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ON EVOLUTION OF FINSLER RICCI SCALAR

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Abstract. Here, we calculate the evolution equation of the reduced hh-curvature and the Ricci scalar along the Finslerian Ricci flow. We prove that Finsler Ricci flow preserves positivity of the reduced hh-curvature on finite time. Next, it is shown that evolution of Ricci scalar is a parabolic-type equation and moreover if the initial Finsler metric is of positive flag curvature, then the flag curvature, as well as the Ricci scalar, remain positive as long as the solution exists. Finally, we present a lower bound for Ricci scalar along Ricci flow.

Introduction

In the last two decades, geometric flows and more notably among them, Ricci flow, have proven to be useful in the study of long-standing conjectures in geometry and topology of Riemannian manifolds. One of its important issues concerns discovering the so-called round metrics (of constant curvature, Einstein, Soliton, etc.) on manifolds by evolving an initial Riemannian metric tensor to make it rounder and draw geometric and topological conclusions from the final round metric. Similarly, several natural questions arise in Finsler geometry, among them is S. S. Chern's question which asks whether there exists a Finsler-Einstein metric on every smooth manifold.

Introducing a similar evolution equation in Finsler geometry involves overcoming a number of new conceptual and fundamental issues in relation to the different definitions of Ricci tensors, existence problem and geometric and physical characterizations of the resulting flows. In [3], D. Bao based on the Akbar-Zadeh's Ricci tensor and in analogy with the Ricci flow in Riemannian geometry has considered the following equation as Ricci flow in Finsler geometry

$$\frac{\partial}{\partial t}\log F = -\mathcal{R}ic, \qquad F(0) = F_0, \tag{1}$$

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where F_0 is the initial Finsler structure. Comparing the definition of Ricci tensor by Hilbert as a critical point of an energy functional and Hamilton's tricks for the definition of normal Ricci flow in Riemannian geometry shows that the definition of D. Bao for Finslerian Ricci flow is quite reasonable. This equation addresses the evolution of the Finsler structure F and seems to make sense, as an un-normalized Ricci flow for Finsler spaces on both the bundle of nonzero tangent vectors TM_0 and the sphere bundle SM. One feature of the equation (1) is its independence to the choice of Cartan, Berwald or Chern connections.

In recent years the interest in Ricci flow in Finsler geometry has grown drastically, where we just cite more recent ones, for instance, [2, 10, 11, 13], etc. The present authors in several joint works, have studied the evolution of Finsler metrics under different Ricci flows. First, the Finslerian Ricci soliton as a self-similar solution to the Finslerian Ricci flow has been introduced and it was shown if there is a Ricci soliton on a compact Finsler manifold then there exists a solution to the Finsler Ricci flow equation and vice-versa, see [9]. Next, as a first step to answering Chern's question, we have considered evolution of a family of Finsler metrics, first under a general flow next under the Finsler Ricci flow and it has been shown that a family of Finsler metrics $g(t)$ which are solutions to the Finsler Ricci flow converge to a smooth limit Finsler metric as t approaches the finite time T, see [15]. Moreover, a Bonnet-Myers type theorem was studied and it is proved that on a Finsler space, a forward complete shrinking Ricci soliton is compact if and only if the corresponding vector field is bounded, using which we have shown a compact shrinking Finsler Ricci soliton has finite fundamental group and hence the first de Rham cohomology group vanishes, see [14].

The existence and uniqueness of solution to the evolution equation (1) in Finsler geometry, is also studied by the present authors in [6,7]. Finally, another significant Ricci flow in Finsler geometry is considered and evolution of Cartan hh-curvature, Ricci tensor and scalar curvature have been obtained in [8].

In the present work, we derive evolution equations for the reduced hh-curvature of Finsler structure $R(X, Z)$ and the Ricci scalar $\mathcal{R}ic$ along the Ricci flow and show that the evolution of Ricci scalar is a parabolic type equation. Next another step to the study of Chern's question is taken and it is shown that if $(M, F(0))$ has positive reduced hh-curvature at the initial time $t = 0$ then its sign remains positive for all $t \in [0, T)$. More precisely, among the others, we prove the following main theorems.

Theorem 1. *Let (M^n, F_0) be a compact Finslerian manifold and $F(t)$ a solution to the evolution equation* (1)*, satisfying a uniform bound for the Ricci tensor on a finite time interval $[0, T)$, where $F(0) = F_0$. If $(M, F(0))$ is of positive flag curvature, then $(M, F(t))$ has positive flag curvature and positive Ricci scalar for all $t \in [0, T)$.*

Theorem 2. *Let* (M^n, F_0) *be a compact Finslerian manifold and* $F(t)$ *a solution to the evolution equation* (1)*, satisfying a uniform bound for the Ricci tensor on a finite time interval* $[0,T)$*, where* $F(0) = F_0$*. If* $(M, F(0))$ *has positive flag curvature and* $\inf_{SM} \mathcal{R}ic_{g(0)} = \alpha > 0$*, then* $\mathcal{R}ic_{g(t)} \geq \frac{\alpha}{1+\alpha t}$ *for all* $t \in [0,T)$.

