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REAL HYPERSURFACES WITH MIAO-TAM CRITICAL METRICS OF COMPLEX SPACE FORMS

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Abstract. Let M be a real hypersurface of a complex space form with constant curvature c. In this paper, we study the hypersurface M admitting Miao-Tam critical metric, i.e., the induced metric g on M satisfies the equation: $-(\Delta_g\lambda)g+\nabla_g^2\lambda-\lambda Ric=g$, where λ is a smooth function on M. At first, for the case where M is Hopf, $c=0$ and $c\neq 0$ are considered respectively. For the non-Hopf case, we prove that the ruled real hypersurfaces of non-flat complex space forms do not admit Miao-Tam critical metrics. Finally, it is proved that a compact hypersurface of a complex Euclidean space admitting Miao-Tam critical metric with $\lambda>0$ or $\lambda<0$ is a sphere and a compact hypersurface of a non-flat complex space form does not exist such a critical metric.

1. Introduction

Recall that on a compact Riemannian manifold (M^n,g), $n>2$ with a smooth boundary ∂M the metric g is referred as *Miao-Tam critical metric* if there exists a smooth function $\lambda : M^n\to\mathbb{R}$ such that

$$-(\Delta_g\lambda)g+\nabla_g^2\lambda-\lambda Ric=g \tag{1}$$

on M and $\lambda=0$ on ∂M, where $\Delta_g, \nabla_g^2\lambda$ are the Laplacian, Hessian operator with respect to the metric g and Ric is the $(0,2)$ Ricci tensor of g. The function λ is known as the potential function. The equation (1) is called as *Miao-Tam equation.* Applying this equation, Miao-Tam in [12] classified Einstein and conformally flat Riemannian manifolds. In particular, they proved that any Riemannian metric g satisfying the equation (1) must have constant scalar curvature. Recently, Patra-Ghosh studied the Miao-Tam equation on certain class of odd dimensional Riemannian manifolds, namely contact metric

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manifolds (see [15, 16]). It was proved that a complete K-contact metric satisfying the Miao-Tam equation is isometric to a unit sphere. Wang-Wang [18] also considered an almost Kenmotsu manifold with Miao-Tam critical metric.

An n-dimensional complex space form is an n-dimensional Kähler manifold with constant sectional curvature c. A complete and simple connected complex space form is complex analytically isometric to a complex projective space $\mathbb{C}P^n$ if $c > 0$, a complex hyperbolic space $\mathbb{C}H^n$ if $c < 0$, a complex Euclidean space $\mathbb{C}^n$ if $c = 0$. The complex projective and complex hyperbolic spaces are called *non-flat complex space forms* and denoted by $\widetilde{M}^n(c)$. Let M be a real hypersurface of a complex space form, then there exists an almost contact structure (ϕ, η, ξ, g) on M induced from the complex space form. In particular, if ξ is an eigenvector of shape operator A, then M is called a *Hopf hypersurface*. Since there are no Einstein real hypersurfaces in non-flat complex space forms ([3, 13]), Cho and Kimura [4, 5] considered a generalization of Einstein metric, called Ricci soliton, which satisfies

$$\frac{1}{2}\mathcal{L}_V g + Ric - \rho g = 0,$$

where V and ρ are the potential vector field and some constant on M, respectively. They proved that a compact contact-type hypersurface with a Ricci soliton in $\mathbb{C}^n$ is a sphere and a compact Hopf hypersurface in a non-flat complex space form does not admit a Ricci soliton.

From the Miao-Tam equation (1), we remark that the Miao-Tam critical metric can also be viewed as a generalization of the Einstein metric since the critical metric will become an Einstein metric if the potential function λ is constant. Thus the above results intrigue us to study the real hypersurfaces admitting Miao-Tam critical metrics of complex space forms. In this article, we mainly study the Hopf hypersurfaces in complex space forms as well as a class of non-Hopf hypersurfaces in non-flat complex space forms. For a compact real hypersurface with Miao-Tam critical metric, we also get a result.

This paper is organized as follows: In Section 2 we recall some basic concepts and related results. In Section 3, we consider respectively the Hopf hypersurfaces with Miao-Tam critical metrics of non-flat complex space forms and complex Euclidean spaces, and one class of non-Hopf hypersurfaces of non-flat complex space forms is considered in Section 4. In the last section we will prove the result of compact real hypersurfaces with Miao-Tam critical metrics.

