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THE TOTAL GRAPH OF NON-ZERO ANNIHILATING IDEALS OF A COMMUTATIVE RING

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Abstract. Assume that R is a commutative ring with non-zero identity which is not an integral domain. An ideal I of R is called an annihilating ideal if there exists a non-zero element $a \in R$ such that $Ia = 0$. S. Visweswaran and H. D. Patel associated a graph with the set of all non-zero annihilating ideals of R, denoted by $\Omega(R)$, as the graph with the vertex-set $\mathrm{A}(R)^*$, the set of all non-zero annihilating ideals of R, and two distinct vertices I and J are adjacent if $I + J$ is an annihilating ideal. In this paper, we study the relations between the diameters of $\Omega(R)$ and $\Omega(R[x])$. Also, we study the relations between the diameters of $\Omega(R)$ and $\Omega(R[[x]])$, whenever R is a Noetherian ring. In addition, we investigate the relations between the diameters of this graph and the zero-divisor graph. Moreover, we study some combinatorial properties of $\Omega(R)$ such as domination number and independence number. Furthermore, we study the complement of this graph.

1. Introduction

In recent years, assigning graphs to algebraic structures has played an important role in the study of algebraic structures, for instance, see [1], [2] and [10]. I. Beck in [3] introduced the idea of a zero-divisor graph of a commutative ring, where he was mainly interested in colorings. D. F. Anderson and P. S. Livingston in [1] introduced the zero-divisor graph of a commutative ring R, denoted by $\Gamma(R)$, as the graph with the vertex-set $Z(R)^*$, the set of all non-zero zero-divisors of R, and two distinct vertices x and y are adjacent if $xy = 0$. They investigate the relations between the ring-theoretic properties of R and graph-theoretic properties of $\Gamma(R)$. S. Visweswaran and H. D. Patel in [10] introduced and studied a graph, denoted by $\Omega(R)$, with the vertex-set $\mathrm{A}(R)^*$, the set of all non-zero annihilating ideals of R, and two distinct vertices I and J are adjacent if $I + J$ is an annihilating ideal.

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Throughout this paper, R is a commutative ring with non-zero identity which is not an integral domain. By a non-trivial ideal of R, we mean a non-zero proper ideal of R. The set of all zero-divisors, nilpotent elements, prime ideals, minimal prime ideals and maximal ideals of R are denoted by $Z(R)$, $\mathrm{Nil}(R)$, $\mathrm{Spec}(R)$, $\mathrm{Min}(R)$ and $\mathrm{Max}(R)$, respectively. Also, $\mathbb{Z}$, $\mathbb{Z}_n$ and $\mathbb{Q}$ are the integers, integers modulo n and rational numbers, respectively. Moreover, the non-zero elements of $X \subseteq R$ will be denoted by X^*. An ideal I of R is called an *annihilating ideal* if there exists $a \in R^*$ such that $Ia = 0$. By $\mathrm{A}(R)$ we mean the set of all annihilating ideals of R and $\mathrm{A}(R)^* := \mathrm{A}(R) \setminus \{0\}$. Given any subset $X \subseteq R$, the annihilator of X is the set $\mathrm{Ann}(X) = \{a \in R \mid aX = 0\}$. A ring R is said to be *reduced* if it has no non-zero nilpotent element. A non-zero ideal I of R is called *essential*, denoted by $I \leq_e R$, if I has a non-zero intersection with any non-zero ideal of R. The *socle* of R, denoted by $\mathrm{soc}(R)$, is the sum of all minimal ideals of R. If R has not a minimal ideal, this sum is defined to be zero. A ring R is said to be *semisimple* if $\mathrm{soc}(R) = R$. The Jacobson radical of R is denoted by $J(R)$.

Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. The *complement* of G, denoted by $\overline{G}$, is the graph with the same vertex-set as G, where two distinct vertices are adjacent whenever they are non-adjacent in G. The *distance* between two vertices in a graph is the number of edges in a shortest path connecting them. The *diameter* of a connected graph G, denoted by $\mathrm{diam}(G)$, is the maximum distance between any pair of the vertices of G ($\mathrm{diam}(G) = \infty$ if G is disconnected). The *girth* of a graph G, denoted by $\mathrm{gr}(G)$, is the length of the shortest cycle in G. A graph with no cycle has infinite girth. Also, for a vertex $v \in V$, the degree of v, denoted by $\deg(v)$, is the number of incident edges. The graph $H = (V_0, E_0)$ is a *subgraph* of G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph* by V_0, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \mid u, v \in V_0\}$. For two vertices u and v in G, the notation $u - v$ means that u and v are adjacent. In a graph G, a set $S \subseteq V(G)$ is a *dominating set* if every vertex not in S has a neighbor in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum size of a dominating set in G. A set $S \subseteq V(G)$ is an *independent set* if the subgraph induced by S contains no edge. The *independence number* $\alpha(G)$ is the maximum size of an independent set in G. A graph G is *complete* if every vertex is adjacent to every other vertex. We denote the complete graph on n vertices by K_n. A *clique* of G is a complete subgraph of G and the number of vertices in a largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G. A *bipartite graph* is one whose vertex-set can be partitioned into two subsets so that no edge has both ends in any one subset. A *complete bipartite graph* is one in which each vertex is joined to every vertex that is not in the same subset. We denote $K_{m,n}$ for the complete bipartite graph with part sizes m and n. A graph is said to be *planar* if it can drawn in the plane so that its edges intersect only at their ends. A *unicyclic graph* is a connected graph with a unique cycle.

Let R be a commutative ring with identity which is not an integral domain. The total graph of non-zero annihilating ideals of R, denoted by $\Omega(R)$, is a graph with the vertex-set $\mathrm{A}(R)^*$, and two distinct vertices I and J are adjacent if $I+J$ is an annihilating ideal. In this paper, we study the relations between the diameters of $\Omega(R)$ and $\Omega(R[x])$. Also, we study the relations between the diameters of $\Omega(R)$ and $\Omega(R[[x]])$, whenever R is a Noetherian ring. In addition, we investigate the relations between the diameters of this graph and the zero-divisor graph. Moreover, we study some combinatorial properties of $\Omega(R)$ such as domination number, independence number and planarity. Among other results, it is proved that the connectivity of the graphs $\Omega(R)$, $\Omega(R[x])$ and $\Omega(R[[x]])$ are equivalent. Furthermore, we study the complement of this graph and we investigate the connectivity of the graphs $\overline{\Omega(R)}$, $\overline{\Omega(R[x])}$ and $\overline{\Omega(R[[x]])}$. Since $\Omega(D)=\emptyset$, where D is an integral domain, we assume that throughout this paper R is a commutative ring with $Z(R)\neq 0$.

2. Main results

In this section, we study the relations between the diameters of $\Omega(R)$ and $\Omega(R[x])$. Also, we study the relations between the diameters of $\Omega(R)$ and $\Omega(R[[x]])$, whenever R is a Noetherian ring. Recall that a prime ideal P of R is said to be *maximal N-prime* of (0) if P is maximal with respect to the property of being contained in $Z(R)$. By [6, Theorem 1], $Z(R)=\cup_{i\in\Theta}P_i$, where $\{P_i\}_{i\in\Theta}$ is the set of all maximal N-primes of (0) in R. Also, by [10, Lemmas 2.3, 3.1, 3.3, 3.4 and 4.1], we have $\mathrm{diam}(\Omega(R))\in\{0,1,2,\infty\}$. In our first result we have the following proposition. Note that xR or Rx is the ideal generated by the element $x\in R$.

Proposition 2.1. *Let R be a ring. Then*

(a) $\mathrm{diam}(\Omega(R[x]))\in\{1,2,\infty\}$.
(b) $\mathrm{gr}(\Omega(R[x]))=3$.
(c) $\mathrm{diam}(\Omega(R[[x]]))\in\{1,2,\infty\}$.
(d) $\mathrm{gr}(\Omega(R[[x]]))=3$.