1. Preliminaries and notations

In order to study evolution equations in Finsler geometry, in analogy with Riemannian geometry, it is more convenient to use global definitions of curvature tensors. In the present work, we use notations and terminologies of [1] and [4]. Here and everywhere in this paper all manifolds are assumed to be closed (compact and without boundary).

Let M be a real n-dimensional manifold of class C^∞. We denote by TM the tangent bundle of tangent vectors, by $\pi : TM_0 \longrightarrow M$ the fiber bundle of non-zero tangent vectors and by $\pi^*TM \longrightarrow TM_0$ the pull back tangent bundle. Let F be a Finsler structure on TM_0 and g the related Finslerian metric. A *Finsler manifold* is denoted here by the pair (M, F). Any point of TM_0 is denoted by $z = (x, y)$, where $x = \pi z \in M$ and $y \in T_{\pi z}M$. We denote by TTM_0, the tangent bundle of TM_0 and by ρ, the canonical linear mapping $\rho : TTM_0 \longrightarrow \pi^*TM$, where $\rho = \pi_*$. For all $z \in TM_0$, let V_zTM be the set of all vertical vectors at z, that is, the set of vectors which are tangent to the fiber through z.

It is well known that TTM_0 can be decomposed on horizontal and vertical subspaces, $TTM_0 = HTM \oplus VTM$. This decomposition permits to write a vector field $\hat{X} \in \mathcal{X}(TM_0)$ into the horizontal and vertical form $\hat{X} = H\hat{X} + V\hat{X}$, uniquely. The corresponding bases are denoted by $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ where $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - G_i^j \frac{\partial}{\partial y^j}$ and G^i are the spray coefficients defined by $G^i = \frac{1}{4} g^{ih} (\frac{\partial^2 F^2}{\partial y^h \partial x^j} y^j - \frac{\partial F^2}{\partial x^h})$. In the sequel, we will denote all the vector fields on TM_0 by $\hat{X}$ and $\hat{Y}$, etc. and the corresponding sections of π^*TM by X and Y, etc. respectively, unless otherwise specified.

Let $\nabla : \mathcal{X}(TM_0) \times \Gamma(\pi^*TM) \longrightarrow \Gamma(\pi^*TM)$ be the Cartan connection. The horizontal and vertical coefficients of Cartan connection are given by $\Gamma^i_{jk} = \frac{1}{2} g^{ih} (\delta_j g_{hk} + \delta_k g_{jh} - \delta_h g_{jk})$ and $C^i_{jk} = \frac{1}{2} g^{ih} \dot{\partial}_h g_{jk}$, respectively where $\delta_k = \frac{\delta}{\delta x^k}$ and $\dot{\partial}_k = \frac{\partial}{\partial y^k}$. The horizontal covariant derivative of a $(0,2)$ tensor T is written as follows.

$$(\nabla_{H\hat{X}} T)(Y, Z) = \nabla_{H\hat{X}} T(Y, Z) - T(\nabla_{H\hat{X}} Y, Z) - T(Y, \nabla_{H\hat{X}} Z). \tag{2}$$

1.1. The *hh*-curvature tensor of Cartan connection

Let us consider the horizontal curvature operator

$$R(X, Y)Z = \nabla_{H\hat{X}} \nabla_{H\hat{Y}} Z - \nabla_{H\hat{Y}} \nabla_{H\hat{X}} Z - \nabla_{[H\hat{X}, H\hat{Y}]} Z,$$

where $X, Y, Z \in \Gamma(\pi^*TM)$ and $\hat{X}, \hat{Y} \in \mathcal{X}(TM_0)$. The *hh-curvature tensor* of Cartan connection ∇ is defined by $R(W, Z, X, Y) := g(R(X, Y)Z, W)$. Replacing W with the local frame $\{e_k\}_{k=1}^n$ we get

$$R(X, Y)Z = \sum_{k=1}^{n} R(e_k, Z, X, Y)e_k. \tag{3}$$

One can check that the hh-curvature of Cartan connection is skew-symmetric with respect to the first two vector fields as well as the last two vector fields, see [1, p. 43]. That is,

$$R(X, Y, Z, W) = -R(Y, X, Z, W),$$
$$R(X, Y, Z, W) = -R(X, Y, W, Z).$$

In a local coordinate system we have

$$R(\partial_i, \partial_j)\partial_k = R^h{}_{kij}\partial_h.$$

Recall that the upper index is placed in the *first* position, that is

$$R_{tkij} := g_{ht}R^h{}_{kij} = g(R(\partial_i, \partial_j)\partial_k, \partial_t).$$

The components of Cartan hh-curvature tensor are given by

$$R^h{}_{kij} = \delta_i\Gamma^h{}_{jk} - \delta_j\Gamma^h{}_{ik} + \Gamma^l{}_{jk}\Gamma^h{}_{il} - \Gamma^l{}_{ik}\Gamma^h{}_{jl} + R^l{}_{ij}C^h_{lk}, \tag{4}$$

where $R^l{}_{ij} = y^pR^l{}_{pij}$. The *reduced hh-curvature* is defined by

$$R(X, Z) := R(X, l, Z, l),$$

where $l := \frac{y^i}{F}\frac{\partial}{\partial x^i}$ is the unitary global section. The reduced hh-curvature is a connection free tensor called also Riemann curvature by certain authors. In the local coordinates the reduced hh-curvature is given by $R^i{}_k := \frac{1}{F^2}y^jR^i{}_{jkm}y^m$ which are entirely expressed in terms of x and y derivatives of spray coefficients G^i as follows.

$$R^i{}_k := -\frac{1}{F^2}(2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}). \tag{5}$$

Note that the components of reduced hh-curvature tensor in (5) are different in a sign by that in [4, p. 66], using Chern connection.

