (R/P)[X]$$

is an algebraic extension, a contradiction. □

The following theorem is the graded version of [6, Theorem].

Theorem 3.8. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then R is a gr-INC-domain if and only if $R \subseteq R_H$ is a gr-primitive extension.*

Proof. Assume that R is a gr-INC-domain. Let T be a homogeneous overring of $\bar{R}$. Thus $R \subseteq T$ is a gr-INC-extension which implies that $\bar{R} \subseteq T$ is a gr-INC-extension by Lemma 3.1(3). Therefore $\bar{R}$ is an integrally closed gr-INC-domain, hence $\bar{R} \subseteq R_H$ is a gr-primitive extension by Proposition 3.7. Now $R \subseteq R_H$ is a gr-primitive extension by Lemma 2.4.

Conversely, assume that $R \subseteq R_H$ is a gr-primitive extension. Then by the proof of Theorem 2.5, $\bar{R}$ is a graded-Prüfer domain. So $\bar{R}$ is a gr-INC-domain by Corollary 3.5. Thus $\bar{R}$ is a gr-residually algebraic domain by Lemma 3.2. Therefore R is a gr-residually algebraic domain by Lemma 3.6, and hence R is a gr-INC-domain by Lemma 3.2. □

Corollary 3.9. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then the following statements are equivalent*:

(1) *R is a gr-quasi-Prüfer domain.*
(2) *R is a gr-residually algebraic domain.*

Proof. It follows by combining Corollary 2.7, Theorem 3.8 and Lemma 3.2. □

The following is the main result of this section.

Theorem 3.10. *Let $\star$ be a homogeneous preserving semistar operation on a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ such that $R^\star \subsetneq R_H$. Then the following statements are equivalent*:

(1) *R is a gr-$\star_f$-INC-domain.*
(2) *$R_{H \setminus P}$ is a gr-INC-domain, for each $P \in h\text{-QSpec}^{\star_f}(R)$.*
(3) *R_P is a quasi-Prüfer domain, for each $P \in h\text{-QSpec}^{\star_f}(R)$.*
(4) *R_P is an INC-domain, for each $P \in h\text{-QSpec}^{\star_f}(R)$.*
(5) *R is a gr-$\star_f$-quasi-Prüfer domain.*

Proof. (1) $\Rightarrow$ (2) Let $P \in h\text{-QSpec}^{\star_f}(R)$ and let T be a homogeneous overring of $R_{H \setminus P}$. Assume that Q_1 and Q_2 are homogeneous prime ideals of T such that $Q_1 \cap R_{H \setminus P} = Q_2 \cap R_{H \setminus P}$. We must show that Q_1 and Q_2 are incomparable. By assumption, T is a gr-$\star_f$-INC-extension of R and $Q_1 \cap R = Q_2 \cap R \subseteq P$ with $P^{\star_f} \subsetneq R^\star$, hence Q_1 and Q_2 are incomparable.

(2) $\Rightarrow$ (1) Let T be a homogeneous overring of R and $Q_1 \subsetneq Q_2$ be homogeneous prime ideals of T such that $P := Q_1 \cap R = Q_2 \cap R \subseteq M$ for some $M \in h\text{-QSpec}^{\star_f}(R)$. Note that $Q_1 T_{H \setminus M} \subsetneq Q_2 T_{H \setminus M}$ and they are prime ideals of $T_{H \setminus M}$ each of which intersects $R_{H \setminus M}$ in $PR_{H \setminus M}$, and so $R_{H \setminus M} \subseteq T_{H \setminus M}$ is not a gr-INC-extension, contradicting (2).

(2) $\Leftrightarrow$ (3) $R_{H \setminus P}$ is a gr-INC-domain for each $P \in h\text{-QSpec}^{\star_f}(R)$ if and only if $R_{H \setminus P} \subseteq R_H$ is a gr-primitive extension by Theorem 3.8, if and only if R_P is a quasi-Prüfer domain, for each $P \in h\text{-QSpec}^{\star_f}(R)$ by Theorem 2.6.

(3) $\Leftrightarrow$ (4) and (3) $\Leftrightarrow$ (5) Follow from [4, Theorem 1.1] and [11, Theorem 2.9], respectively. □

Corollary 3.11. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then the following statements are equivalent*:

(1) *R is a gr-INC-domain.*
(2) *$R_{H \setminus P}$ is a gr-INC-domain, for each $P \in h\text{-Spec}(R)$.*
(3) *R_P is an INC-domain, for each $P \in h\text{-Spec}(R)$.*
(4) *R is a gr-quasi-Prüfer domain.*

Proof. Set $\star = d$ in Theorem 3.10. □

Corollary 3.12. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then the following statements are equivalent*:

(1) *R is a gr-t-INC-domain.*
(2) *$R_{H \setminus P}$ is a gr-INC-domain, for each $P \in h\text{-Spec}^t(R)$.*
(3) *R_P is an INC-domain, for each $P \in h\text{-Spec}^t(R)$.*
(4) *R is a UMt-domain.*

Proof. Follows from Theorem 3.10 by setting $\star = v$ and [11, Theorem 3.2]. □

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