2. Some basic concepts and related results

Let $(\widetilde{M}^n, \widetilde{g})$ be a complex n-dimensional Kähler manifold and M be an immersed, without boundary, real hypersurface of $\widetilde{M}^n$ with the induced metric g. Denote by J the complex structure on $\widetilde{M}^n$. There exists a local defined unit normal vector field N on M and we write $\xi := -JN$ by the structure vector field of M. An induced one-form η is defined by $\eta(\cdot) = \widetilde{g}(J\cdot, N)$, which is dual to ξ. For any vector field X on M the tangent part of JX is denoted by $\phi X = JX - \eta(X)N$. Moreover, the following identities hold:

$$\phi^2 = -Id + \eta \otimes \xi, \quad \eta \circ \phi = 0, \quad \phi \circ \xi = 0, \quad \eta(\xi) = 1, \tag{2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{3}$$

$$g(X, \xi) = \eta(X), \tag{4}$$

where $X, Y \in \mathfrak{X}(M)$. By (2)-(4), we know that (ϕ, η, ξ, g) is an almost contact metric structure on M.

Denote by ∇, A the induced Riemannian connection and the shape operator on M, respectively. Then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \widetilde{\nabla}_X N = -AX, \tag{5}$$

where $\widetilde{\nabla}$ is the connection on $\widetilde{M}^n$ with respect to $\widetilde{g}$. Also, we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX. \tag{6}$$

In particular, M is said to be a *Hopf hypersurface* if the structure vector field ξ is an eigenvector of A.

From now on we always assume that the sectional curvature of $\widetilde{M}^n$ is constant c. When $c = 0$, $\widetilde{M}^n$ is complex Euclidean space $\mathbb{C}^n$. When $c \neq 0$, $\widetilde{M}^n$ is a non-flat complex space form, denoted by $\widetilde{M}^n(c)$, then from (5), we know that the curvature tensor R of M is given by

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}\Big(&g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &+ 2g(X, \phi Y)\phi Z\Big) + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{7}$$

and the shape operator A satisfies

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\Big(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\Big) \tag{8}$$

for any vector fields X, Y, Z on M. From (7), we get the Ricci tensor Q of type $(1, 1)$:

$$QX = \frac{c}{4}\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X, \tag{9}$$

where h denotes the mean curvature of M (i.e., $h = trace(A)$). We denote by S the scalar curvature of M, i.e., $S = trace(Q)$.

If M is a Hopf hypersurface of $\widetilde{M}^n(c)$, $A\xi = \alpha\xi$, where $\alpha = g(A\xi, \xi)$. Due to [14, Theorem 2.1], α is constant. Remark that when $c = 0$, α is also constant (see the proof of [5, Lemma 1]). Using the equation (8), we obtain

$$(\nabla_\xi A)X = \alpha\phi AX - A\phi AX + \frac{c}{4}\phi X \tag{10}$$

for any vector field X. Since $\nabla_\xi A$ is self-adjoint, by taking the anti-symmetry part of (10), we get the relation:

$$2A\phi AX - \frac{c}{2}\phi X = \alpha(\phi A + A\phi)X. \tag{11}$$

As the tangent bundle TM can be decomposed as $TM = \mathbb{R}\xi \oplus \mathfrak{D}$, where $\mathfrak{D} = \{X \in TM : X \perp \xi\}$, the condition $A\xi = \alpha\xi$ implies $A\mathfrak{D} \subset \mathfrak{D}$, thus we can pick up $X \in \mathfrak{D}$ such that $AX = fX$ for some function f on M. Then from (11) we obtain

$$(2f - \alpha)A\phi X = \Big(f\alpha + \frac{c}{2}\Big)\phi X. \tag{12}$$

If $2f = \alpha$, then $c = -4f^2$, which shows that M is locally congruent to a horosphere in $\mathbb{C}H^n$(see [2]).

Next we recall an important lemma for a Riemannian manifold satisfying Miao-Tam equation (1).

Lemma 2.1 ([7]). *Let a Riemannian manifold (M^n, g) satisfies the Miao-Tam equation. Then the curvature tensor R can be expressed as*

$$R(X,Y)\nabla\lambda = X(\lambda)QY - Y(\lambda)QX + \lambda\{(\nabla_X Q)Y - (\nabla_Y Q)X\} + X(\beta)Y - Y(\beta)X$$

for any vector fields X, Y on M and $\beta = -\frac{S\lambda+1}{n-1}$.

Applying this lemma we obtain:

Lemma 2.2. *For a Hopf real hypersurface M^{2n-1} with Miao-Tam critical metric of a complex space form, the following equation holds*:

$$\lambda\alpha\Big[X(h) - \xi(h)\eta(X)\Big] = \mu\Big(\xi(\lambda)\eta(X) - X(\lambda)\Big) + \alpha^2\xi(\lambda)\eta(X) - \alpha AX(\lambda), \tag{13}$$

where $\mu = \frac{c}{4}(2n-1) + \alpha h - \alpha^2 - \frac{S}{2n-2}$.