1.2. Flag curvature and Ricci scalar

Consider the vector field l, called the flagpole, and the unit vector $V = V^i\frac{\partial}{\partial x^i} \in \Gamma(\pi^*TM)$, called the transverse edge, which is perpendicular to the flagpole, the *flag curvature* is defined by

$$K(x, y, l \wedge V) := V^j(l^iR_{jikl}l^l)V^k =: V^jR_{jk}V^k.$$

If the transverse edge V is orthogonal to the flagpole but not necessarily of unit length, then

$$K(x, y, l \wedge V) = \frac{V^jR_{jk}V^k}{g(V, V)}. \tag{6}$$

The case in which V is neither of unit length nor orthogonal to l is treated in [4, p. 191]. The *Ricci scalar* is defined as trace of the flag curvature, i.e.,

$$\mathcal{R}ic := \sum_{\alpha=1}^{n-1} K(x, y, l \wedge e_\alpha), \tag{7}$$

where $\{e_1, \ldots, e_{n-1}, l\}$ is considered as a g-orthonormal basis for T_xM. Equivalently,

$$\mathcal{R}ic = g^{ik} R_{ik} = R^i{}_i,$$

where $R^i{}_k$ are defined by (5).

1.3. A Riemannian connection on the indicatrix

For a fixed point $x_0 \in M$, the fiber $\pi^{-1}(x_0) = T_{x_0}M$ is a submanifold of TM_0 with the Riemannian metric $\tilde{g}(X, Y) = g_{ij}(x_0, y) dy^i dy^j (X, Y)$ determined by the vertical part of Sasakian metric on TM, where $X, Y \in V_z T_{x_0}M$. The hyper-surface $S_{x_0}M = \{y \in T_{x_0}M : F(x_0, y) = 1\}$ of $T_{x_0}M$ is called *indicatrix* in $x_0 \in M$. On the other hand a hyper-surface $S_{x_0}M$ can be expressed in local coordinates by the coordinate functions

$$y^i = y^i(t^\alpha),$$

where the Greek letters $\alpha, \beta, \gamma, \ldots$ run over the range $1, \ldots, n-1$ and the Latin letters $i, j, k, \ldots$ run over the range $1, \ldots, n$. Let f be a real function defined on $S_{x_0}M$. By chain rule we have $df(y(t)) = \partial_\alpha f dt^\alpha$, where

$$\partial_\alpha = y^i_\alpha F \dot{\partial}_i, \quad y^i_\alpha = \frac{\partial y^i}{\partial t^\alpha}. \tag{8}$$

Hence ∂_α, define $(n-1)$ tangent vectors on $S_{x_0}M$. The induced Riemannian metric tensor $g_{\alpha\beta}$ on $S_{x_0}M$ is given by

$$g_{\alpha\beta} = g_{ij} y^i_\alpha y^j_\beta,$$

where $g_{ij}(x_0, y)$ are the components of Riemannian metric tensor on $T_{x_0}M$. Let $\dot{y} = y^j \dot{\partial}_j$ be a vector field tangent to the fiber through $z = \pi^{-1}(x_0)$. Partial derivatives of $F^2(x, y) = 1$ with respect to y^i, yields

$$g_{ij} y^j y^i_\alpha = g(\partial_\alpha, \dot{y}) = 0. \tag{9}$$

Therefore, $\dot{y}$ is normal to the $(n-1)$ tangent vectors y^i_α of $S_{x_0}M$ and hence the pair $(y^i_\alpha, \dot{y})$ defines n linearly independent tangent vector fields on $T_{x_0}M$. We denote by $\dot{D}_{\dot{\partial}_k}$ the corresponding Riemannian covariant derivative on $(T_{x_0}M, \tilde{g})$, where the coefficients are given by

$$\dot{D}_{\dot{\partial}_k} \dot{\partial}_j = C^i_{jk}(x_0, y) \dot{\partial}_i.$$

Let $\dot{\nabla}$ be the induced connection on $(S_{x_0}M, g_{\alpha\beta})$. Relation between $\dot{D}$ and $\dot{\nabla}$ is given by the Gaussian formula

$$\dot{D}_Y X = \dot{\nabla}_Y X - \tilde{g}(X, Y) \dot{y},$$

where $X, Y \in T_z(S_{x_0}M)$. Replacing X and Y by the basis fields ∂_α and ∂_β yields

$$\dot{\nabla}_\beta y^i_\alpha = -A^i_{jk} y^j_\alpha y^k_\beta - g_{\alpha\beta} y^i, \tag{10}$$

where $A^i_{jk} = FC^i_{jk}$, see [1, pp. 147–149].

