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NOTES ON THE MINKOWSKI MEASURE, THE MINKOWSKI SYMMETRAL, AND THE BANACH-MAZUR DISTANCE

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Abstract. In this paper we derive some basic inequalities connecting the Minkowski measure of symmetry, the Minkowski symmetral and the Banach-Mazur distance. We then explore the geometric contents of these inequalities and shed light on the structure of the quotient $\mathfrak{B}/\mathrm{Aff}$ of the space of convex bodies modulo the affine transformations.

1. Preliminaries and statement of results

Initiated by an early work of Minkowski, measures of asymmetry (or symmetry) for convex bodies comprise an ever popular subject in convex geometry. Many kinds of measures of asymmetry, predominantly in maximum-minimum problems, have been proposed and studied (see [1–3, 7–10] and the references therein).

Let $\mathbb{R}^n$, $n \geq 2$, denote the n-dimensional Euclidean space. Throughout this paper we will work in an n-dimensional real vector space $\mathcal{X}$ which can be identified with $\mathbb{R}^n$ by specifying an orthonormal basis in $\mathcal{X}$. A bounded closed convex set $\mathcal{C} \subset \mathcal{X}$ is called a convex body if it has non-empty interior: $\mathrm{int}\,\mathcal{C} \neq \emptyset$. The space of all convex bodies in $\mathcal{X}$ will be denoted by $\mathfrak{B} = \mathfrak{B}_{\mathcal{X}}$.

An affine self-map of $\mathcal{X}$ is a map $\varphi : \mathcal{X} \to \mathcal{X}$ such that $\varphi(X) = A \cdot X + Z$, where $A : \mathcal{X} \to \mathcal{X}$ is a linear endomorphism of $\mathcal{X}$ and $Z \in \mathcal{X}$. Note that an affine map is linear if and only if $Z = 0$.

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An affine map φ is an affine transformation if A is nonsingular, that is, A is a linear isomorphism, an element of the general linear group $GL(\mathcal{X})$. The set of all affine transformations of $\mathcal{X}$ is denoted by $\mathrm{Aff} = \mathrm{Aff}(\mathcal{X})$. Associating to $\varphi \in \mathrm{Aff}(\mathcal{X})$ the pair $(A, Z) \in GL(\mathcal{X}) \times \mathcal{X}$ with $\varphi(X) = A \cdot X + Z$, $A \in GL(\mathcal{X})$, $Z \in \mathcal{X}$, establishes an isomorphism between the affine group Aff and the semidirect product $GL(\mathcal{X}) \ltimes \mathcal{T}(\mathcal{X})$, where $\mathcal{T}(\mathcal{X}) \cong \mathcal{X}$ is the group of translations of $\mathcal{X}$. The natural projection $\mathrm{Aff}(\mathcal{X}) \to GL(\mathcal{X})$, $\varphi \mapsto \varphi_0$ associating to the affine transformation φ above the linear isomorphism φ_0 with $\varphi_0(X) = A \cdot X$, $X \in \mathcal{X}$, will play an important part in the sequel.

The affine group Aff acts naturally on $\mathfrak{B}$. Two convex bodies $\mathcal{C}, \mathcal{C}' \in \mathfrak{B}$ are called affine equivalent if they are on the same Aff-orbit, that is, $\mathcal{C}' = \varphi(\mathcal{C})$ for some $\varphi \in \mathrm{Aff}(\mathcal{X})$. Clearly, all simplices are affine equivalent and so are all ellipsoids.

The present paper is centered around three fundamental concepts of convex geometry: (1) The Minkowski measure of symmetry; (2) The Minkowski symmetral; and (3) The Banach-Mazur distance. These are defined as follows.

Minkowski Measure of Symmetry: Given $\mathcal{C} \in \mathfrak{B}$, for $\mathcal{O} \in \mathrm{int}\,\mathcal{C}$ and a hyperplane $\mathcal{H}$ with $\mathcal{O} \in \mathcal{H}$, let $\mathcal{H}'$ and $\mathcal{H}''$ be the two supporting hyperplanes of $\mathcal{C}$ parallel to $\mathcal{H}$. Let $\mathcal{R}(\mathcal{H}, \mathcal{O}) = \mathcal{R}_{\mathcal{C}}(\mathcal{H}, \mathcal{O})$ be the support ratio, the ratio, not less than 1, in which $\mathcal{H}$ divides the distance between $\mathcal{H}'$ and $\mathcal{H}''$. Letting