Proof. (a) Let $a\in Z(R)^*$. Then $aR[x]$ and $axR[x]$ are distinct vertices of $\Omega(R[x])$ and hence $\mathrm{diam}(\Omega(R[x]))\in\{1,2,\infty\}$.

(b) Let $a\in Z(R)^*$. Then $aR[x]-axR[x]-ax^2R[x]-aR[x]$ is a cycle of length three in $\Omega(R[x])$.

By a similar way as used in the proof of (a) and (b), one can prove the items (c) and (b). □

In the next theorem, we show that the connectivity of the graphs $\Omega(R)$, $\Omega(R[x])$ and $\Omega(R[[x]])$ are equivalent. Before that, the following lemma is necessary.

Lemma 2.2. *Let R be a ring. Then $\Omega(R)$ is disconnected if and only if R is a reduced ring with exactly two minimal prime ideals.*

Proof. By [10, Lemmas 2.3, 3.1, 3.3, 3.4 and 4.1], $\Omega(R)$ is disconnected if and only if $P_1 \cap P_2 = 0$, where P_1 and P_2 are maximal N-primes of (0). Thus $\Omega(R)$ is disconnected if and only if R is a reduced ring with exactly two minimal prime ideals. □

Theorem 2.3. *Let R be a ring. Then the following statements are equivalent*:

(a) $\Omega(R)$ *is disconnected.*
(b) $\Omega(R[x])$ *is disconnected.*
(c) $\Omega(R[[x]])$ *is disconnected.*

Proof. (a) $\Longleftrightarrow$ (b) Suppose that $\Omega(R)$ is disconnected. Then R is reduced and $\mathrm{Min}(R) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$. We show that $R[x]$ is reduced and $|\mathrm{Min}(R[x])| = 2$. It is clear that $\mathfrak{p}_1[x]$ and $\mathfrak{p}_2[x]$ are prime ideals of $R[x]$. Since $\mathfrak{p}_1 \cap \mathfrak{p}_2 = 0$, we have $\mathfrak{p}_1[x] \cap \mathfrak{p}_2[x] = (\mathfrak{p}_1 \cap \mathfrak{p}_2)[x] = 0$. Thus $R[x]$ is a reduced ring and $|\mathrm{Min}(R[x])| = 2$. Therefore, $\Omega(R[x])$ is disconnected.

Conversely, assume that $\Omega(R[x])$ is disconnected. Then $R[x]$ is a reduced ring with exactly two minimal prime ideals. Now by [9, Remarks 3.27(ii)] and [9, Exercise 2.43(iii)], one can see that R is a reduced ring with exactly two minimal prime ideals. Thus $\Omega(R)$ is disconnected.

(a) $\Longleftrightarrow$ (c) follows similarly. □

Example 2.4. Let $R = (\mathbb{Z}_2 \times \mathbb{Z}_2)[x]$. Since $\Omega(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is disconnected, $\Omega((\mathbb{Z}_2 \times \mathbb{Z}_2)[x])$ is disconnected. Now assume that $R_1 = (\mathbb{Z}_2 \times \mathbb{Z}_2)[x, y]$. Then since $R_1 = R[y]$, $\Omega((\mathbb{Z}_2 \times \mathbb{Z}_2)[x, y])$ is disconnected.

Let f be a zero-divisor of $R[x]$. It is well known that there exists $c \in R^*$ such that $cf = 0$ (see [9, Exercise 1.36(iii)]). In the following lemma we generalize this statement.

Lemma 2.5. *Let I be an annihilating ideal of $R[x]$. Then there exists $c \in R^*$ such that $cI = 0$.*

Proof. Let I be an annihilating ideal of $R[x]$. If $I = 0$, then the statement is clear. Thus we let $I \neq 0$. Then there exists $g \in R[x]^*$ such that $gI = 0$. Without loss of generality, we may assume that $g = a_0 + a_1x + \cdots + a_nx^n$ is a polynomial of least degree n such that $gI = 0$. Assume that $f = b_0 + b_1x + \cdots + b_mx^m \in I^*$. Since $gf = 0$, we have $a_nb_m = 0$. Thus $b_mgI = 0$. Now since g is a polynomial of least degree n such that $gI = 0$, $b_ma_i = 0$ for $i = 0, 1, \ldots, n$. Similarly, if $a_nb_k = 0$ for some $k \in \{0, 1, \ldots, m\}$, then $a_ib_k = 0$ for $i = 0, 1, \ldots, n$. Now suppose that $j \in \{0, 1, \ldots, m\}$ is maximum such that $a_nb_j \neq 0$. Then since $gf = 0$, we have $(a_nb_j + a_{n-1}b_{j+1} + \cdots)x^{n+j} = 0$. Thus $a_nb_j = 0$ which is a contradiction. Hence $a_nb_j = 0$ for all $j \in \{0, 1, \ldots, m\}$. Now we conclude that $a_nf = 0$, for all $f \in I$. Therefore, $a_nI = 0$. □

In the following proposition we study the relations between the diameters of $\Omega(R)$ and $\Omega(R[x])$.

Proposition 2.6. *Let R be a ring. Then we have the following statements*:

(a) $\mathrm{diam}(\Omega(R)) \in \{0,1\}$ *if and only if* $\mathrm{diam}(\Omega(R[x])) = 1$.
(b) $\mathrm{diam}(\Omega(R)) = 2$ *if and only if* $\mathrm{diam}(\Omega(R[x])) = 2$.

Proof. (a) Suppose that $\mathrm{diam}(\Omega(R)) \in \{0,1\}$. Let I and J be two distinct vertices of $\Omega(R[x])$. Put

$$\Delta := \{\text{the set of all coefficients of elements of } I\}$$

and

$$\Lambda := \{\text{the set of all coefficients of elements of } J\}.$$

Then it is easy to see that Δ and Λ are annihilating ideals of R, by Lemma 2.5. Since $\mathrm{diam}(\Omega(R)) \in \{0,1\}$, $\Delta + \Lambda$ is an annihilating ideal of R. Thus there exists $c \in R^*$ such that $c(I + J) = 0$. Then I is adjacent to J in $\Omega(R[x])$ and hence $\mathrm{diam}(\Omega(R[x])) = 1$.

Conversely, suppose that $\mathrm{diam}(\Omega(R[x])) = 1$. If $Z(R)$ is a minimal ideal of R, then we have $\mathrm{diam}(\Omega(R)) = 0$. Otherwise, assume that I and J are two distinct vertices of $\Omega(R)$. Then $I[x]$ and $J[x]$ are distinct vertices of $\Omega(R[x])$. Since $I[x]$ and $J[x]$ are adjacent in $\Omega(R[x])$, by Lemma 2.5 there exists an element $c \in R^*$ such that $c(I[x] + J[x]) = 0$. Hence $c(I + J) = 0$ and so I is adjacent to J in $\Omega(R)$. Thus $\mathrm{diam}(\Omega(R)) = 1$.

(b) By Theorem 2.3 and item (a), it is straightforward. □

We use the following two lemmas in the sequel.

Lemma 2.7. *Let R be a Noetherian ring. Then $Z(R[[x]]) = \cup_{i=1}^{n} P_i[[x]]$, where $P_i = \mathrm{Ann}_R(r_i) \in \mathrm{Spec}(R)$ and $r_i \in R^*$ for $i = 1, 2, \ldots, n$. In particular, if $Z(R)$ is an ideal, then $Z(R[[x]]) = \mathrm{Ann}_R(r)[[x]]$, where $Z(R) = \mathrm{Ann}_R(r)$ for some $r \in R^*$.*

Proof. Since R is a Noetherian ring, (0) has a minimal primary decomposition by [9, Corollary 4.35]. Then by [4, Theorem 4], [9, Proposition 8.19] and [9, Proposition 8.22], one can see that $Z(R[[x]]) = \cup_{i=1}^{n} P_i[[x]]$, where $P_i = \mathrm{Ann}_R(r_i) \in \mathrm{Spec}(R)$ and $r_i \in R^*$ for $i = 1, 2, \ldots, n$. The "in particular" statement follows similarly (see [6, Theorem 81]). □

Lemma 2.8. *Let R be a Noetherian ring. Then $Z(R) = \mathrm{Ann}(r)$ for some $r \in Z(R)^*$ if and only if $\mathrm{diam}(\Omega(R)) \in \{0,1\}$.*

Proof. Assume that $Z(R) = \mathrm{Ann}(r)$ for some $r \in Z(R)^*$. If $|\mathrm{A}(R)^*| = 1$, then $\mathrm{diam}(\Omega(R)) = 0$. Thus we can suppose that I and J are two distinct vertices of $\Omega(R)$. Then $r(I + J) = 0$ and hence I is adjacent to J. Thus $\mathrm{diam}(\Omega(R)) = 1$.