1.4. Local basis on the unitary sphere bundle SM

Consider the sphere bundle $SM := TM/\sim$ as a quotient space, where the equivalent relation is defined by $y \sim y'$ if and only if $y = \lambda y'$ for some $\lambda > 0$. Given any $(x, y) \in TM$, we shall denote its equivalence class as a point in SM by $(x, [y]) \in SM$. The natural projection $p : SM \longrightarrow M$ pulls back the tangent bundle TM to an n-dimensional vector bundle p^*TM over the $2n-1$ dimensional base SM. Given local coordinates (x^i) on M, we shall economize on notation and regard the corresponding collections $\{\frac{\partial}{\partial x^i}\}$, $\{dx^i\}$ as local bases for the pull back bundle p^*TM and its dual p^*T^*M, respectively.

Let $\{e_a = u^i_a \frac{\partial}{\partial x^i}\}$ be a local orthonormal frame for p^*TM and $\{\omega^a = v^a_i dx^i\}$ its co-frame, where $\omega^a(e_b) = \delta^a_b$. Clearly we have $e_n := l$, where $l = \frac{y^i}{F}\frac{\partial}{\partial x^i}$ is the distinguished global section and $\omega^n = \frac{\partial F}{\partial y^i} dx^i$. Also we have $\frac{\partial}{\partial x^i} = v^a_i e_a$ and $dx^i = u^i_a \omega^a$, where relation between (u^i_a) and (v^a_i) are given by $v^a_i u^i_b = \delta^a_b$ and $u^i_a v^a_j = \delta^i_j$. For convenience, we shall also regard the e_a's and ω^a's as local vector fields and 1-forms, respectively on SM, see [5]. Let us define

$$\hat{e}_a = u^i_a \frac{\delta}{\delta x^i}, \quad \hat{e}_{n+\alpha} = u^i_\alpha F \frac{\partial}{\partial y^i},$$
$$\omega^a = v^a_i dx^i, \quad \omega^{n+\alpha} = v^\alpha_i \frac{\delta y^i}{F}.$$

It can be shown that $\{\hat{e}_a, \hat{e}_{n+\alpha}\}$ and $\{\omega^a, \omega^{n+\alpha}\}$ are local basis for the tangent bundle TSM and the cotangent bundle T^*SM, respectively, where the Latin indices $a, b, \ldots,$ run over the range $1, \ldots, n$ and the Greek indices run over the range $1, \ldots, n-1$. Tangent vectors on SM which are annihilated by all $\{\omega^{n+\alpha}\}$'s form the horizontal sub-bundle HSM of TSM. The fibers of HSM are n-dimensional and $\{\hat{e}_a\}$ is a local basis for the fibers of HSM. On the other hand, let $VSM := \cup_{x\in M} T(S_xM)$ be the vertical sub-bundle of TSM. Its fibers are $n-1$ dimensional and $\{\hat{e}_{n+\alpha}\}$ is a local basis for the fibers of VSM. Here, $\hat{e}_{n+\alpha}$ coincide with ∂_α previously mentioned in Subsection 1.3. The decomposition $TSM = HSM \oplus VSM$ holds well because HSM and VSM are directly summed, see [5].

1.5. Ricci tensors and Ricci flows in Finsler space

There are several well known definitions for Ricci tensor in Finsler geometry. For instance, H. Akbar-Zadeh has considered two Ricci tensors on Finsler manifolds in his works namely, one is defined by $Ric_{ij} := [\frac{1}{2}F^2\mathcal{R}ic]_{y^iy^j}$ and another by $Rc_{ij} := \frac{1}{2}(\mathsf{R}_{ij} + \mathsf{R}_{ji})$, where R_{ij} is the trace of hh-curvature of Cartan connection defined by $\mathsf{R}_{ij} = R^l{}_{ilj}$. D. Bao based on the first definition of Ricci tensor has considered the following Ricci flow in Finsler geometry,

$$\frac{\partial}{\partial t} g_{jk}(t) = -2Ric_{jk}, \qquad g_{(t=0)} = g_0, \tag{11}$$

where $g_{jk}(t)$ is a family of Finslerian metrics defined on $\pi^*TM \times [0, T)$. Contracting (11) with $y^j y^k$, via Euler's theorem, leads to $\frac{\partial}{\partial t} F^2 = -2F^2 \mathcal{R}ic$. That is,

$$\frac{\partial}{\partial t} \log F(t) = -\mathcal{R}ic, \qquad F_{(t=0)} = F_0, \tag{12}$$

where F_0 is the initial Finsler structure, see [3]. It can be easily verified that (11) and (12) are equivalent. This Ricci flow is used in [6, 9, 10, 14, 15]. Throughout the present work, we consider the first Akbar-Zadeh's definition of Ricci tensor and the related Ricci flow studied by D. Bao.

One of the advantages of the Ricci quantity used here is its independence on the choice of Cartan, Berwald or Chern(Rund) connections. Another feature of this Ricci tensor is the parabolic form of the evolution of its Ricci scalar in the sense of Proposition 3.1.

We say that the Ricci tensor has a *uniform bound* if there is a constant K such that $\|Ric_{(x,y,t)}\|_{g(t)} \leq K$, where $\|.\|_{g(t)}$ is the norm defined by $g(t)$.