Proof. Replacing Z in (7) by $\nabla\lambda$, we have

$$\begin{aligned} R(X,Y)\nabla\lambda = \frac{c}{4}\Big(&Y(\lambda)X - X(\lambda)Y + \phi Y(\lambda)\phi X - \phi X(\lambda)\phi Y \\ &+ 2g(X,\phi Y)\phi\nabla\lambda\Big) + AY(\lambda)AX - AX(\lambda)AY. \end{aligned} \tag{14}$$

By combining with Lemma 2.1, we get

$$\begin{aligned} &X(\lambda)QY - Y(\lambda)QX + \lambda\{(\nabla_X Q)Y - (\nabla_Y Q)X\} \\ &= \Big(\frac{c}{4} - \frac{S}{2n-2}\Big)\Big(Y(\lambda)X - X(\lambda)Y\Big) + \frac{c}{4}\Big(\phi Y(\lambda)\phi X - \phi X(\lambda)\phi Y \\ &\quad + 2g(X,\phi Y)\phi\nabla\lambda\Big) + AY(\lambda)AX - AX(\lambda)AY. \end{aligned} \tag{15}$$

Now making use of (9), for any vector fields X, Y we first compute

$$(\nabla_Y Q)X = \frac{c}{4}\{-3(\nabla_Y\eta)(X)\xi - 3\eta(X)\nabla_Y\xi\} + Y(h)AX + h(\nabla_Y A)X$$

$$\begin{aligned}
&-(\nabla_Y A)AX - A(\nabla_Y A)X\\
=& -\frac{3c}{4}\{g(\phi AY, X)\xi + \eta(X)\phi AY\} + Y(h)AX + h(\nabla_Y A)X\\
&-(\nabla_Y A)AX - A(\nabla_Y A)X.
\end{aligned}$$

By (8), we thus obtain

(16)

$$\begin{aligned}
&(\nabla_X Q)Y - (\nabla_Y Q)X\\
=& -\frac{3c}{4}\{g(\phi AX + A\phi X, Y)\xi + \eta(Y)\phi AX - \eta(X)\phi AY\}\\
&+ X(h)AY - Y(h)AX + \frac{hc}{4}\Big(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\Big)\\
&-(\nabla_X A)AY + (\nabla_Y A)AX - \frac{c}{4}\Big(\eta(X)A\phi Y - \eta(Y)A\phi X - 2g(\phi X, Y)A\xi\Big).
\end{aligned}$$

Therefore, taking the product of (15) with ξ and using (16), we conclude that

$$\begin{aligned}
&-\frac{3c}{4}g(\phi AX + A\phi X, Y) + \alpha X(h)\eta(Y) - \alpha Y(h)\eta(X)\\
&-g((\nabla_X A)AY + (\nabla_Y A)AX, \xi) - \frac{hc - \alpha c}{2}g(\phi X, Y)\\
=& \frac{\mu}{\lambda}\Big(Y(\lambda)\eta(X) - X(\lambda)\eta(Y)\Big) + \frac{\alpha}{\lambda}AY(\lambda)\eta(X) - \frac{\alpha}{\lambda}AX(\lambda)\eta(Y),
\end{aligned} \tag{17}$$

where $\mu = \frac{c}{4}(2n-1) + \alpha h - \alpha^2 - \frac{S}{2n-2}$. Moreover, using (11) we compute

$$\begin{aligned}
&g((\nabla_X A)AY - (\nabla_Y A)AX, \xi)\\
=& g(\frac{\alpha}{2}(\phi AX - A\phi X) - \frac{c}{4}\phi X, AY) - g(\frac{\alpha}{2}(\phi AY - A\phi Y) - \frac{c}{4}\phi Y, AX).
\end{aligned}$$

Substituting this into (17) we arrive at

$$\begin{aligned}
&-\frac{c + 2\alpha^2}{4}(\phi AX + A\phi X) + \alpha X(h)\xi - \alpha\eta(X)\nabla h\\
&+\frac{\alpha}{2}(A^2\phi X + \phi A^2 X) - \frac{2hc - \alpha c}{4}\phi X\\
=& \frac{\mu}{\lambda}\Big(\eta(X)\nabla\lambda - X(\lambda)\xi\Big) + \frac{\alpha}{\lambda}\eta(X)A\nabla\lambda - \frac{\alpha}{\lambda}AX(\lambda)\xi.
\end{aligned} \tag{18}$$