Conversely, assume that $\mathrm{diam}(\Omega(R)) \in \{0,1\}$. If $\mathrm{diam}(\Omega(R)) = 0$, then $Z(R) = \mathrm{Ann}(r)$ for some $r \in Z(R)^*$. Thus we can suppose that $\mathrm{diam}(\Omega(R)) = 1$. Let $x, y \in Z(R)$. Then since $\mathrm{diam}(\Omega(R)) = 1$, $a(Rx + Ry) = 0$ for some $a \in Z(R)^*$. Thus $Z(R)$ is an ideal. Now since R is a Noetherian ring, by [9, Proposition 8.19], [9, Proposition 8.22] and [6, Theorem 81], it is easy to see that $Z(R) = \mathrm{Ann}(r)$ for some $r \in Z(R)^*$. □

In the next proposition we study the relations between the diameters of $\Omega(R)$ and $\Omega(R[[x]])$, when R is a Noetherian ring.

Proposition 2.9. *Let R be a Noetherian ring. Then we have the following statements*:

(a) $\mathrm{diam}(\Omega(R)) \in \{0, 1\}$ *if and only if* $\mathrm{diam}(\Omega(R[[x]])) = 1$.
(b) $\mathrm{diam}(\Omega(R)) = 2$ *if and only if* $\mathrm{diam}(\Omega(R[[x]])) = 2$.

Proof. (a) Suppose that $\mathrm{diam}(\Omega(R)) \in \{0, 1\}$. Since R is Noetherian, by Lemmas 2.7 and 2.8, we have $Z(R[[x]]) = \mathrm{Ann}_R(r)[[x]]$ for some $r \in Z(R)^*$. Let I and J be two distinct vertices of $\Omega(R[[x]])$. Then $I + J \subseteq Z(R[[x]])$. Now since $Z(R[[x]]) = \mathrm{Ann}_R(r)[[x]]$ for some $r \in Z(R)^*$, we have $r(I + J) = 0$. Thus I is adjacent to J in $\Omega(R[[x]])$. Then we conclude that $\mathrm{diam}(\Omega(R[[x]])) = 1$.

Conversely, suppose that $\mathrm{diam}(\Omega(R[[x]])) = 1$. If $Z(R)$ is a minimal ideal of R, then we have $\mathrm{diam}(\Omega(R)) = 0$. Otherwise, let I and J be two distinct vertices of $\Omega(R)$. Then it is easy to see that $I[[x]]$ and $J[[x]]$ are two distinct vertices of $\Omega(R[[x]])$. Since $Z(R[[x]]) = \cup_{i=1}^n P_i[[x]]$, where $P_i = \mathrm{Ann}_R(r_i) \in \mathrm{Spec}(R)$ for some $r_i \in Z(R)^*$, and $\mathrm{diam}(\Omega(R[[x]])) = 1$, we have $I[[x]] + J[[x]] \subseteq \cup_{i=1}^n P_i[[x]]$. Thus by [6, Theorem 81], $I[[x]] + J[[x]] \subseteq P_j[[x]]$ for some $j \in \{1, 2, \ldots, n\}$. Then there exists $d \in Z(R)^*$ such that $d(I[[x]] + J[[x]]) = 0$. Thus $d(I + J) = 0$ and hence I is adjacent to J in $\Omega(R)$. Then we conclude that $\mathrm{diam}(\Omega(R)) = 1$.

(b) By Theoremy 2.3 and item (a), it is straightforward. □

3. Some combinatorial properties of $\Omega(R)$

In this section, we investigate some combinatorial properties of $\Omega(R)$ such as domination number and independence number. Moreover, we investigate the relations between the diameters of this graph and the zero-divisor graph. In the next proposition, we determine the domination number of $\Omega(R)$. Before that, we need the following two lemmas.

Lemma 3.1. *Let R be a ring and I an ideal of R. Then $I + \mathrm{Ann}(I)$ is an essential ideal of R.*

Proof. Assume to the contrary that $I + \mathrm{Ann}(I)$ is not an essential ideal of R. Then there exists a non-zero ideal J of R such that $J \cap (I + \mathrm{Ann}(I)) = 0$. Thus $J \cap I = 0$ and hence $J \subseteq \mathrm{Ann}(I)$ which is impossible. Therefore, $I + \mathrm{Ann}(I)$ is an essential ideal of R. □

Lemma 3.2. *Assume that I and J are two distinct vertices of $\Omega(R)$. Then I is adjacent to J if and only if $\mathrm{Ann}(I) \cap \mathrm{Ann}(J) \neq 0$.*

Proof. Assume that I is adjacent to J. Then $x(I + J) = 0$, for some $x \in R^*$. Thus $\mathrm{Ann}(I) \cap \mathrm{Ann}(J) \neq 0$. Conversely, assume that $\mathrm{Ann}(I) \cap \mathrm{Ann}(J) \neq 0$. Now we choose a non-zero element $y \in \mathrm{Ann}(I) \cap \mathrm{Ann}(J)$. Thus $y(I + J) = 0$ and hence I is adjacent to J. □

Proposition 3.3. *Let R be a ring and I a vertex of $\Omega(R)$. Then the set $\{I, \mathrm{Ann}(I)\}$ is a dominating set. In particular, $\gamma(\Omega(R)) \leq 2$.*

Proof. Suppose that $J \in \mathrm{A}(R)^* \setminus \{I, \mathrm{Ann}(I)\}$. If $I = \mathrm{Ann}(I)$, then by Lemma 3.1 we have $\mathrm{Ann}(I) \cap \mathrm{Ann}(J) \neq 0$ and hence J is adjacent to I, by Lemma 3.2. Now assume that $I \neq \mathrm{Ann}(I)$. Suppose that J is not adjacent to I. Then $\mathrm{Ann}(I) \cap \mathrm{Ann}(J) = 0$. Thus $\mathrm{Ann}(J) \subseteq \mathrm{Ann}(\mathrm{Ann}(I))$. Therefore, J is adjacent to $\mathrm{Ann}(I)$. □

Example 3.4. Let $R = \mathbb{Z} \times \mathbb{Z}$. Then the set $\{\mathbb{Z} \times 0, 0 \times \mathbb{Z}\}$ is a dominating set in $\Omega(R)$.

Corollary 3.5. *Let R be a ring. Then*

(a) *R is non-reduced if and only if $\gamma(\Omega(R)) = 1$.*
(b) *R is reduced if and only if $\gamma(\Omega(R)) = 2$.*

Proof. (a) Let R be non-reduced. Then there exists $I \in \mathrm{A}(R)^*$ such that $I^2 = 0$. Now by Lemmas 3.1 and 3.2, we conclude that I is adjacent to every other vertex and so $\gamma(\Omega(R)) = 1$.

Conversely, assume that $\gamma(\Omega(R)) = 1$. Then there exists a vertex of $\Omega(R)$, say I, such that I is adjacent to every other vertex. If $I = \mathrm{Ann}(I)$, then R is non-reduced. Otherwise, we can suppose that $I \neq \mathrm{Ann}(I)$. Now since I is adjacent to $\mathrm{Ann}(I)$, there exists $x \in R^*$ such that $x(I + \mathrm{Ann}(I)) = 0$. Thus $x^2 = 0$ and so R is non-reduced.