1.6. Statement of the maximum principle

We recall here the weak maximum principle states that the extremum of solutions to elliptic equations are dominated by their extremum on the boundary, more intuitively we have the following theorem.

Theorem A ([12], Weak maximum principle for scalars)**.** *Let M be a closed manifold. Assume, for $t \in [0, T]$, where $0 < T < \infty$, that $g(t)$ is a smooth family of metrics on M, and $X(t)$ is a smooth family of vector fields on M. Let $f : \mathbb{R} \times [0, T] \longrightarrow \mathbb{R}$ be a smooth function. Suppose that $u \in C^\infty(M \times [0, T], \mathbb{R})$ solves*

$$\frac{\partial u}{\partial t} \leq \Delta_{g(t)} u + \langle X(t), \nabla u \rangle + f(u, t).$$

Suppose further that $\phi : [0, T] \longrightarrow \mathbb{R}$ solves

$$\begin{cases} \frac{d\phi}{dt} = f(\phi(t), t), \\ \phi(0) = \alpha \in \mathbb{R}. \end{cases}$$

If $u(\cdot, 0) \leq \alpha$, then $u(\cdot, t) \leq \phi(t)$ for all $t \in [0, T]$.

By applying this result when the signs of u, ϕ and α are reversed and f is appropriately modified, we find the following modification:

Corollary B ([12], Weak minimum principle)**.** *Theorem A also holds with the sense of all three inequalities reversed, that is, replacing all three instances of $\leq$ by $\geq$.*

2. Evolution of the reduced curvature tensor

In this section, we derive evolution equation for the reduced hh-curvature $R(X,Z)$ along the Ricci flow and show that if $(M,F(0))$ has positive reduced hh-curvature at the initial time, namely, $R_{g(0)}>0$, then $(M,F(t))$ has positive reduced hh-curvature $R_{g(t)}>0$ for all $t\in[0,T)$. Let X and Y be two fixed sections of the pulled back bundle π^*TM in the sense that X and Y are independent of t and define $A(X,Y):=\frac{\partial}{\partial t}(\nabla_{H\hat{X}}Y)$. Now we are in a position to prove the following proposition.

Proposition 2.1. *Let $Z,X\in\Gamma(\pi^*TM)$ be two fixed vector fields on TM_0. Then*

$$\frac{\partial}{\partial t}(F^2R(Z,X))=-2\sum_{k=1}^{n}F^2R(e_k,X)Ric(e_k,Z), \tag{13}$$

where $R(Z,X)$ is the reduced hh-curvature and $\{e_k\}_{k=1}^n$ is an orthonormal basis for π^*TM.

Proof. Let $W,Z\in\Gamma(\pi^*TM)$ and $\hat{X},\hat{Y}\in\mathcal{X}(TM_0)$ be fixed vector fields on TM. By definition of the hh-curvature tensor and the equations (3) and (11) we have

$$\begin{aligned}
&\frac{\partial}{\partial t}(R(Z,W,X,Y))\\
&=\frac{\partial}{\partial t}(g(R(X,Y)W,Z))\\
&=(\frac{\partial}{\partial t}g)(R(X,Y)W,Z)+g(\frac{\partial}{\partial t}R(X,Y)W,Z)\\
&=-2Ric(\sum_{k=1}^{n}R(e_k,W,X,Y)e_k,Z)\\
&\quad+g\Big(\frac{\partial}{\partial t}(\nabla_{H\hat{X}}\nabla_{H\hat{Y}}W-\nabla_{H\hat{Y}}\nabla_{H\hat{X}}W-\nabla_{[H\hat{X},H\hat{Y}]}W),Z\Big).
\end{aligned}$$

Using the notation $A(X,Y)=\frac{\partial}{\partial t}(\nabla_{H\hat{X}}Y)$ leads

$$\begin{aligned}
\frac{\partial}{\partial t}(R(Z,W,X,Y))=&-2\sum_{k=1}^{n}R(e_k,W,X,Y)Ric(e_k,Z)+g\Big(A(X,\nabla_{H\hat{Y}}W),Z\Big)\\
&+g\Big(\nabla_{H\hat{X}}(A(Y,W)),Z\Big)-g\Big(A(Y,\nabla_{H\hat{X}}W),Z\Big)\\
&-g\Big(\nabla_{H\hat{Y}}(A(X,W)),Z\Big)-g\Big(A(\rho[H\hat{X},H\hat{Y}],W),Z\Big).
\end{aligned}$$

By means of the horizontal torsion freeness of Cartan connection, see [1], we have $\nabla_{H\hat{X}}W-\nabla_{H\hat{W}}X=\rho[H\hat{X},H\hat{W}]$. Applying the horizontal covariant derivative (2) to A, the above equation leads to

$$\frac{\partial}{\partial t}(R(Z,W,X,Y)) = -2\sum_{k=1}^{n} R(e_k,W,X,Y)Ric(e_k,Z) + g\Big((\nabla_{H\hat{X}}A)(Y,W),Z\Big) - g\Big((\nabla_{H\hat{Y}}A)(X,W),Z\Big). \tag{14}$$