$$\mathcal{R}(\mathcal{O}) = \mathcal{R}_{\mathcal{C}}(\mathcal{O}) := \sup_{\mathcal{H} \ni \mathcal{O}} \mathcal{R}_{\mathcal{C}}(\mathcal{H}, \mathcal{O}), \quad \mathcal{O} \in \mathrm{int}\,\mathcal{C},$$

the Minkowski measure of symmetry of $\mathcal{C}$ is defined as

$$\mathfrak{m}^* = \mathfrak{m}^*_{\mathcal{C}} := \inf_{\mathcal{O} \in \mathrm{int}\,\mathcal{C}} \mathcal{R}_{\mathcal{C}}(\mathcal{O}).$$

It is an elementary fact that the infimum is attained. Clearly, $\mathfrak{m}^*$ is an affine invariant function on $\mathfrak{B}$, that is, we have

$$\mathfrak{m}^*_{\varphi(\mathcal{C})} = \mathfrak{m}^*_{\mathcal{C}}, \quad \mathcal{C} \in \mathfrak{B}, \quad \varphi \in \mathrm{Aff}(\mathcal{X})$$

(see [1, 10]).

For our purposes, a more concise definition of $\mathfrak{m}^*$ is needed as follows:

$$\mathfrak{m}^*_{\mathcal{C}} = \inf\{\lambda > 0 \mid \mathcal{C} + X \subset -\lambda \mathcal{C} \text{ for some } X \in \mathcal{X}\}. \tag{1}$$

The equivalence of this with the Minkowski measure of symmetry defined by the support ratio above is a simple calculation (see [8, 9] and also [10, Lemma 3.2.3]). In addition, there are a few other equivalent classical formulations of $\mathfrak{m}^*$ such as the definition based on the chord-ratio, or the definition using affine functions in the affine dual space (see [1, 2]).

The most important and classical inequality for the Minkowski measure of symmetry is

$$1 \leq \mathfrak{m}^*_{\mathcal{C}} \leq n, \quad \mathcal{C} \in \mathfrak{B}.$$

Moreover, we have $\mathfrak{m}^*_{\mathcal{C}} = 1$ if and only if $\mathcal{C}$ is centrally symmetric, and $\mathfrak{m}^*_{\mathcal{C}} = n$ if and only if $\mathcal{C}$ is a simplex (see [7]). These show that the Minkowski measure of symmetry is a measure of how symmetric or anti-symmetric a convex body is.

Minkowski Symmetral: For $\mathcal{C} \in \mathfrak{B}$, we define the Minkowski symmetral of $\mathcal{C}$ as $\tilde{\mathcal{C}} = (\mathcal{C} - \mathcal{C})/2$. Clearly, $\tilde{\mathcal{C}}$ is centrally symmetric with respect to the origin. The Minkowski symmetral has the following properties:

(MS1) $(\mathcal{C} + \mathcal{C}')\tilde{} = \tilde{\mathcal{C}} + \tilde{\mathcal{C}}'$, $\mathcal{C}, \mathcal{C}' \in \mathfrak{B}$;
(MS2) $\widetilde{\lambda\mathcal{C}} = \lambda\tilde{\mathcal{C}}$, $\mathcal{C} \in \mathfrak{B}$, $\lambda \in \mathbb{R}$;
(MS3) $\mathcal{C} = \tilde{\mathcal{C}}$ if and only if $\mathcal{C}$ is centrally symmetric with respect to the origin; in particular $\tilde{\tilde{\mathcal{C}}} = \tilde{\mathcal{C}}$;
(MS4) $\widetilde{\varphi(\mathcal{C})} = \varphi_0(\tilde{\mathcal{C}})$, where $\varphi \mapsto \varphi_0$ is the projection $\mathrm{Aff}(\mathcal{X}) \to GL(\mathcal{X})$.