Finally, taking an inner product of (18) with ξ gives (13). □

3. Hopf real hypersurfaces of complex space forms

First of all, we assume $c \neq 0$, i.e., M^{2n-1} is a Hopf real hypersurface of non-flat complex space form $\widetilde{M}^n(c)$. We first consider $\alpha = 0$, i.e., $A\xi = 0$, then the relation (13) yields

$$\Big(-\frac{S}{2n-2} + \frac{c}{4}(2n-1)\Big)\Big(\xi(\lambda)\xi - \nabla\lambda\Big) = 0. \tag{19}$$

If $-\frac{S}{2n-2}+\frac{c}{4}(2n-1)=0$, i.e., $S=\frac{1}{2}c(n-1)(2n-1)$. Then from (18) we find

$$\frac{c}{4}(\phi AX+A\phi X)=0, \tag{20}$$

which yields $\phi AX+A\phi X=0$ for all vector field X. This is contradictory with [14, Corollary 2.12]. Thus $S\neq\frac{c}{2}(n-1)(2n-1)$, and it follows from (19) that $\nabla\lambda=\xi(\lambda)\xi$. Differentiating this along X gives

$$\nabla_X\nabla\lambda=X(\xi(\lambda))\xi+\xi(\lambda)\phi AX. \tag{21}$$

On the other hand, from (1) we can obtain

$$\nabla_X\nabla\lambda=(1+\Delta\lambda)X+\lambda QX. \tag{22}$$

Comparing (21) and (22), we have

$$X(\xi(\lambda))\xi+\xi(\lambda)\phi AX=(1+\Delta\lambda)X+\lambda QX. \tag{23}$$

Moreover, by (9), putting $X=\xi$ gives

$$\xi(\xi(\lambda))=1+\Delta\lambda+\frac{\lambda c}{2}(n-1). \tag{24}$$

Choose a local orthonormal frame $\{e_i\}$ such that $e_{2n-1}=\xi$ and $e_{n-1+i}=\phi e_i$ for $i=1,\dots,n-1$. Using the frame to contract over X in (23), we also derive that

$$\xi(\xi(\lambda))=(1+\Delta\lambda)(2n-1)+\lambda S.$$

Comparing with (24), we find

$$(2n-2)(1+\Delta\lambda)+\lambda S=\frac{\lambda c}{2}(n-1). \tag{25}$$

Furthermore, by taking the trace of Miao-Tam equation (1), we get

$$(2-2n)\Delta\lambda-\lambda S=2n-1, \tag{26}$$

which, together with (25), yields

$$\frac{\lambda c}{2}(n-1)+1=0. \tag{27}$$

This shows that λ is constant. Thus M is Einstein, but as is well-known that there are no Einstein hypersurfaces in a non-flat complex space form as in introduction, hence we immediately obtain:

Proposition 3.1. *A real hypersurface with $A\xi=0$ of a non-flat complex space form does not admit Miao-Tam critical metric.*

Next we consider the case where $\alpha\neq 0$. If for every $X\in\mathfrak{D}$ such that $AX=\frac{\alpha}{2}X$, as before we know that M is locally congruent a horosphere in $\mathbb{C}H^n$ and $c=-\alpha^2$. Moreover, the mean curvature $h=n\alpha$ is constant. Then from (18) we can obtain $nc=-\frac{\alpha^2}{2}$. This implies $2n=1$. It is impossible.

Now choose $X \in \mathfrak{D}$ such that $AX = fX$ with $f \neq \frac{\alpha}{2}$, so from (18) we have

$$\begin{aligned}&-\frac{c+2\alpha^2}{4}(f\phi X + \widetilde{f}\phi X) + \alpha X(h)\xi + \frac{\alpha}{2}(\widetilde{f}^2\phi X + f^2\phi X) - \frac{2hc-\alpha c}{4}\phi X\\ =&-\frac{\mu}{\lambda}X(\lambda)\xi - \frac{\alpha}{\lambda}AX(\lambda)\xi.\end{aligned}$$

Here we have used $A\phi X = \widetilde{f}\phi X$ with $\widetilde{f} = \frac{f\alpha + \frac{c}{2}}{2f-\alpha}$ followed from (12). Since $\phi X \in \mathfrak{D}$, we further derive

$$-(c+2\alpha^2)(f+\widetilde{f}) + 2\alpha(\widetilde{f}^2 + f^2) - (2hc - \alpha c) = 0. \tag{28}$$