Let $u = y^i\frac{\partial}{\partial x^i}$ be the canonical section. Since its horizontal derivative vanishes, namely $\nabla_{H\hat{X}}u = 0$, we have

$$g((\nabla_{H\hat{X}}A)(u,u),Z) = g((\nabla_{\hat{u}}A)(X,u),Z) = 0,$$

where $\hat{u} = y^i\frac{\delta}{\delta x^i}$. Therefore, letting $Y = W = u$ and using $R(Z,u,X,u) = F^2R(Z,X)$ the equation (14) reduces to

$$\frac{\partial}{\partial t}(F^2R(Z,X)) = -2\sum_{k=1}^{n} F^2R(e_k,X)Ric(e_k,Z).$$

This completes the proof. □

If we put $\bar{R}(Z,X) := F^2R(Z,X)$, then (13) reads

$$\frac{\partial}{\partial t}\bar{R}(Z,X) = -2\sum_{k=1}^{n}\bar{R}(e_k,X)Ric(e_k,Z). \tag{15}$$

Proposition 2.2. *Let $(M^n, F(t))$ be a family of solutions to the Finslerian Ricci flow. If there is a constant K such that $\|Ric\|_{g(t)} \leq K$ on the time interval $[0,T)$, and the reduced hh-curvature $R_{g(0)}$ of $F(0)$ is positive that is, $R_{g(0)}(V,V) > 0$ for all $V \in \Gamma(\pi^*TM)$ perpendicular to the distinguished global section l, then there exists a positive constant $C(n)$ such that*

$$e^{-2KCT}\bar{R}_{(x,y,0)}(V,V) \leq \bar{R}_{(x,y,t)}(V,V) \leq e^{2KCT}\bar{R}_{(x,y,0)}(V,V)$$

for all $(x,y) \in TM$ and $t \in [0,T)$.

Proof. Let $(x,y) \in TM$, $t_0 \in [0,T)$ and $V \in \Gamma(\pi^*TM)$ be a nonzero arbitrary section perpendicular to the distinguished global section $l := \frac{y^i}{F}\frac{\partial}{\partial x^i}$. We have

$$\begin{aligned}\|\log(\frac{\bar{R}_{(x,y,t_0)}(V,V)}{\bar{R}_{(x,y,0)}(V,V)})\| &= \|\int_0^{t_0}\frac{\partial}{\partial t}[\log\bar{R}_{(x,y,t)}(V,V)]dt\| \\ &= \|\int_0^{t_0}\frac{\frac{\partial}{\partial t}\bar{R}_{(x,y,t)}(V,V)}{\bar{R}_{(x,y,t)}(V,V)}dt\|. \end{aligned}\tag{16}$$

By means of (15) we have

$$\|\int_0^{t_0}\frac{\frac{\partial}{\partial t}\bar{R}_{(x,y,t)}(V,V)}{\bar{R}_{(x,y,t)}(V,V)}dt\| = \|\int_0^{t_0}\frac{-2\sum_{k=1}^n\bar{R}_{(x,y,t)}(e_k,V)Ric_{(x,y,t)}(e_k,V)}{\bar{R}_{(x,y,t)}(V,V)}dt\|.$$

Therefore, (16) becomes

$$\|\log(\frac{\bar{R}_{(x,y,t_0)}(V,V)}{\bar{R}_{(x,y,0)}(V,V)})\| = \|\int_0^{t_0}\frac{-2\sum_{k=1}^n\bar{R}_{(x,y,t)}(e_k,V)Ric_{(x,y,t)}(e_k,V)}{\bar{R}_{(x,y,t)}(V,V)}dt\|$$

$$= \| \int_0^{t_0} \frac{2\langle \bar{R}_{(x,y,t)}(V), Ric_{(x,y,t)}(V)\rangle}{\bar{R}_{(x,y,t)}(V,V)} dt\|$$
$$\leq \int_0^{t_0} \| \frac{2\langle \bar{R}_{(x,y,t)}(V), Ric_{(x,y,t)}(V)\rangle}{\bar{R}_{(x,y,t)}(V,V)} \| dt.$$

By means of Cauchy-Schwarz inequality we have

$$\|\langle \bar{R}_{(x,y,t)}(V), Ric_{(x,y,t)}(V)\rangle\| \leq \|\bar{R}_{(x,y,t)}(V)\|\|Ric_{(x,y,t)}(V)\|.$$

Therefore, we obtain

$$\|\log(\frac{\bar{R}_{(x,y,t_0)}(V,V)}{\bar{R}_{(x,y,0)}(V,V)})\| \leq \int_0^{t_0} 2\frac{\|\bar{R}_{(x,y,t)}(V)\|\|Ric_{(x,y,t)}(V)\|}{\|\bar{R}_{(x,y,t)}(V,V)\|} dt. \tag{17}$$

There exists a positive constant C, depending only on n such that

$$\|\bar{R}_{(x,y,t)}(V)\|\|Ric_{(x,y,t)}(V)\| \leq C\|\bar{R}_{(x,y,t)}(V,V)\|\|Ric_{(x,y,t)}(V,V)\|. \tag{18}$$