Banach-Mazur Distance: The (extended) Banach-Mazur distance function $d_{BM} : \mathfrak{B} \times \mathfrak{B} \to \mathbb{R}$ is defined for $\mathcal{C}, \mathcal{C}' \in \mathfrak{B}$ as

$$d_{BM}(\mathcal{C}, \mathcal{C}') = \inf\{\lambda \geq 1 \mid \mathcal{C}' \subset \varphi(\mathcal{C}) \subset \lambda\mathcal{C}' + X \text{ with } \varphi \in \mathrm{Aff}(\mathcal{X}) \text{ and } X \in \mathcal{X}\}.$$

(By compactness of $\mathcal{C}$ and $\mathcal{C}'$, the infimum is clearly attained.)

d_{BM} satisfies the following properties:

(BM1) $d_{BM}(\mathcal{C}, \mathcal{C}') = 1$ for $\mathcal{C}, \mathcal{C}' \in \mathfrak{B}$ if and only if $\mathcal{C}' = \varphi(\mathcal{C})$ for some $\varphi \in \mathrm{Aff}(\mathcal{X})$;
(BM2) $d_{BM}(\mathcal{C}, \mathcal{C}') = d_{BM}(\mathcal{C}', \mathcal{C})$ for any $\mathcal{C}, \mathcal{C}' \in \mathfrak{B}$;
(BM3) $d_{BM}(\mathcal{C}, \mathcal{C}') \leq d_{BM}(\mathcal{C}, \mathcal{C}'') \cdot d_{BM}(\mathcal{C}'', \mathcal{C}')$ for any $\mathcal{C}, \mathcal{C}', \mathcal{C}'' \in \mathfrak{B}$;
(BM4) $d_{BM}(\varphi(\mathcal{C}), \psi(\mathcal{C}')) = d_{BM}(\mathcal{C}, \mathcal{C}')$ for any $\mathcal{C}, \mathcal{C}' \in \mathfrak{B}$ and $\varphi, \psi \in \mathrm{Aff}(\mathcal{X})$.

Clearly, the natural logarithm $\ln d_{BM}$ is a pseudo-metric on $\mathfrak{B}$. (By (BM1), affine equivalent convex bodies have zero distance.) But $\ln d_{BM}$ is actually a metric on $\mathfrak{B}/\mathrm{Aff}$, the quotient of $\mathfrak{B}$ by the action of the affine group Aff. It is well-known that, equipped with the Banach-Mazur distance, $\mathfrak{B}/\mathrm{Aff}$ becomes a compact metric space. (For a recent account on these facts, see [10].)

The following basic inequality is our starting point:

Proposition 1. *We have*

$$1 \leq \max\left(\frac{\mathfrak{m}^*_{\mathcal{C}}}{\mathfrak{m}^*_{\mathcal{C}'}}, \frac{\mathfrak{m}^*_{\mathcal{C}'}}{\mathfrak{m}^*_{\mathcal{C}}}\right) \leq d_{BM}(\mathcal{C}, \mathcal{C}'), \quad \mathcal{C}, \mathcal{C}' \in \mathfrak{B}. \tag{2}$$

The proposition implies

$$| \ln \mathfrak{m}^*_{\mathcal{C}} - \ln \mathfrak{m}^*_{\mathcal{C}'} | \leq \ln d_{BM}(\mathcal{C}, \mathcal{C}'), \quad \mathcal{C}, \mathcal{C}' \in \mathfrak{B}.$$

In other words, with respect to the Banach-Mazur distance $\ln d_{BM}$, the function $\ln \mathfrak{m}^*$ is Lipschitz with Lipschitz constant equal to 1.

By simple calculus we have

$$\frac{| x - y |}{n} \leq | \ln(x) - \ln(y) |, \ 1 \leq x, y \leq n, \quad \ln(z) \leq z - 1, \ z > 0.$$

These yield

$$| \mathfrak{m}^*_{\mathcal{C}} - \mathfrak{m}^*_{\mathcal{C}'} | \leq n\,(d_{BM}(\mathcal{C}, \mathcal{C}') - 1), \quad \mathcal{C}, \mathcal{C}' \in \mathfrak{B}.$$

If $\mathcal{C}' \in \mathfrak{B}$ is centrally symmetric then $\mathfrak{m}^*_{\mathcal{C}'} = 1$, and (2) gives the following:

Corollary 1. *For $\mathcal{C} \in \mathfrak{B}$ and centrally symmetric $\mathcal{C}' \in \mathfrak{B}$, we have*

$$\mathfrak{m}^*_{\mathcal{C}} \leq d_{BM}(\mathcal{C}, \mathcal{C}'). \tag{3}$$

Remark. For $\mathcal{C} \in \mathfrak{B}$ and centrally symmetric $\mathcal{C}' \in \mathfrak{B}$, the upper estimate

$$d_{BM}(\mathcal{C}, \mathcal{C}') \leq 2n - 1, \quad \mathcal{C} \in \mathfrak{B},$$

is a classical result due to Lassak [6].