(b) By Proposition 3.3 and item (a), it is straightforward. □

Proposition 3.6. *Let R_1 and R_2 be rings and $R = R_1 \times R_2$. Then $\gamma(\Omega(R)) = 1$ if and only if $\gamma(\Omega(R_1)) = 1$ or $\gamma(\Omega(R_2)) = 1$.*

Proof. By Corollary 3.5, $\gamma(\Omega(R)) = 1$ if and only if R is non-reduced. On the other hand, R is non-reduced if and only if R_1 or R_2 is non-reduced. Thus we conclude that $\gamma(\Omega(R)) = 1$ if and only if $\gamma(\Omega(R_1)) = 1$ or $\gamma(\Omega(R_2)) = 1$. □

An *annihilator prime* for a ring R is any prime ideal P which equals the annihilator of some non-zero ideal of R. It is easy to see that any ideal maximal among the annihilators of non-zero ideals of a ring R is prime. We use $\mathcal{A}(R)$ to denote the set of all maximal annihilators of a ring R. Note that if R is a Noetherian ring, then $\mathcal{A}(R) \neq \emptyset$. In the next proposition, we study the independence number of $\Omega(R)$.

Proposition 3.7. *Let R be a ring such that $1 \leq |\mathcal{A}(R)| < \infty$ and $Z(R) = \cup_{P \in \mathcal{A}(R)} P$. Then $\alpha(\Omega(R)) = |\mathcal{A}(R)|$.*

Proof. If $|\mathcal{A}(R)| = 1$, then $Z(R)$ is an annihilator ideal. Thus $\Omega(R)$ is a complete graph and hence $\alpha(\Omega(R)) = 1$. Now suppose that $n = |\mathcal{A}(R)| \geq 2$ and $\mathcal{A}(R) = \{P_1, P_2, \ldots, P_n\}$. First we show that $\mathcal{A}(R)$ is an independent set. To see this, assume that P_1 is adjacent to P_2. Then there exists $x \in R^*$ such that $x(P_1 + P_2) = 0$. Since $Z(R) = \cup_{P \in \mathcal{A}(R)} P$, by [6, Teorem 81] there exists $P_i \in \mathcal{A}(R)$ such that $P_1 + P_2 \subseteq P_i$ which is a contradiction. Hence $\alpha(\Omega(R)) \geq |\mathcal{A}(R)|$. Now assume that $S := \{I_1, I_2, \ldots, I_{n+1}\}$ is an independent set with $n+1$ vertices. Thus since $Z(R) = \cup_{P\in\mathcal{A}(R)}P$, by [6, Theorem 81] there exist distinct $i, j \in \{1, 2, \ldots, n\}$ and $P \in \mathcal{A}(R)$ such that $I_i + I_j \subseteq P$. Then I_i is adjacent to I_j which is a contradiction. Hence $\alpha(\Omega(R)) = |\mathcal{A}(R)|$. □

Corollary 3.8. *Let R be a ring. Then*

(a) *If R is Noetherian, then $\alpha(\Omega(R)) < \infty$.*
(b) *If R is reduced and $|\mathrm{Min}(R)| < \infty$, then $\alpha(\Omega(R)) = |\mathrm{Min}(R)|$.*

Proof. (a) Assume that R is a Noetherian ring. Then by [6, Theorem 80], $Z(R) = \cup_{P\in\mathcal{A}(R)}P$ and $1 \leq |\mathcal{A}(R)| < \infty$. Thus $\alpha(\Omega(R)) < \infty$.

(b) Assume that R is a reduced ring and $\mathrm{Min}(R) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$. Then $\mathrm{Ann}(\mathfrak{p}_i) \neq 0$ for $i = 1, 2, \ldots, n$. We show that $\mathrm{Min}(R) = \mathcal{A}(R)$. To see this, let $\mathfrak{p} \in \mathrm{Min}(R)$. If $\mathfrak{p} \notin \mathcal{A}(R)$, then there exists $x \in R^*$ such that $\mathfrak{p} \subsetneqq \mathrm{Ann}(x)$. Now since R is reduced, we have $\mathrm{Ann}(x) \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in \mathrm{Min}(R)$ which is impossible. Thus $\mathrm{Min}(R) \subseteq \mathcal{A}(R)$. Now assume that $P \in \mathcal{A}(R)$. Then by [5, Corollary 2.4] and [6, Theorem 81], we have $P \in \mathrm{Min}(R)$. Thus we conclude that $\mathrm{Min}(R) = \mathcal{A}(R)$ and hence $\alpha(\Omega(R)) = |\mathrm{Min}(R)|$. □

In the next proposition we study the case that the independence number of $\Omega(R)$ is finite.

Proposition 3.9. *Let R be a reduced ring such that every prime ideal contained in $Z(R)$ is a subset of a finite union of annihilator prime ideals. If $\alpha(\Omega(R)) < \infty$, then the number of annihilator ideals of R is at most $2^{\alpha(\Omega(R))}$.*

Proof. Suppose that P is a prime ideal contained in $Z(R)$ and $P \subseteq \cup_{i=1}^{n}\mathrm{Ann}(X_i)$, where $X_i \subseteq R$ and $\mathrm{Ann}(X_i)$ is an annihilator prime ideal for $i = 1, 2, \ldots, n$. Then by [6, Theorem 81], we have $P \subseteq \mathrm{Ann}(X_i)$ for some $i \in \{1, 2, \ldots, n\}$. Hence we conclude that $\mathrm{Ann}(\mathfrak{p}) \neq 0$, for all $\mathfrak{p} \in \mathrm{Min}(R)$. Now since $\alpha(\Omega(R)) < \infty$, we can suppose that $\alpha(\Omega(R)) = n$. We show that $|\mathrm{Min}(R)| = n$. To see this, let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_{n+1}$ be distinct minimal prime ideals of R. Now if $\mathfrak{p}_1$ is adjacent to $\mathfrak{p}_2$, then there exists $a \in R^*$ such that $a(\mathfrak{p}_1 + \mathfrak{p}_2) = 0$. Hence since R is reduced, we have $\mathfrak{p}_1 + \mathfrak{p}_2 \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in \mathrm{Min}(R)$ which is impossible. Thus by Corollary 3.8, we may assume that $\mathrm{Min}(R) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$. Now suppose that $X \subseteq R$. Then one can see that $\mathrm{Ann}(X)\mathrm{Ann}(\mathrm{Ann}(X)) = 0$ and hence $\mathrm{Ann}(X) \subseteq \mathrm{Ann}(\mathrm{Ann}(\mathrm{Ann}(X)))$. On the other hand, $X\mathrm{Ann}(X) = 0$ and so $X \subseteq \mathrm{Ann}(\mathrm{Ann}(X))$. Thus we conclude that $\mathrm{Ann}(\mathrm{Ann}(\mathrm{Ann}(X))) \subseteq \mathrm{Ann}(X)$. Hence $\mathrm{Ann}(\mathrm{Ann}(\mathrm{Ann}(X))) = \mathrm{Ann}(X)$. Now by our assumptions, we have $\mathrm{Ann}(X) \subseteq \cap_{i\in I}\mathfrak{p}_i$ and $\mathrm{Ann}(\cap_{j\in J}\mathfrak{p}_j) \subseteq \mathrm{Ann}(X)$, where $I := \{i \,|\, \mathrm{Ann}(X) \subseteq \mathfrak{p}_i\}$ and $J := \{i \,|\, \mathrm{Ann}(\mathrm{Ann}(X)) \subseteq \mathfrak{p}_i\}$. Since $\mathrm{Ann}(X)\mathrm{Ann}(\mathrm{Ann}(X)) = 0$, $I \cup J = \{1, 2 \ldots, n\}$. On the other hand, since R is a reduced ring, $(\cap_{i\in I}\mathfrak{p}_i)\cap(\cap_{j\in J}\mathfrak{p}_j) = 0$. Now we have $\cap_{i\in I}\mathfrak{p}_i \subseteq \mathrm{Ann}(\cap_{j\in J}\mathfrak{p}_j)$ (see [9, Remarks 2.28(i)]). Thus we conclude that the number of annihilator ideals of R is at most $2^{\alpha(\Omega(R))}$ (note that we have $\alpha(\Omega(R)) = |\mathrm{Min}(R)|$). □

Remark 3.10. Let R be an Artinian ring. Then by [7, Theorem 4.12], $\mathrm{J}(R)$ is nilpotent. Also, by [9, Exercise 8.50] we have $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is a local Artinian ring for $i = 1, 2, \ldots, n$. Moreover, $|\mathrm{Max}(R)| = n$ and $Z(R) = \cup_{i=1}^{n} \mathfrak{m}_i$, where $\mathfrak{m}_i = \mathrm{Ann}(x_i)$ is a maximal ideal of R and $x_i \in R^*$ for $i = 1, 2, \ldots, n$. We use this fact in the sequel.