By means of (17) and (18) and using the fact that $\|T(U,U)\| \leq \|T\|_{g(t)}$ for any 2-tensor T and the unit vector U, we have

$$\begin{aligned}\|\log(\frac{\bar{R}_{(x,y,t_0)}(V,V)}{\bar{R}_{(x,y,0)}(V,V)})\| &\leq \int_0^{t_0} 2C\|Ric_{(x,y,t)}(V,V)\|dt \\ &\leq \int_0^{t_0} 2C\|Ric_{(x,y,t)}\|_{g(t)}dt \\ &\leq \int_0^{t_0} 2CKdt \\ &\leq 2CKT.\end{aligned}$$

By assumption $R_{(x,y,0)}(V,V) > 0$ and hence $\bar{R}_{(x,y,0)}(V,V) > 0$. Therefore, the uniform bound on $\bar{R}_{(x,y,t)}(V,V)$ follows from exponentiation, namely,

$$e^{-2KCT}\bar{R}_{(x,y,0)}(V,V) \leq \bar{R}_{(x,y,t)}(V,V) \leq e^{2KCT}\bar{R}_{(x,y,0)}(V,V)$$

for all $(x,y) \in TM$ and $t \in [0,T)$. This completes the proof. □

Proposition 2.2 implies that if $(M^n, F(t))$ is a family of solutions to the Finslerian Ricci flow satisfying a uniform Ricci tensor bound on a finite time interval $[0,T)$, then positive reduced hh-curvature is preserved under the Ricci flow. More precisely,

Proposition 2.3. *Let $(M^n, F(t))$ be a family of solutions to the Finslerian Ricci flow with $F(0) = F_0$. If there is a constant K such that $\|Ric\|_{g(t)} \leq K$ on the time interval $[0,T)$ and the reduced hh-curvature $R_{g(0)}$ of $F(0)$ is positive, that is, $R_{g(0)}(V,V) > 0$ for all $V \in \Gamma(\pi^*TM)$ perpendicular to the distinguished global section l, then the reduced hh-curvature $R_{g(t)}$ of $F(t)$ remains positive in short time, namely, $R_{g(t)}(V,V) > 0$ for all $t \in [0,T)$.*

Proof of Theorem 1. By assumption $(M, F(0))$ has positive flag curvature. Definition of the flag curvature (6) implies that $R_{g_0} > 0$. By means of Proposition 2.3, $R_{g(t)} > 0$ for all $t \in [0, T)$. Using the definition of the flag curvature (6) once more shows that $F(t)$ has positive flag curvature, as long as the solution exists. By means of this fact and definition of the Ricci scalar (7) we have $\mathcal{R}ic_{g(t)} > 0$ for all $t \in [0, T)$. This completes the proof of Theorem 1. □

3. Evolution of the Ricci scalar $\mathcal{R}ic$

Proposition 3.1. *The Ricci scalar of $g(t)$ satisfies the evolution equation*

$$\frac{\partial}{\partial t}\mathcal{R}ic = -F^2 R^{ij}\frac{\partial^2}{\partial y^i \partial y^i}\mathcal{R}ic. \tag{19}$$

Proof. By means of (13) and taking the trace over Z and X we obtain

$$\frac{\partial}{\partial t}(\sum_{l=1}^{n} F^2 R(e_l, e_l)) = -2F^2 \sum_{k,l=1}^{n} R(e_k, e_l) Ric(e_k, e_l). \tag{20}$$

In the natural basis, (20) becomes

$$\frac{\partial}{\partial t}(F^2\mathcal{R}ic) = -2F^2 R^{ij} Ric_{ij}. \tag{21}$$

By means of the chain rule and the definition of Ricci tensor, (21) is written as follows.

$$\frac{\partial}{\partial t}\mathcal{R}ic = -F^2 R^{ij}\frac{\partial^2}{\partial y^i \partial y^i}\mathcal{R}ic - 2(tr_g R)\mathcal{R}ic + 2\mathcal{R}ic^2.$$

Since $tr_g R = \mathcal{R}ic$, we have

$$\frac{\partial}{\partial t}\mathcal{R}ic = -F^2 R^{ij}\frac{\partial^2}{\partial y^i \partial y^i}\mathcal{R}ic.$$

This completes the proof. □

In the remainder of this section, we discuss one implication of Proposition 3.1.

Proof of Theorem 2. By means of Proposition 3.1, the Ricci scalar satisfies the evolution equation (19). One can rewrite (19) with respect to the basis of TSM. By means of (8) we have

$$\partial_\beta \mathcal{R}ic = F y^j_\beta \frac{\partial \mathcal{R}ic}{\partial y^j}.$$

The vertical covariant derivative leads

$$\begin{aligned} \dot{\nabla}_\alpha \partial_\beta \mathcal{R}ic &= \dot{\nabla}_\alpha (F y^j_\beta \frac{\partial \mathcal{R}ic}{\partial y^j}) \\ &= (\dot{\nabla}_\alpha F) y^j_\beta \frac{\partial \mathcal{R}ic}{\partial y^j} + F(\dot{\nabla}_\alpha y^j_\beta)\frac{\partial \mathcal{R}ic}{\partial y^j} + F y^j_\beta (\dot{\nabla}_\alpha \frac{\partial \mathcal{R}ic}{\partial y^j}). \end{aligned} \tag{22}$$