Moreover, inserting $\widetilde{f} = \frac{f\alpha + \frac{c}{2}}{2f-\alpha}$ into the equation (28), we have

$$\begin{aligned}&8\alpha f^4 - 4(c+4\alpha^2)f^3 + (6\alpha c + 8\alpha^3 - 8hc)f^2\\ &+ (8hc\alpha - 4\alpha^2 c - c^2)f + \alpha c^2 + 2\alpha^3 c - 2hc\alpha^2 = 0.\end{aligned} \tag{29}$$

Now we denote the roots of the polynomial by f_1, f_2, f_3, f_4, then from the relation between the roots and coefficients we obtain

$$\begin{cases} f_1 + f_2 + f_3 + f_4 = \frac{c+4\alpha^2}{2\alpha}, \\ f_1f_2 + f_1f_3 + f_1f_4 + f_2f_3 + f_2f_4 + f_3f_4 = \frac{3\alpha c + 4\alpha^3 - 4hc}{4\alpha}, \\ f_1f_2f_3 + f_1f_2f_4 + f_2f_3f_4 = -\frac{8hc\alpha - 4\alpha^2 c - c^2}{8\alpha}, \\ f_1f_2f_3f_4 = \frac{c^2 + 2\alpha^2 c - 2hc\alpha}{8}. \end{cases} \tag{30}$$

As the proof of [5, Lemma 4.2], we can also get the following.

Lemma 3.2. *The mean curvature h is constant.*

Hence from (13) we conclude

$$A\nabla\lambda = \frac{\mu}{\alpha}\phi^2\nabla\lambda + \alpha\xi(\lambda)\xi.$$

By taking the inner product with the principal vector $X \in \mathfrak{D}$, we obtain

$$(f + \frac{\mu}{\alpha})X(\lambda) = 0.$$

If $X(\lambda) = 0$ for all $X \in \mathfrak{D}$, then $\nabla\lambda = \xi(\lambda)\xi$. As the proof of Proposition 3.1, we see that M is Einstein, which is impossible.

If $X(\lambda) \neq 0$ for all $X \in \mathfrak{D}$, then $f + \frac{\mu}{\alpha} = 0$, i.e., M has only two distinct constant principal curvatures $\alpha, -\frac{\mu}{\alpha}$. Further, we see from (12) that

$$2f^2 - 2\alpha f - \frac{c}{2} = 0. \tag{31}$$

Since the hypersurface M has two distinct constant principle curvatures: α of multiplicity 1 and f of multiplicity $2n-2$, it is easy to get that the mean curvature $h = \alpha + (2n-2)f$ and the scalar curvature $S = c(n^2-1) + 2\alpha(2n-2)f + (2n-2)(2n-3)f^2$. Thus

$$\mu = -\frac{3c}{4} + (2n-4)\alpha f - (2n-3)f^2.$$

Inserting this into the relation $f + \frac{\mu}{\alpha} = 0$, we obtain

$$(2n-3)(\alpha f - f^2) = \frac{3c}{4}. \tag{32}$$

Combining (31) with (32), we find $nc = 0$, which is a contradiction.

If $X(\lambda) \neq 0$ for some principle vector $X \in \mathfrak{D}$, and without loss general, we suppose $e_1(\lambda) \neq 0$, then $Ae_1 = -\frac{\mu}{\alpha} e_1$ and $A\phi e_1 = \frac{\alpha\mu - \frac{c}{2}\alpha}{2\mu + \alpha^2} \phi e_1$.

Notice that if the hypersurface M of $\mathbb{C}H^n$ has constant principal curvatures, the classification is as follows:

Theorem 3.3 ([2]). *Let M be a Hopf real hypersurface in $\mathbb{C}H^n (n \geq 2)$ with constant principal curvatures. Then M is locally congruent to the following*:

(1) A_2 : *Tubes around a totally geodesic $\mathbb{C}H^{n-1} \subset \mathbb{C}H^n$.*
(2) B : *Tubes of radius r around a totally geodesic real hyperbolic space $\mathbb{R}H^n \subset \mathbb{C}H^n$.*
(3) N : *Horospheres in $\mathbb{C}H^n$.*

Since the horospheres have two distinct principal curvatures, it is impossible. By Theorems 3.9 and 3.12 in [14], the Type A_2, B hypersurfaces have three distinct principal curvatures: $\lambda_1 = \frac{1}{r}\tanh(u), \lambda_2 = \frac{1}{r}\coth(u)$ and $\alpha = \frac{2}{r}\tanh(2u)$. Then $h = \alpha + (n-1)(\lambda_1 + \lambda_2) = \alpha + \frac{2(n-1)}{r}\coth(2u)$. On the other hand, from Corollary 2.3(ii) in [14], we also have $\frac{1}{r^2} = \frac{\lambda_1+\lambda_2}{2}\alpha + \frac{c}{4}$, i.e., $c = -\frac{4}{r^2}$. This implies from the last relation in (30) that

$$\frac{1}{r^4} = \frac{c^2 + 2\alpha^2 c - 2hc\alpha}{8} = \frac{4n-2}{r^4}.$$

Thus $n = \frac{3}{4}$, that is impossible.