In the following proposition we study the case that $\Omega(R)$ is unicyclic.

Proposition 3.11. *Let R be a ring. Then $\Omega(R)$ is unicyclic if and only if R is a local ring with exactly three non-trivial ideals or $R \cong \mathbb{F} \times S$, where $\mathbb{F}$ is a field and S is a ring with exactly one non-trivial ideal.*

Proof. Assume that $\Omega(R)$ is unicyclic. Then every vertex of $\Omega(R)$ contains at most three non-zero ideals. Thus R contains a minimal ideal, say Rx. Hence $Rx \cong \frac{R}{\mathrm{Ann}(x)}$ as an R-module isomorphism and hence $\frac{R}{\mathrm{Ann}(x)}$ is an Artinian R-module. Also, since $\mathrm{Ann}(x)$ is a vertex of $\Omega(R)$, $\mathrm{Ann}(x)$ satisfies the descending chain condition on R-submodules and so $\mathrm{Ann}(x)$ is an Artinian R-module. Now since $0 \to \mathrm{Ann}(x) \to R \to \frac{R}{\mathrm{Ann}(x)} \to 0$ is an exact sequence, R is an Artinian ring by [9, Corollary 7.19]. Thus $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is a local Artinian ring for $i = 1, 2, \ldots, n$. First assume that R is not decomposable. Then since $\mathrm{J}(R)$ is nilpotent, we conclude that the number of non-trivial ideals of R is exactly three. Now assume that R is decomposable. We show that $n = 2$. To see this, let $n = 3$. Then $R_1 \times R_2 \times 0 - R_1 \times 0 \times 0 - 0 \times R_2 \times 0 - R_1 \times R_2 \times 0$ and $0 \times R_2 \times R_3 - 0 \times 0 \times R_3 - 0 \times R_2 \times 0 - 0 \times R_2 \times R_3$ are cycles in $\Omega(R)$ which is a contradiction. If $n > 3$, then by a similar method one can see that $\Omega(R)$ is not unicyclic. Thus we may assume that $n = 2$. Now if R_1 and R_2 are not fields, then $R_1 \times 0 - I_1 \times 0 - I_1 \times I_2 - R_1 \times 0$ and $0 \times R_2 - 0 \times I_2 - I_1 \times I_2 - 0 \times R_2$, where I_i is a non-trivial ideal of R_i for $i = 1, 2$, are cycles in $\Omega(R)$ which is a contradiction. Now without loss of generality, we can suppose that R_1 is a field. If R_2 is a ring with two distinct non-trivial ideals J_1 and J_2, then $R_1 \times 0 - R_1 \times J_1 - 0 \times J_1 - R_1 \times 0$ and $0 \times R_2 - 0 \times J_1 - 0 \times J_2 - 0 \times R_2$ are cycles in $\Omega(R)$ which is impossible. Therefore, R_2 is a ring with exactly one non-trivial ideal.

The converse is clear. □

Proposition 3.12. *Let R be a ring. If for all vertices of the form $I = \mathrm{Ann}(X)$, where $X \subseteq R$, $\deg(I) < \infty$, then $\Omega(R)$ is a finite graph.*

Proof. Suppose that $I = \mathrm{Ann}(X)$, where $X \subseteq R$, is a vertex of $\Omega(R)$. Since $\deg(I) < \infty$, I satisfies the descending chain condition on R-submodules. Now by a method similar to that we used in the proof of Proposition 3.11, we conclude that R is an Artinian ring. Then $|\mathrm{Max}(R)| = n$ and $Z(R) = \cup_{i=1}^{n} \mathfrak{m}_i$, where $\mathfrak{m}_i = \mathrm{Ann}(x_i)$ is a maximal ideal of R and $x_i \in R^*$ for $i = 1, 2, \ldots, n$. Now since $\deg(\mathfrak{m}_i) < \infty$ for $i = 1, 2, \ldots, n$, the number of ideals contained in $\mathfrak{m}_i$ for $i = 1, 2, \ldots, n$, is finite. Hence we conclude that $\Omega(R)$ is a finite graph. □

Example 3.13. Assume that $R = \mathbb{Z}_2 \times \mathbb{Z}$. Then one can see that $\deg(\mathbb{Z}_2 \times 0) = 0$. Also, $\Omega(R)$ is not a finite graph.

Proposition 3.14. *Let R be a reduced ring. If $\Omega(R)$ contains a vertex of finite degree, then $R \cong \mathbb{F} \times S$, where $\mathbb{F}$ is a field and S is a reduced ring.*

Proof. Let I be a vertex of $\Omega(R)$ such that $\deg(I) < \infty$. Then I satisfies the descending chain condition on R-submodules. Thus R contains a minimal ideal, say J. Now by [7, Lemma 10.22], we have $J = Rx$, where $x^2 = x$. Hence by [7, Exercise 1.7], it is easy to see that $R \cong \mathbb{F} \times S$, where $\mathbb{F}$ is a field and S is a reduced ring. □

Recall that a graph is said to be planar if it can drawn in the plane so that its edges intersect only at their ends. A *subdivision* of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. By Kuratowski's Theorem, there is a characterization for planar graphs that say a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ (see [12, Theorem 6.2.2]). In the next proposition we study the planarity of $\Omega(R)$.

Proposition 3.15. *Let R be a ring such that $\Omega(R)$ is planar.*

(a) *If R is not decomposable, then the number of non-trivial ideals of R is at most four.*

(b) *If R is decomposable, then R is isomorphic to one of the following rings:*

$$\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3,\ \mathbb{F} \times S$$

where $\mathbb{F}$ and $\mathbb{F}_i$ are fields for $i = 1, 2, 3$ and S is a ring with at most one non-trivial ideal.

Proof. (a) Suppose that I is a vertex of $\Omega(R)$. Since $\Omega(R)$ is planar, by Kuratowski's Theorem the number of non-zero ideals contained in I is at most four. Now by a method similar to that we used in the proof of Proposition 3.11, we conclude that R is an Artinian ring. Thus $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is a local Artinan ring for $i = 1, 2, \ldots, n$. Since R is not decomposable, R is a local Artinian ring. On the other hand, $\mathrm{J}(R)$ is nilpotent. Then the number of non-trivial ideals of R is at most four.