It is well-known that we have

$$d_{BM}(\mathcal{C}, \tilde{\mathcal{C}}) \leq \mathfrak{m}^*_{\mathcal{C}}, \quad \mathcal{C} \in \mathfrak{B}$$

(see [8] or [10, Proposition 3.2.2]). This, combined with Corollary 1, gives the following:

Corollary 2. *We have*

$$d_{BM}(\mathcal{C}, \tilde{\mathcal{C}}) = \mathfrak{m}^*_{\mathcal{C}}, \quad \mathcal{C} \in \mathfrak{B}. \tag{4}$$

Remark. The inequality in (2) and its consequences have been stated in [9] with short proofs. In Section 2 we will give detailed proofs.

Let $\mathfrak{S} \subset \mathfrak{B}$ be the set of all centrally symmetric convex bodies, and $\mathfrak{S}_0 \subset \mathfrak{S}$ the set of convex bodies that are centrally symmetric with respect to the origin. Clearly, we have $\mathfrak{S} = \mathcal{T}(\mathfrak{S}_0)$, where $\mathcal{T} = \mathcal{T}(\mathcal{X})$ is the translation group. Actually, for $\mathcal{C} \in \mathfrak{S}$, we have $\mathcal{C} = \mathcal{C}_0 + Z$ with unique $\mathcal{C}_0 \in \mathfrak{S}_0$ and $Z \in \mathcal{X}$. Associating to $\mathcal{C} \in \mathfrak{S}$ the pair $(\mathcal{C}_0, Z) \in \mathfrak{S}_0 \times \mathcal{X}$ gives the identification $\mathfrak{S} = \mathfrak{S}_0 \times \mathcal{X}$.

The affine group Aff leaves $\mathfrak{S}$ invariant, and the general linear group $GL(\mathcal{X})$ leaves $\mathfrak{S}_0$ invariant.

Corollaries 1-2 have a nice geometric interpretation as follows: Given $\mathcal{C} \in \mathfrak{B}$, the (minimal) Banach-Mazur distance (d_{BM}) of $\mathcal{C}$ from $\mathfrak{S}(\subset \mathfrak{B})$ is realized by the Minkowski symmetral $\tilde{\mathcal{C}}$, and this distance is equal to the Minkowski measure of symmetry $\mathfrak{m}^*_{\mathcal{C}}$. Equivalently, we have

$$d_{BM}(\mathcal{C}, \mathfrak{S}) = \inf_{\mathcal{C}' \in \mathfrak{S}} d_{BM}(\mathcal{C}, \mathcal{C}') = d_{BM}(\mathcal{C}, \tilde{\mathcal{C}}) = \mathfrak{m}^*_{\mathcal{C}}, \quad \mathcal{C} \in \mathfrak{B}. \tag{5}$$

The Minkowski symmetral at which the infimum in (5) is attained is by no means unique. To elaborate on this, for $\mathcal{C} \in \mathfrak{B}$, we let

$$\mathfrak{S}_{\mathcal{C}} = \{\mathcal{C}' \in \mathfrak{S} \mid d_{BM}(\mathcal{C}, \mathcal{C}') = \mathfrak{m}^*_{\mathcal{C}}\}.$$

Clearly, we have $\tilde{\mathcal{C}} \in \mathfrak{S}_{\mathcal{C}}$.

Example. Consider $\mathfrak{S}_\Delta$, where $\Delta \subset \mathbb{R}^2$ is a triangle. Since all triangles are affinely equivalent, we may assume that Δ is equilateral. Since $\mathfrak{m}^*_\Delta = 2$, we have

$$\mathfrak{S}_\Delta = \{\mathcal{C} \in \mathfrak{S} \mid \Delta \subset \varphi(\mathcal{C}) \subset 2\Delta + X \text{ with } \varphi \in \text{Aff}(\mathcal{X}) \text{ and } X \in \mathcal{X}\}.$$

By performing affine eqivalences in the defining chain of inclusions, we may assume that $X = 0$. Thus, a very transparent picture of $\mathfrak{S}_\Delta$ emerges; it consists of (the affine images of) those centrally symmetric convex bodies $\mathcal{C} \in \mathfrak{S}$ for which we have

$$\Delta \subset \mathcal{C} \subset 2\Delta.$$

In particular, the regular hexagon $\tilde{\Delta}$ is one of these, and it is inscribed in the common circumcircle of Δ and incircle of $2\tilde{\Delta} = (\Delta - \Delta)$.