For the case of $\mathbb{C}P^n$, the classification is as follow:

Theorem 3.4 ([9, 17]). *Let M be a Hopf hypersurface in $\mathbb{C}P^n (n \geq 2)$ with constant principal curvatures. Then M is an open part of*

(1) A_2 : *a tuber over a totally geodesic complex projective space $\mathbb{C}P^k$ of radius $\frac{\pi r}{4}$ for $0 \leq k \leq n-1$, where $r = \frac{2}{\sqrt{c}}$, or*
(2) B : *a tuber over a complex quadric Q_{n-1} and $\mathbb{R}P^n$, or*
(3) C : *a tube around the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^k$ into $\mathbb{C}P^{2k+1}$ for some $k \leq 2$, or*
(4) D : *a tube around the Plücker embedding into $\mathbb{C}P^9$ of the complex Grassmann manifold $G_2(\mathbb{C}^5)$ of complex 2-planes in $\mathbb{C}^5$, or*
(5) E : *a tube around the half spin embedding into $\mathbb{C}P^{15}$ of the Hermitian symmetric space $SO(10) = U(5)$.*

The Type A_2 and B hypersurfaces have three distinct principal curvatures: $\lambda_1 = -\frac{1}{r}\cot(u), \lambda_2 = \frac{1}{r}\tan(u), \alpha = \frac{2}{r}\tan(2u)$ (see [14, Theorems 3.14 and 3.15]). From the first relation of (30), we have

$$\lambda_1 + \lambda_2 = \frac{c + 4\alpha^2}{4\alpha} \Rightarrow -\frac{16}{r^2} = c + 4\alpha^2.$$

It gives a contradiction since $c > 0$.

For the Type C, D and E hypersurfaces, they have five distinct principal curvatures (see [14, Theorems 3.16, 3.17, and 3.18]). We compute

$$\frac{1}{r}\Big(-\cot(u)+\tan(u)+\cot(\frac{\pi}{4}-u)+\cot(\frac{3\pi}{4}-u)\Big)=\frac{2}{r}(1+\cot^2(2u)).$$

Thus the first relation of (30) implies

$$-\frac{24}{r^2}\cot^2(2u)=c+\frac{8}{r^2}.$$

It is impossible since $c > 0$. So the hypersurfaces of Type C, D, E do not admit Miao-Tam critical metrics.

Summarizing the above discussion, we thus assert the following:

Proposition 3.5. *A real hypersurface with $A\xi = \alpha\xi, \alpha \neq 0$ in a non-flat complex space form does not admit Miao-Tam critical metric.*

Together Proposition 3.1 with Proposition 3.5, we prove:

Theorem 3.6. *There exist no Hopf real hypersurfaces with Miao-Tam critical metric in non-flat complex space forms.*

In the following we always assume $c = 0$. That is to say that M is a real hypersurface of complex Euclidean space $\mathbb{C}^n$. First of all, if $A\xi = 0$, we obtain from (19)

$$S\Big(\xi(\lambda)\xi-\nabla\lambda\Big)=0.$$

If $S \neq 0$, we have $\nabla\lambda = \xi(\lambda)\xi$. As before we can also lead to (27), but it yields a contradiction since $c = 0$. Thus the scalar curvature $S = 0$, and the relation (26) implies $\Delta\lambda = -\frac{2n-1}{2n-2}$. Actually, $\lambda = -\frac{2n-1}{4n-4}|x|^2$ on $\mathbb{R}^{2n-1}$. Since $R(\xi, X, \xi, X) = 0$ for all X, the sectional curvature of M is also zero. By Hartman and Nirenberg's theorem in [8], M is a hyperplane or a cylinder, hence we have the following:

Theorem 3.7. *Let M^{2n-1} be a complete real hypersurface with $A\xi = 0$ of complex Euclidean space $\mathbb{C}^n$. If M admits Miao-Tam critical metric, it is a generalized cylinder $\mathbb{R}^{2n-1-p} \times \mathbb{S}^p$ or $\mathbb{R}^{2n-1}$.*