(b) Assume that R is decomposable. Then since $\Omega(R)$ is planar, we have $R \cong R_1 \times R_2 \times \cdots \times R_n$, where $n \neq 1$ and R_i is a local Artinan ring for $i = 1, 2, \ldots, n$. We claim that $n \leq 3$. To see this, let $R \cong R_1 \times R_2 \times R_3 \times R_4$. Then one can easily see that the set $\{R_1 \times R_2 \times R_3 \times 0, R_1 \times R_2 \times 0 \times 0, R_1 \times 0 \times 0 \times 0, 0 \times R_2 \times 0 \times 0, 0 \times 0 \times R_3 \times 0\}$ forms K_5 and hence $\Omega(R)$ is not planar which is a contradiction. If $n > 4$, then by a similar method one can see that $\Omega(R)$ is not planar. Now suppose that $n = 3$. If $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$, where $\mathbb{F}_i$ is a field for $i = 1, 2, 3$, then it is easy to see that $\Omega(R)$ is planar. Otherwise, without loss of generality we may assume that R_1 contains a non-trivial ideal as I_1. Then the set $\{R_1 \times R_2 \times 0, R_1 \times 0 \times 0, I_1 \times R_2 \times 0, I_1 \times 0 \times 0, 0 \times R_2 \times 0\}$ forms K_5 and hence $\Omega(R)$ is not planar. Now suppose that $n = 2$. Assume that R_1 and R_2 are not fields. If I_i is a non-trivial ideal of R_i for $i = 1, 2$, then the set $\{R_1 \times I_2, R_1 \times 0, I_1 \times I_2, I_1 \times 0, 0 \times I_2\}$ forms K_5 and hence $\Omega(R)$ is not planar which is a contradiction. Thus without loss of generality, we may assume that R_1 is a field. If J_1 and J_2 are two distinct non-trivial ideals of R_2, then the set $\{R_1 \times 0, 0 \times J_1, 0 \times J_2, R_1 \times J_1, R_1 \times J_2\}$ forms K_5 and hence $\Omega(R)$ is not planar which is a contradiction. Thus the number of non-trivial ideals of R_2 must be at most one. □

Corollary 3.16. *Let R be a ring. If $\Omega(R)$ is planar, then R is an Artinian ring.*

In the next proposition we study the case that $\Omega(R)$ is a connected bipartite graph.

Proposition 3.17. *Let R be a ring and $\mathrm{diam}(\Omega(R)) \neq 0$. Then $\Omega(R)$ is a connected bipartite graph if and only if $\Omega(R) = K_2$.*

Proof. Suppose that $\Omega(R)$ is bipartite. Then by [12, Theorem 1.2.18], $\Omega(R)$ contains no cycle of odd length. Thus every vertex of $\Omega(R)$ contains at most two non-zero ideals. Then by a similar way as used in the proof of Proposition 3.11, one can see that R is an Artinian ring. Hence every proper ideal of R is an annihilating ideal (see Remark 3.10). We claim that R is indecomposable. To see this, assume that $R \cong R_1 \times R_2$, where R_1 and R_2 are rings. If R_1 and R_2 are fields, then $\Omega(R)$ is disconnected which is a contradiction. Otherwise, let I_1 be a non-trivial ideal of R_1. Then $0 \times R_2 - I_1 \times 0 - I_1 \times R_2 - 0 \times R_2$ is a cycle of length three in $\Omega(R)$ which is a contradiction. Thus R is indecomposable and hence by Remark 3.10, R is a local ring with exactly two non-trivial ideals. Therefore, $\Omega(R) = K_2$.

The converse is clear. □

In the next proposition we study the case that $\mathrm{J}(R)$ is a vertex of $\Omega(R)$.

Proposition 3.18. *Let R be a ring with finitely many maximal ideals. Then $\mathrm{J}(R)$ is a vertex of $\Omega(R)$ if and only if R contains a minimal ideal and $\mathrm{J}(R) \neq 0$.*

Proof. Suppose that $|\mathrm{Max}(R)| < \infty$. We claim that $\mathrm{soc}(R) = \mathrm{Ann}(\mathrm{J}(R))$. Since $|\mathrm{Max}(R)| < \infty$, by [9, Exercise 3.60] we have $\frac{R}{\mathrm{J}(R)} \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_n$, where $\mathbb{F}_i$ is a field for $i = 1, 2, \ldots, n$. Thus $\frac{R}{\mathrm{J}(R)}$ is an Artinian ring. Therefore, $\mathrm{soc}(R) = \mathrm{Ann}(\mathrm{J}(R))$ by [7, Exercise 4.18]. Now $\mathrm{J}(R)$ is a vertex of $\Omega(R)$ if and only if R contains a minimal ideal and $\mathrm{J}(R) \neq 0$. □

Proposition 3.19. *Let R be a ring. If for all vertices of the form $I = \mathrm{Ann}(X)$, where $X \subseteq R$, $\deg(I) < \infty$, then $\mathrm{J}(R)^{\omega(\Omega(R))+1} = 0$.*

Proof. Suppose that for all vertices of the form $I = \mathrm{Ann}(X)$, where $X \subseteq R$, we have $\deg(I) < \infty$. Then by Proposition 3.12, $\Omega(R)$ is a finite graph. Thus $\omega(\Omega(R)) < \infty$. Hence every vertex of $\Omega(R)$ satisfies the descending chain condition on R-submodules. Now by a method similar to that we used in the proof of

Proposition 3.11, we conclude that R is an Artinian ring. Thus by Remark 3.10, $\mathrm{J}(R)$ is nilpotent. If $\mathrm{J}(R)^{\omega(\Omega(R))+1} \neq 0$, then the subgraph induced by vertices $\{\mathrm{J}(R), \mathrm{J}(R)^2, \ldots, \mathrm{J}(R)^{\omega(\Omega(R))+1}\}$ is a clique of $\Omega(R)$ with $\omega(\Omega(R))+1$ vertices which is impossible. Therefore, we conclude that $\mathrm{J}(R)^{\omega(\Omega(R))+1} = 0$. □

Proposition 3.20. *Let R be an Artinian ring such that $\mathrm{J}(R) \neq 0$. Then $\mathrm{J}(R)$ is adjacent to every other vertex.*

Proof. Assume that R is an Artiniain ring. Then by Remark 3.10, $\mathrm{J}(R)$ is nilpotent and hence $\mathrm{J}(R)$ is a vertex of $\Omega(R)$. Now by [7, Exercise 4.18], we have $\mathrm{soc}(R) = \mathrm{Ann}(\mathrm{J}(R))$. Since each non-zero ideal of R contains a minimal ideal, we conclude that $\mathrm{Ann}(\mathrm{J}(R)) \leq_e R$. Thus by Lemma 3.2, $\mathrm{J}(R)$ is adjacent to every other vertex. □

The zero-divisor graph of a commutative ring R, denoted by $\Gamma(R)$, is a graph with the vertex-set $Z(R)^*$ and two distinct vertices x and y are adjacent if $xy = 0$. By [1, Theorem 2.3], $\Gamma(R)$ is connected and $\mathrm{diam}(\Gamma(R)) \leq 3$. In the next proposition, we study the relations between the diameters of $\Omega(R)$ and $\Gamma(R)$. Before that, we need the following theorem.

Theorem 3.21 ([8, Theorem 2.6])**.** *Let R be a ring.*

(a) $\mathrm{diam}(\Gamma(R)) = 0$ *if and only if either* $R \cong \mathbb{Z}_4$ *or* $R \cong \frac{\mathbb{Z}_2[x]}{(x^2)}$.
(b) $\mathrm{diam}(\Gamma(R)) = 1$ *if and only if* $xy = 0$ *for all distinct* $x, y \in Z(R)$ *and* $|Z(R)| \geq 3$.
(c) $\mathrm{diam}(\Gamma(R)) = 2$ *if and only if either R is reduced with exactly two minimal prime ideals and at least three non-zero zero-divisors, or $Z(R)$ is an ideal whose square is not zero and each pair of distinct zero-divisors has a non-zero annihilator.*
(d) $\mathrm{diam}(\Gamma(R)) = 3$ *if and only if there are distinct* $x, y \in Z(R)^*$ *such that* $\mathrm{Ann}(x) \cap \mathrm{Ann}(y) = 0$ *and either R is a reduced ring with more than two minimal prime ideals, or R is non-reduced.*

Note that by [10], $\mathrm{diam}(\Omega(R)) \in \{0, 1, 2, \infty\}$ (see [10, Lemmas 2.3, 3.1, 3.3, 3.4 and 4.1]).