For $\mathcal{C}_0, \mathcal{C}_1 \in \mathfrak{B}$, the curve

$$\lambda \mapsto \mathcal{C}_\lambda = (1-\lambda)\mathcal{C}_0 + \lambda\mathcal{C}_1, \quad \lambda \in [0,1],$$

consists of convex bodies, that is, this curve lies entirely in $\mathfrak{B}$. The subset $\mathfrak{S} \subset \mathfrak{B}$ is "convex" in the sense that, if $\mathcal{C}_0, \mathcal{C}_1 \in \mathfrak{S}$ implies $\mathcal{C}_\lambda \in \mathfrak{S}$ for all $\lambda \in [0,1]$.

The next proposition asserts convexity of the minimal level-set set $\mathfrak{S}_{\mathcal{C}}$ as follows:

Proposition 2. *For $\mathcal{C}_0, \mathcal{C}_1 \in \mathfrak{S}_{\mathcal{C}}$, we have*

$$\mathcal{C}_\lambda = (1-\lambda)\mathcal{C}_0 + \lambda\mathcal{C}_1 \in \mathfrak{S}_{\mathcal{C}}, \quad \lambda \in [0,1].$$

Due to convexity of $\mathfrak{S}$, based on classical analogy, it is natural to expect that the "distance function" $\mathcal{C} \mapsto d_{BM}(\mathcal{C}, \mathfrak{S}) = \mathfrak{m}^*_{\mathcal{C}}$, $\mathcal{C} \in \mathfrak{B}$, is convex:

$$\mathfrak{m}^*_{(1-\lambda)\mathcal{C}_0+\lambda\mathcal{C}_1} \leq (1-\lambda)\mathfrak{m}^*_{\mathcal{C}_0} + \lambda\mathfrak{m}^*_{\mathcal{C}_1}, \quad \mathcal{C}_0, \mathcal{C}_1 \in \mathfrak{B},\ \lambda \in [0,1]. \tag{6}$$

This, however, fails for a large class of convex bodies. To illustrate this first, we begin with the following:

Example. Let $\Delta \subset \mathcal{X}$ be a regular simplex with altitude length equal to $n+1$, and $\bar{\mathcal{B}}$ the closed unit ball. Given $r > 0$, we first observe that the convex body $(1-\lambda)r\Delta + \lambda\bar{\mathcal{B}}$, $\lambda \in (0,1)$, is the λ-neighborhood of the convex body $(1-\lambda)r\Delta$. Since the centroid of Δ splits an altitude line to the ratio n to 1, the Minkowski measure can be readily computed as

$$\mathfrak{m}^*_{(1-\lambda)r\Delta+\lambda\bar{\mathcal{B}}} = \frac{(1-\lambda)rn + \lambda}{(1-\lambda)r + \lambda}.$$

In particular, we have

$$\lim_{r\to\infty} \mathfrak{m}^*_{(1-\lambda)r\Delta+\lambda\bar{\mathcal{B}}} = n.$$

On the other hand, since the Minkowski measure is affine invariant, we have

$$(1-\lambda)\mathfrak{m}^*_{r\Delta} + \lambda\mathfrak{m}^*_{\bar{\mathcal{B}}} = (1-\lambda)n + \lambda < n.$$

We obtain that, for given $\lambda \in (0,1)$ and r large, (6) cannot hold for $\mathcal{C}_0 = r\Delta$ and $\mathcal{C}_1 = \bar{\mathcal{B}}$.

The next proposition assests that this phenomenon holds in a much more general setting:

Proposition 3. *Let $\mathcal{C}_0, \mathcal{C}_1 \in \mathfrak{B}$ be convex bodies such that $\mathfrak{m}^*_{\mathcal{C}_0} > \mathfrak{m}^*_{\mathcal{C}_1}$. Then, for $r > 0$ large, the function $\lambda \mapsto \mathfrak{m}^*_{(1-\lambda)r\mathcal{C}_0+\lambda\mathcal{C}_1}$, $\lambda \in [0,1]$, is not convex.*