When $\alpha \neq 0$. Let us choose $X \in \mathfrak{D}$ such that $AX = \beta X$ for a smooth function β, then we know $\beta \neq \frac{\alpha}{2}$, otherwise, if $\beta = \frac{\alpha}{2}$, then $-4\beta^2 = c = 0$ from (12), i.e., $\beta = 0$. This is a contradiction with $\alpha \neq 0$. Further, from (12) we have

$$A\phi X=\frac{\beta\alpha}{2\beta-\alpha}\phi X. \tag{33}$$

Therefore we find that the equation (29) holds, and for $c = 0$ and $f = \beta$ it becomes

$$(\beta^2-\alpha\beta)^2=0.$$

So $\beta^2 = \alpha\beta$, that means that β is constant and further h is also constant. If $\alpha = \beta$, from (33) we see that the shape operator can be expressed as $A = \alpha I$, where I denotes the identity map. In this case, M is locally congruent to a sphere.

If $\beta = 0$, $A = \alpha\eta \otimes \xi$, as the proof of [11, Theorem 1.1], we know that M is $\mathbb{S}^1 \times \mathbb{R}^{2n-2}$. Therefore we assert the following:

Theorem 3.8. *Let M^{2n-1} be a complete real hypersurface with $A\xi = \alpha\xi$, $\alpha \neq 0$, of complex Euclidean space $\mathbb{C}^n$. If M admits Miao-Tam critical metric, it is locally congruent to a sphere, or $\mathbb{S}^1 \times \mathbb{R}^{2n-2}$.*

4. Ruled hypersurfaces of non-flat complex space forms

In this section we study a class of non-Hopf hypersurfaces with Miao-Tam critical metric of non-flat complex space forms. Let $\gamma : I \to \widetilde{M}^n(c)$ be any regular curve. For $t \in I$, let $\widetilde{M}^n_{(t)}(c)$ be a totally geodesic complex hypersurface through the point $\gamma(t)$ which is orthogonal to the holomorphic plane spanned by $\gamma'(t)$ and $J\gamma'(t)$. Write $M = \{\widetilde{M}^n_{(t)}(c) : t \in I\}$. Such a construction asserts that M is a real hypersurface of $\widetilde{M}^n(c)$, which is called a *ruled hypersurface*. It is well-known that the shape operator A of M is written as:

$$\begin{aligned} A\xi &= \alpha\xi + \beta W (\beta \neq 0), \\ AW &= \beta\xi, \\ AZ &= 0 \text{ for any } Z \perp \xi, W, \end{aligned}$$

where W is a unit vector field orthogonal to ξ, and α, β are differentiable functions on M. From (9), we have

$$Q\xi = (\frac{1}{2}(n-1)c - \beta^2)\xi, \tag{34}$$

$$QW = (\frac{1}{4}(2n+1)c - \beta^2)W, \tag{35}$$

$$QZ = (\frac{1}{4}(2n+1)c)Z \text{ for any } Z \perp \xi, W. \tag{36}$$

From these equations we know the scalar curvature $S = (n^2-1)c - 2\beta^2$. Since S is constant, this shows that β is also constant. Further, the following relation $\nabla\beta = (\beta^2 + c/4)\phi W$ is valid (see [10]), which yields

$$\beta^2 + c/4 = 0 \quad \text{and} \quad S = -(4n^2 - 2)\beta^2. \tag{37}$$

Further, the following lemma holds:

Lemma 4.1 ([10])**.** *For all $Z \in \{X \in TM : \eta(X) = g(X, W) = g(X, \phi W) = 0\}$, we have the following relations*:

$$\begin{aligned} &\nabla_W \phi W = -2\beta W, \quad \nabla_W W = (\beta + \beta^2)\phi W, \\ &\nabla_Z \phi W = -\beta Z, \quad \nabla_Z W = \beta\phi Z, \end{aligned}$$

$$\nabla_{\phi W}\phi W = 0.$$

Now putting $Y = \xi$ and $X = W$ in (15) yields

$$W(\lambda)(\frac{1}{2}(n-1)c - \beta^2)\xi - \xi(\lambda)(\frac{1}{4}(2n+1)c - \beta^2)W + \lambda\{(\nabla_W Q)\xi - (\nabla_\xi Q)W\} = \Big(\frac{c}{4} - \frac{S}{2n-2}\Big)\Big(\xi(\lambda)W - W(\lambda)\xi\Big) + A\xi(\lambda)AW - AW(\lambda)A\xi. \tag{38}$$

Because β is constant, from (35) and (34), by Lemma 4.1 we compute

$$\begin{aligned}(\nabla_W Q)\xi - (\nabla_\xi Q)W &= \nabla_W(Q\xi) - Q\nabla_W\xi - \nabla_\xi(QW) + Q\nabla_\xi W \\ &= -W(\beta^2)\xi + \xi(\beta^2)W = 0.\end{aligned}$$