Proposition 3.22. *Let R be a ring. Then we have the following statements*:

(a) *If* $\mathrm{diam}(\Gamma(R)) = 0$, *then* $\mathrm{diam}(\Omega(R)) = 0$.
(b) *If* $\mathrm{diam}(\Gamma(R)) = 1$ *and* $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$, *then* $\mathrm{diam}(\Omega(R)) \in \{0, 1\}$.
(c) *If* $\mathrm{diam}(\Gamma(R)) = 2$, *then* $\mathrm{diam}(\Omega(R)) \in \{1, 2, \infty\}$.
(d) *If* $\mathrm{diam}(\Gamma(R)) = 3$, *then* $\mathrm{diam}(\Omega(R)) = 2$.
(e) *If* $\mathrm{diam}(\Omega(R)) = 0$, *then* $\mathrm{diam}(\Gamma(R)) \in \{0, 1\}$.
(f) *If* $\mathrm{diam}(\Omega(R)) = 1$, *then* $\mathrm{diam}(\Gamma(R)) \in \{1, 2\}$.
(g) *If* $\mathrm{diam}(\Omega(R)) = 2$, *then* $\mathrm{diam}(\Gamma(R)) \in \{2, 3\}$.
(h) *If* $\mathrm{diam}(\Omega(R)) = \infty$ *and* $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$, *then* $\mathrm{diam}(\Gamma(R)) = 2$.

Proof. (a) Assume that $\mathrm{diam}(\Gamma(R)) = 0$. Then by Theorem 3.21(a), either $R \cong \mathbb{Z}_4$ or $R \cong \frac{\mathbb{Z}_2[x]}{(x^2)}$ and hence $\mathrm{diam}(\Omega(R)) = 0$.

(b) Assume that $\mathrm{diam}(\Gamma(R)) = 1$. Since $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$, by [1, Theorem 2.8] we have $xy = 0$ for all $x, y \in Z(R)$. Thus, $\mathrm{diam}(\Omega(R)) \in \{0, 1\}$.

(c) Assume that $\mathrm{diam}(\Gamma(R)) = 2$. Then by Theorem 3.21(c), either R is reduced with exactly two minimal prime ideals and at least three non-zero zero-divisors, or $Z(R)$ is an ideal whose square is not zero and each pair of distinct zero-divisors has a non-zero annihilator. Now if R is a reduced ring with exactly two minimal prime ideals, then by Lemma 2.2 we have $\mathrm{diam}(\Omega(R)) = \infty$. Otherwise, we may assume that $Z(R)$ is an ideal whose square is not zero and each pair of distinct zero-divisors has a non-zero annihilator. Then we have $\mathrm{diam}(\Omega(R)) \in \{1, 2\}$.

(d) Assume that $\mathrm{diam}(\Gamma(R)) = 3$. Then by Theorem 3.21(d), there are distinct $x, y \in Z(R)^*$ such that $\mathrm{Ann}(x) \cap \mathrm{Ann}(y) = 0$ and either R is a reduced ring with more than two minimal prime ideals, or R is non-reduced. Now by Lemma 2.2, we have $\mathrm{diam}(\Omega(R)) = 2$.

(e) Assume that $\mathrm{diam}(\Omega(R)) = 0$. Then $Z(R)$ is an ideal whose square is zero. Hence by Theorem 3.21(c) and (d), we have either $\mathrm{diam}(\Gamma(R)) = 0$ or $\mathrm{diam}(\Gamma(R)) = 1$.

(f) Assume that $\mathrm{diam}(\Omega(R)) = 1$. Then the subgraph induced by the principal ideals of R is complete and hence one can easily see that $Z(R)$ is an ideal. If $Z(R) = \mathrm{Ann}(x)$ for some non-zero $x \in Z(R)$, then $\mathrm{diam}(\Gamma(R)) \in \{1, 2\}$. Otherwise, if $Z(R)$ is not an annihilator ideal, then $Z(R)$ is an ideal whose square is not zero. Since $\mathrm{diam}(\Omega(R)) = 1$, each pair of distinct zero-divisors has a non-zero annihilator (see Lemma 3.2). Hence by Theorem 3.21(c), we have $\mathrm{diam}(\Gamma(R)) = 2$.

(g) Assume that $\mathrm{diam}(\Omega(R)) = 2$. Then by [1, Theorem 2.8], one can easily see that $\mathrm{diam}(\Gamma(R)) \in \{2, 3\}$.

(h) Assume that $\mathrm{diam}(\Omega(R)) = \infty$. Then by Lemma 2.2, R is reduced with exactly two minimal prime ideals. Hence since $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$, by Theorem 3.21(c) we have $\mathrm{diam}(\Gamma(R)) = 2$. $\square$

Corollary 3.23. *Let R be a ring. If $\mathrm{diam}(\Omega(R)) = 0$, then $\mathrm{diam}(\Omega(R[[x]])) = 1$.*

Proof. Suppose that $\mathrm{diam}(\Omega(R)) = 0$. Then by Proposition 3.22, we have $\mathrm{diam}(\Gamma(R)) \in \{0, 1\}$. Now by [2, Theorem 3], $\mathrm{diam}(\Gamma(R[[x]])) = 1$. Therefore, by Propositions 3.22 and 2.1, we have $\mathrm{diam}(\Omega(R[[x]])) = 1$. $\square$

In the next proposition we study the relation between the clique number of $\Gamma(R)$ and the independence number of $\Omega(R)$.

Proposition 3.24. *Let R be a reduced ring. If $\omega(\Gamma(R)) < \infty$, then we have $\alpha(\Omega(R)) < \infty$.*

Proof. Suppose that R is a reduced ring and $\omega(\Gamma(R)) < \infty$. Then by [3, Theorem 3.7], $|\mathrm{Min}(R)| < \infty$. Therefore by Corollary 3.8, $\alpha(\Omega(R)) < \infty$. $\square$

4. The complement of $\Omega(R)$

The complement of $\Omega(R)$, introduced and studied in [11], is the graph with the same vertex-set as $\Omega(R)$, where two distinct vertices I and J are adjacent if and only if $\mathrm{Ann}(I+J)=0$. We use $\overline{\Omega(R)}$ to denote the complement of $\Omega(R)$. Note that by Corollary 3.5, if R is non-reduced and $|\mathrm{A}(R)^*|\geq 2$, then $\overline{\Omega(R)}$ is not connected. Moreover, by [11, Proposition 2.4], if R is a reduced ring, then $\mathrm{diam}(\overline{\Omega(R)})\leq 3$. Thus we have the following lemma.

Lemma 4.1. *Let R be a ring. Then $\overline{\Omega(R)}$ is connected if and only if R is a reduced ring or $|\mathrm{A}(R)^*|=1$.*

Proof. See [11, Proposition 2.4] and Corollary 3.5. □

In the next theorem we study the relations between the connectivity of the graphs $\overline{\Omega(R)}$, $\overline{\Omega(R[x])}$ and $\overline{\Omega(R[[x]])}$.

Theorem 4.2. *Let R be a ring. Then the following statements are equivalent:*
(a) *$\overline{\Omega(R)}$ is disconnected or $|\mathrm{A}(R)^*|=1$.*
(b) *$\overline{\Omega(R[x])}$ is disconnected.*
(c) *$\overline{\Omega(R[[x]])}$ is disconnected.*

Proof. (a) $\Longleftrightarrow$ (b) Suppose that $\overline{\Omega(R)}$ is disconnected or $|\mathrm{A}(R)^*|=1$. Then by Lemma 4.1, R is non-reduced. Hence $R[x]$ is non-reduced and so $\overline{\Omega(R[x])}$ is disconnected.

Conversely, assume that $\overline{\Omega(R[x])}$ is disconnected. Then $R[x]$ is non-reduced and hence by [9, Exercise 1.36], we conclude that R is non-reduced. Thus $\overline{\Omega(R)}$ is disconnected or $|\mathrm{A}(R)^*|=1$.