Note that a property weaker than convexity is known to hold:

$$\mathfrak{m}^*_{(1-\lambda)\mathcal{C}_0+\lambda\mathcal{C}_1} \leq \max(\mathfrak{m}^*_{\mathcal{C}_0}, \mathfrak{m}^*_{\mathcal{C}_1}), \quad \mathcal{C}_0, \mathcal{C}_1 \in \mathfrak{B},\ \lambda \in [0,1];$$

and it is an easy consequence of "super-additivity" (and affine invariance) of the Minkowski measure

$$\mathfrak{m}^*_{\mathcal{C}_0+\mathcal{C}_1} \leq \max(\mathfrak{m}^*_{\mathcal{C}_0}, \mathfrak{m}^*_{\mathcal{C}_1}), \quad \mathcal{C}_0, \mathcal{C}_1 \in \mathfrak{B}.$$

(The former follows directly from latter by making affine conbinations; see also [1].)

Remark. In the opposite end of the spectrum of $\mathfrak{m}^*$ one can define the (affine) measure of asymmetry of $\mathcal{C} \in \mathfrak{B}$ as $d_{BM}(\mathcal{C}, \Delta)$, where Δ is a simplex in $\mathcal{X}$.

Our next result is the following:

Proposition 4. *For $\mathcal{C}, \mathcal{C}' \in \mathfrak{B}$, we have*

$$d_{BM}(\widetilde{\mathcal{C}}, \widetilde{\mathcal{C}'}) \leq d_{BM}(\mathcal{C}, \mathcal{C}') \leq \mathfrak{m}^*_{\mathcal{C}} \mathfrak{m}^*_{\mathcal{C}'} \cdot d_{BM}(\tilde{\mathcal{C}}, \tilde{\mathcal{C}}'). \tag{7}$$

Remark. The upper bounds for the Banach-Mazur distances in (7) are classical. We have

$$d_{BM}(\mathcal{C}, \mathcal{C}') \leq (n-1)\min(\mathfrak{m}_{\mathcal{C}}, \mathfrak{m}_{\mathcal{C}'}) + n \leq n^2, \quad \mathcal{C}, \mathcal{C}' \in \mathfrak{B}, \tag{8}$$

$$d_{BM}(\mathcal{C}, \mathcal{C}') \leq n, \quad \mathcal{C}, \mathcal{C}' \in \mathfrak{S}. \tag{9}$$

The first inequality in (8) is due to Qi Guo in [4] (note also that Lassak's inequality cited above also follows from this), the second inequality in (8) as well as (9) are the celebrated results of Fritz John [5].

The Minkowski symmetral defines a map $\mathcal{S} : \mathfrak{B} \to \mathfrak{S}_0$ by $\mathcal{S}(\mathcal{C}) = \tilde{\mathcal{C}} = (\mathcal{C} - \mathcal{C})/2$, $\mathcal{C} \in \mathfrak{B}$. By the first inequality in (7), $\mathcal{S}$ is continuous. Clearly, the restriction of $\mathcal{S}$ to $\mathfrak{S}$ acts as translations (translating the center of symmetry to the origin), and the restriction of $\mathcal{S}$ to $\mathfrak{S}_0$ is the identity on $\mathfrak{S}_0$. Taking quotients, $\mathcal{S}$ gives rise to a retraction[1] $\Sigma : \mathfrak{B}/\mathrm{Aff} \to \mathfrak{S}/\mathrm{Aff}$. For the image we have $\mathfrak{S}/\mathrm{Aff} = \mathfrak{S}_0/GL(\mathcal{X})$.

Remark. The quotient $\mathfrak{S}/\mathrm{Aff} = \mathfrak{S}_0/GL(\mathcal{X})$ is the so-called Banach-Mazur compactum. It is usually given by calibrating the John's ellipsoid in each convex body to the unit ball, and thereby reducing the acting group Aff or $GL(\mathcal{X})$ to the orthogonal group $O(\mathcal{X})$. The topology of the Banach-Mazur compactum is subtle.

We now claim that $\mathfrak{S}/\mathrm{Aff}$ is a strong deformation retract[2] of $\mathfrak{B}/\mathrm{Aff}$; in particular, these two spaces are homotopy equivalent. The homotopy for the

[1]Given a topological space X and a subspace A, a continuous map $r : X \to A$ is called a retraction if the restriction $r|A$ is the identity of A.