Inserting this into (38), we conclude that

$$\begin{cases} W(\lambda)\Big[(\frac{1}{4}(2n-1)c - 2\beta^2 - \frac{S}{2n-2}\Big] & = 0, \\ \xi(\lambda)\Big[\frac{1}{2}(n+1)c - 2\beta^2 - \frac{S}{2n-2}\Big] & = 0. \end{cases} \tag{39}$$

From (39), we get $\xi(\lambda) = W(\lambda) = 0$ since $\frac{1}{2}(n+1)c - 2\beta^2 - \frac{S}{2n-2} \neq 0$, which is followed from (37).

Putting $Y = \xi$ and $X = Z$ in (15), we have

$$Z(\lambda)(\frac{1}{2}(n-1)c - \beta^2)\xi - \xi(\lambda)(\frac{1}{4}(2n+1)c)Z + \lambda\{(\nabla_Z Q)\xi - (\nabla_\xi Q)Z\} = \Big(\frac{c}{4} - \frac{S}{2n-2}\Big)\Big(\xi(\lambda)Z - Z(\lambda)\xi\Big). \tag{40}$$

By Lemma 4.1, we also obtain

$$(\nabla_Z Q)\xi - (\nabla_\xi Q)Z = -Z(\beta^2)\xi + \xi(\beta^2)Z = 0.$$

Since $\xi(\lambda) = 0$, the relation (40) becomes

$$Z(\lambda)\Big[\frac{1}{4}(2n-1)c - \beta^2 - \frac{S}{2n-2}\Big] = 0.$$

Thus $Z(\lambda) = 0$ since $\frac{1}{4}(2n-1)c - \beta^2 - \frac{S}{2n-2} \neq 0$ as before.

By taking $X = \phi W$ and $Y = \xi$ in (15), a similar computation gives

$$-\lambda\beta(\frac{1}{2}(n+2)c + \beta^2) = \Big(-\frac{S}{2n-2} + \frac{1}{4}(2n-1)c - \beta^2\Big)\phi W(\lambda). \tag{41}$$

Inserting (37) into (41), we find

$$\phi W(\lambda) = \frac{\lambda\beta(2n+3)(n-1)}{2n-1}.$$

Consequently, we obtain

$$\nabla\lambda = \frac{\lambda\beta(2n+3)(n-1)}{2n-1}\phi W. \tag{42}$$

On the other hand, as we known $\nabla_X \nabla \lambda = \lambda QX + (1 + \Delta\lambda)X$ by Miao-Tam equation (1). When $X = Z$ and W respectively, by Lemma 4.1 it follows respectively from (35), (36) and (42) that

$$-\frac{\lambda\beta^2(2n+3)(n-1)}{2n-1} = -\lambda(2n+1)\beta^2 + (1+\Delta\lambda),$$
$$-2\frac{\lambda\beta^2(2n+3)(n-1)}{2n-1} = -\lambda(2n+2)\beta^2 + (1+\Delta\lambda).$$

It will give $\lambda\beta^2 = 0$, which is a contradiction with $\lambda, \beta \neq 0$. Hence the following theorem is proved.

Theorem 4.2. *There exist no ruled hypersurfaces with Miao-Tam critical metrics of non-flat complex space forms.*

5. Compact hypersurfaces of complex space forms

For the case where M is compact, we immediately obtain the following result:

Theorem 5.1. *Let M^{2n-1} be a compact real hypersurface admitting Miao-Tam critical metric with $\lambda > 0$ or $\lambda < 0$ of complex Euclidean space $\mathbb{C}^n$, then M is a sphere. In the compact real hypersurfaces of a non-flat complex space form $\widetilde{M}^n(c)$ there does not exist such a critical metric.*

Proof. Write $\mathring{Ric} = Ric - \frac{S}{2n-1}g$. It is proved the following relation(see the proof of [1, Lemma 5]):

$$\operatorname{div}(\mathring{Ric}(\nabla\lambda)) = \lambda|\mathring{Ric}|^2.$$

Thus integrating it over M gives $\mathring{Ric} = 0$ if $\lambda > 0$ or $\lambda < 0$, that means that $Ric = \frac{S}{2n-1}g$. Namely M is Einstein. For the case of complex Euclidean space $\mathbb{C}^n$, it is proved that M is a sphere, a hyperplane, or a hypercylinder over a complete plane curve (cf. [6]). But the latter two cases are not compact. For $c \neq 0$, it is impossible since there are no Einstein hypersurfaces in a non-flat complex space form. Therefore we complete the proof. □

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