On the other hand

$$\dot{\nabla}_\alpha F = \dot{\nabla}_{Fy^i_\alpha \frac{\partial}{\partial y^i}} F = Fy^i_\alpha F_{y^i} = Fy^i_\alpha l_i = g_{ij} y^j y^i_\alpha. \tag{23}$$

By means of (9) and (23) we have $\dot{\nabla}_\alpha F = 0$. Using (10), equation (22) becomes

$$\begin{aligned}\dot{\nabla}_\alpha \partial_\beta \mathcal{R}ic &= F(-A^j_{kl} y^k_\alpha y^l_\beta - g_{\alpha\beta} y^j)\frac{\partial \mathcal{R}ic}{\partial y^j} + Fy^j_\beta(\dot{\nabla}_\alpha \frac{\partial \mathcal{R}ic}{\partial y^j}) \\ &= -FA^j_{kl} y^k_\alpha y^l_\beta \frac{\partial \mathcal{R}ic}{\partial y^j} + Fy^j_\beta(\dot{\nabla}_{Fy^i_\alpha \frac{\partial}{\partial y^i}} \frac{\partial \mathcal{R}ic}{\partial y^j}) \\ &= -FA^j_{kl} y^k_\alpha y^l_\beta \frac{\partial \mathcal{R}ic}{\partial y^j} + F^2 y^i_\alpha y^j_\beta \dot{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial \mathcal{R}ic}{\partial y^j}. \qquad (24)\end{aligned}$$

By vertical covariant derivative, equation (24) is written

$$\begin{aligned}\dot{\nabla}_\alpha \partial_\beta \mathcal{R}ic &= -FA^j_{kl} y^k_\alpha y^l_\beta \frac{\partial \mathcal{R}ic}{\partial y^j} + F^2 y^i_\alpha y^j_\beta (\frac{\partial^2 \mathcal{R}ic}{\partial y^i \partial y^j} - C^k_{ij} \frac{\partial \mathcal{R}ic}{\partial y^k}) \\ &= F^2 y^i_\alpha y^j_\beta \frac{\partial^2 \mathcal{R}ic}{\partial y^i \partial y^j} - 2FA^k_{ij} y^i_\alpha y^j_\beta \frac{\partial \mathcal{R}ic}{\partial y^k}. \qquad (25)\end{aligned}$$

Converting (25) in $R^{\alpha\beta} = F^{-2} R^{ij} y^\alpha_i y^\beta_j$ yields

$$R^{\alpha\beta} \dot{\nabla}_\alpha \partial_\beta \mathcal{R}ic = R^{ij} \frac{\partial^2 \mathcal{R}ic}{\partial y^i \partial y^j} - 2F^{-1} A^k_{ij} R^{ij} \frac{\partial \mathcal{R}ic}{\partial y^k}. \tag{26}$$

Using (8) we have $\dot{\partial}_k = F^{-1} y^\lambda_k \partial_\lambda$ and from which $\frac{\partial \mathcal{R}ic}{\partial y^k} = F^{-1} y^\lambda_k \partial_\lambda \mathcal{R}ic$. Hence, replacing in (26) we obtain

$$R^{ij} \frac{\partial^2 \mathcal{R}ic}{\partial y^i \partial y^j} = R^{\alpha\beta} \dot{\nabla}_\alpha \partial_\beta \mathcal{R}ic + 2F^{-2} A^k_{ij} R^{ij} y^\lambda_k \partial_\lambda \mathcal{R}ic.$$

Putting $H^\lambda := -2A^k_{ij} R^{ij} y^\lambda_k$, we can rewrite (19) on SM as follows.

$$\frac{\partial}{\partial t} \mathcal{R}ic = -F^2 R^{\alpha\beta} \dot{\nabla}_\alpha \partial_\beta \mathcal{R}ic + H^\lambda \partial_\lambda \mathcal{R}ic. \tag{27}$$

By means of (27) one can write the following inequality

$$\frac{\partial}{\partial t} \mathcal{R}ic \geq -F^2 R^{\alpha\beta} \dot{\nabla}_\alpha \partial_\beta \mathcal{R}ic + H^\lambda \partial_\lambda \mathcal{R}ic - \mathcal{R}ic^2. \tag{28}$$

By assumption $(M, F(0))$ has positive flag curvature. Definition of the flag curvature (6) shows that $R_{g_0} > 0$. Hence, Proposition 2.3 implies that $R^{\alpha\beta}(t)$ is positive definite for all $t \in [0, T)$. Therefore, inequality (28) is an inequality of parabolic type. Let ϕ be a solution to the ODE

$$\frac{d}{dt}\phi = -\phi^2, \tag{29}$$

with initial value $\phi(0) = \inf_{SM} \mathcal{R}ic_{g(0)} = \alpha$. Equation (29) is a Bernoulli equation and its exact solution is

$$\phi(t) = \frac{\alpha}{1 + \alpha t}.$$

Using the weak minimum principle, in the sense of Corollary B, and the inequality (28) we conclude that $\mathcal{R}ic_{g(t)} \geq \frac{\alpha}{1+\alpha t}$. This completes the proof of Theorem 2. $\square$

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