[2]Continuing with the previous footnote, a retraction $r : X \to A$ is called a strong deformation retraction if there exists a homotopy $F : X \times [0,1] \to X$ such that $F(x,0) = x$, $x \in X$, $F(x,1) = r(x)$, $x \in X$, and $F(a,t) = a$, $a \in A$, $t \in [0,1]$. We call A a strong deformation retract of X. It is well-known that in this case the spaces X and A have the same homotopy type, that is they are homotopy equivalent.

map $\mathcal{S} : \mathfrak{B} \to \mathfrak{S}_0 \subset \mathfrak{S}$ realizing this is $\Phi : \mathfrak{B} \times [0,1] \to \mathfrak{B}$ defined by

$$\Phi(\mathcal{C}, \lambda) = (1-\lambda)\mathcal{C} + \lambda\tilde{\mathcal{C}}, \quad \mathcal{C} \in \mathfrak{B},\ \lambda \in [0,1].$$

We summarize this in the following:

Proposition 5. *The Minkowski symmetral gives rise to a continuous deformation retraction* $\Sigma : \mathfrak{B}/\mathrm{Aff} \to \mathfrak{S}/\mathrm{Aff}$*, and with this* $\mathfrak{S}/\mathrm{Aff}$ *is a strong deformation retract of* $\mathfrak{B}/\mathrm{Aff}$*.*

2. Proofs

Proof of Proposition 1. Let $\delta \geq 0$ and $\mathcal{C}, \mathcal{C}' \in \mathfrak{B}$ such that $d_{BM}(\mathcal{C}, \mathcal{C}') \leq 1 + \delta$. By the definition of the Banach-Mazur distance

$$\mathcal{C}' \subset \varphi(\mathcal{C}) \subset (1+\delta)\mathcal{C}' + Y \tag{10}$$

for some $\varphi \in \mathrm{Aff}(\mathcal{X})$ and $Y \in \mathcal{X}$. Since $\mathfrak{m}^*$ and d_{BM} are affine invariant, without loss of generality, we may assume that φ is the identity.

Let $\lambda > 0$ such that $\mathfrak{m}^*_{\mathcal{C}} \leq \lambda$. Then (1) gives

$$\mathcal{C} + X \subset -\lambda\mathcal{C} \tag{11}$$

for some $X \in \mathcal{X}$. Combining (10) and (11), we obtain

$$\mathcal{C}' + X + \lambda Y \subset \mathcal{C} + X + \lambda Y \subset -\lambda(\mathcal{C} - Y) \subset -\lambda(1+\delta)\mathcal{C}'.$$

Now, applying (1) to $\mathcal{C}'$ we obtain $\mathfrak{m}^*_{\mathcal{C}'} \leq \lambda(1+\delta)$. Thus, we get

$$\mathfrak{m}^*_{\mathcal{C}'} \leq \mathfrak{m}^*_{\mathcal{C}} \cdot d_{BM}(\mathcal{C}, \mathcal{C}'). \tag{12}$$

The same inequality holds with the roles of $\mathcal{C}, \mathcal{C}'$ in (12) interchanged. These together give (2). □

Proof of Proposition 2. Given $\mathcal{C} \in \mathfrak{B}$ and $\mathcal{C}_0, \mathcal{C}_1 \in \mathfrak{S}_{\mathcal{C}}$ we let $\mathcal{C}_\lambda = (1-\lambda)\mathcal{C}_0 + \lambda\mathcal{C}_1$, $\lambda \in [0,1]$. Clearly, $\mathcal{C}_\lambda \in \mathfrak{S}$, $\lambda \in [0,1]$. Since

$$d_{BM}(\mathcal{C}, \mathcal{C}_0) = d_{BM}(\mathcal{C}, \mathcal{C}_1) = \mathfrak{m}^*_{\mathcal{C}},$$

we have

$$\mathcal{C}_0 \subset \varphi_0(\mathcal{C}) \subset \mathfrak{m}^*_{\mathcal{C}}\, \mathcal{C}_0 + X_0$$
$$\mathcal{C}_1 \subset \varphi_1(\mathcal{C}) \subset \mathfrak{m}^*_{\mathcal{C}}\, \mathcal{C}_1 + X_1$$

for some $\varphi_0, \varphi_1 \in \mathrm{Aff}$ and $X_0, X_1 \in \mathcal{X}$. Combining these, we obtain

$$\begin{aligned}(1-\lambda)\mathcal{C}_0 + \lambda\mathcal{C}_1 &\subset (1-\lambda)\varphi_0(\mathcal{C}) + \lambda\varphi_1(\mathcal{C}) \\ &\subset \mathfrak{m}^*_{\mathcal{C}}\ ((1-\lambda)\mathcal{C}_0 + \lambda\mathcal{C}_1) + (1-\lambda)X_0 + \lambda X_1.\end{aligned}$$