(a) $\Longleftrightarrow$ (c) follows similarly. □

In the following lemma we provide a shorter proof for [11, Proposition 2.11].

Lemma 4.3. *Let R be a ring. Then $\overline{\Omega(R)}$ is complete if and only if either $Z(R)$ is a minimal ideal of R or $R\cong\mathbb{F}_1\times\mathbb{F}_2$, where $\mathbb{F}_i$ is a field for $i=1,2$.*

Proof. Suppose that $\overline{\Omega(R)}$ is complete. If R is non-reduced, then $Z(R)$ is a minimal ideal of R, by Corollary 3.5. Now assume that R is reduced. Let I be a vertex of $\overline{\Omega(R)}$. Since $\overline{\Omega(R)}$ is complete, I is a minimal ideal of R. Hence $I=Re$, where $e^2=e$, by [7, Lemma 10.22]. Thus $R\cong R_1\times R_2$, where R_i is a field for $i=1,2$ (see [7, Exercise 1.7]).

The converse is clear. □

By Lemma 4.3, if $\mathrm{diam}(\overline{\Omega(R)})=1$, then $R\cong\mathbb{F}_1\times\mathbb{F}_2$, where $\mathbb{F}_i$ is a field for $i=1,2$. Also, by Proposition 2.1, we have $\mathrm{diam}(\overline{\Omega(R[x])})\neq 0$ and $\mathrm{diam}(\overline{\Omega(R[[x]])})\neq 0$. Now by [11, Proposition 2.4] and Theorem 4.2, we have the following corollary.

Corollary 4.4. *Let R be a reduced ring. Then*

(a) $\operatorname{diam}(\overline{\Omega(R[x])}) \in \{2,3\}$.
(b) $\operatorname{diam}(\overline{\Omega(R[[x]])}) \in \{2,3\}$.

In next two propositions, we investigate the relations between the diameters of the graphs $\overline{\Omega(R)}$, $\overline{\Omega(R[x])}$ and $\overline{\Omega(R[[x]])}$.

Proposition 4.5. *Let R be a reduced ring. Then*

(a) $\operatorname{diam}(\overline{\Omega(R)}) \in \{1,2\}$ *if and only if* $\operatorname{diam}(\overline{\Omega(R[x])}) = 2$.
(b) $\operatorname{diam}(\overline{\Omega(R)}) = 3$ *if and only if* $\operatorname{diam}(\overline{\Omega(R[x])}) = 3$.

Proof. (a) Suppose that $\operatorname{diam}(\overline{\Omega(R)}) \in \{1,2\}$. Then by [11, Remark 2.19], R has exactly two minimal prime ideals. Let $\operatorname{Min}(R) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$. Thus, $\mathfrak{p}_1[x]$ and $\mathfrak{p}_2[x]$ are prime ideals of $R[x]$. Now since $\mathfrak{p}_1 \cap \mathfrak{p}_2 = 0$, we have $\mathfrak{p}_1[x] \cap \mathfrak{p}_2[x] = (\mathfrak{p}_1 \cap \mathfrak{p}_2)[x] = 0$. Thus $R[x]$ is a reduced ring and $|\operatorname{Min}(R[x])| = 2$. Therefore, $\operatorname{diam}(\overline{\Omega(R[x])}) = 2$, by [11, Remark 2.19] and Corollary 4.4.

Conversely, assume that $\operatorname{diam}(\overline{\Omega(R[x])}) = 2$. Then by [11, Remark 2.19], $R[x]$ is a reduced ring with exactly two minimal prime ideals. Now by [9, Remarks 3.27(ii)] and [9, Exercise 2.43(iii)], it is easy to see that R is a reduced ring with exactly two minimal prime ideals. Thus $\operatorname{diam}(\overline{\Omega(R)}) \in \{1,2\}$, by [11, Remark 2.19].

(b) By [11, Proposition 2.4], Corollary 4.4 and item (a), it is clear. □

Now by a method similar to that we used in the proof of Proposition 4.5, we have the following proposition.

Proposition 4.6. *Let R be a reduced ring. Then*

(a) $\operatorname{diam}(\overline{\Omega(R)}) \in \{1,2\}$ *if and only if* $\operatorname{diam}(\overline{\Omega(R[[x]])}) = 2$.
(b) $\operatorname{diam}(\overline{\Omega(R)}) = 3$ *if and only if* $\operatorname{diam}(\overline{\Omega(R[[x]])}) = 3$.

In the next proposition we study the case that $\overline{\Omega(R)}$ is unicyclic.

Proposition 4.7. *Let R be a ring such that $|\operatorname{Min}(R)| < \infty$ and x is unit for all $x \in R \setminus Z(R)$. Then $\overline{\Omega(R)}$ is unicyclic if and only if $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$, where $\mathbb{F}_i$ is a field for $i = 1,2,3$.*

Proof. Assume that $\overline{\Omega(R)}$ is unicyclic. Then since $\overline{\Omega(R)}$ is connected, R is a reduced ring (see Lemma 4.1). Now by [5, Corollary 2.4] we have $Z(R) = \cup_{\mathfrak{p} \in \operatorname{Min}(R)} \mathfrak{p}$. Moreover, by [6, Theorem 81] we have $\operatorname{Min}(R) = \operatorname{Max}(R)$. Then it is easy to see that the subgraph induced by vertices of $\operatorname{Min}(R)$ is complete. Hence we conclude that $|\operatorname{Min}(R)| \leq 3$. Now assume that $|\operatorname{Min}(R)| = 3$. Then by [9, Exercise 3.60], we have $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$, where $\mathbb{F}_i$ is a field for $i = 1,2,3$. Thus one can see that $\overline{\Omega(R)}$ is unicyclic. Now assume that $|\operatorname{Min}(R)| = 2$. Then by [9, Exercise 3.60], we have $R \cong \mathbb{F}_1 \times \mathbb{F}_2$, where $\mathbb{F}_i$ is a field for $i = 1,2$. Thus $\overline{\Omega(R)}$ is not unicyclic which is a contradiction.

The converse is clear. □

In the following proposition we find a dominating set in $\overline{\Omega(R)}$.

Proposition 4.8. *Let R be a reduced ring such that $|\mathrm{Min}(R)| < \infty$. Then $\mathrm{Min}(R)$ is a dominating set of $\overline{\Omega(R)}$.*

Proof. Since R is a reduced ring with $|\mathrm{Min}(R)| < \infty$, $\mathrm{Ann}(\mathfrak{p}) \neq 0$ for all $\mathfrak{p} \in \mathrm{Min}(R)$. Let $\mathrm{Min}(R) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$. Now assume that I is a vertex of $\overline{\Omega(R)} \setminus \mathrm{Min}(R)$. Since R is a reduced ring, there exists $\mathfrak{p}_i \in \mathrm{Min}(R)$ such that $I \nsubseteq \mathfrak{p}_i$. We show that I is adjacent to $\mathfrak{p}_i$ in $\overline{\Omega(R)}$. To see this, let $y \in R^*$ such that $y(I + \mathfrak{p}_i) = 0$. Then since R is a reduced ring, we have $y \in \cap_{j=1, j\neq i}^{n} \mathfrak{p}_j$. Hence $I \subseteq \mathfrak{p}_i$ which is impossible. Therefore, $\mathrm{Ann}(I + \mathfrak{p}_i) = 0$ and so I is adjacent to $\mathfrak{p}_i$ in $\overline{\Omega(R)}$. Thus $\mathrm{Min}(R)$ is a dominating set of $\overline{\Omega(R)}$. □

Example 4.9. Assume that $R = \frac{\mathbb{Q}[x,y]}{(x^2,xy,y^2)}$. Then it is easy to see that R is a local ring and $\mathrm{J}(R)$ is nilpotent. Thus $\deg(I) = 0$, for all vertices I of $\overline{\Omega(R)}$. Therefore, $\mathrm{A}(R)^*$ is a dominating set in $\overline{\Omega(R)}$.

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