We write this as

$$\mathcal{C}_\lambda \subset \varphi_\lambda(\mathcal{C}) \subset \mathfrak{m}^*_{\mathcal{C}}\, \mathcal{C}_\lambda + X_\lambda, \tag{13}$$

where $\varphi_\lambda = (1-\lambda)\varphi_0 + \lambda\varphi_1$ and $X_\lambda = (1-\lambda)X_0 + \lambda X_1 \in \mathcal{X}$. Clearly, $\varphi_\lambda : \mathcal{X} \to \mathcal{X}$ is an affine map. In addition, it must be nonsingular since, by

(13), the image of φ_λ contains $\mathcal{C}_\lambda$ and the latter has nonempty interior. Thus $\varphi_\lambda \in \mathrm{Aff}$, an affine transformation.

By (13) and the definition of the Banach-Mazur distance, we have

$$d_{BM}(\mathcal{C}, \mathcal{C}_\lambda) \leq \mathfrak{m}^*_{\mathcal{C}}.$$

By (5), $\mathfrak{m}^*_{\mathcal{C}}$ is the minimal distance, therefore equality holds here. Hence, $\mathcal{C}_\lambda \in \mathfrak{S}_{\mathcal{C}}$, $\lambda \in [0,1]$, and the proposition follows. □

Proof of Proposition 3. Let $\lambda \in (0,1)$ be fixed and $r > 0$. Since the Minkowski measure is affine invariant, we have

$$\lim_{r\to\infty} \mathfrak{m}^*_{(1-\lambda)r\mathcal{C}_0+\lambda\mathcal{C}_1} = \lim_{r\to\infty} \mathfrak{m}^*_{(1-\lambda)\mathcal{C}_0+\lambda(1/r)\mathcal{C}_1} = \mathfrak{m}^*_{(1-\lambda)\mathcal{C}_0} = \mathfrak{m}^*_{\mathcal{C}_0}.$$

On the other hand, we have

$$(1-\lambda)\mathfrak{m}^*_{\mathcal{C}_0} + \lambda\mathfrak{m}^*_{\mathcal{C}_1} = \mathfrak{m}^*_{\mathcal{C}_0} + \lambda(\mathfrak{m}^*_{\mathcal{C}_1} - \mathfrak{m}^*_{\mathcal{C}_0}) < \mathfrak{m}^*_{\mathcal{C}_0}$$

since, by assumption, $\mathfrak{m}^*_{\mathcal{C}_0} > \mathfrak{m}^*_{\mathcal{C}_1}$. Comparing these two, we see that, for large $r > 0$, the function $\lambda \mapsto \mathfrak{m}^*_{(1-\lambda)r\mathcal{C}_0+\lambda\mathcal{C}_1}$ cannot be convex. The proposition follows. □

Proof of Proposition 4. We need a simple lemma as follows:

Lemma. *Let $\mathcal{A}, \mathcal{A}', \mathcal{B}, \mathcal{B}' \in \mathcal{X}$. If $\mathcal{A} \subset \mathcal{A}', \mathcal{B} \subset \mathcal{B}'$, then*

$$\mathcal{A} - \mathcal{B} \subset \mathcal{A}' - \mathcal{B}';$$

in particular, if $\mathcal{C}, \mathcal{C}' \in \mathfrak{B}$ and $\mathcal{C} \subset \mathcal{C}'$, then $\tilde{\mathcal{C}} \subset \tilde{\mathcal{C}}'$.

Proof. Let $A - B \in \mathcal{A} - \mathcal{B}$, where $A \in \mathcal{A}, B \in \mathcal{B}$, then $A \in \mathcal{A}', B \in \mathcal{B}'$, which implies $A - B \in \mathcal{A}' - \mathcal{B}'$. The lemma follows. □

Returning to the proof of Proposition 4, assume $d_{BM}(\mathcal{C}, \mathcal{C}') \leq \lambda$ for some $\lambda \geq 1$. By definition, we have

$$\mathcal{C} \subset \varphi(\mathcal{C}') \subset \lambda\,\mathcal{C} + X$$

for some $\varphi \in \mathrm{Aff}$ and $X \in \mathcal{X}$. By the lemma above, we obtain

$$\frac{1}{2}(\mathcal{C} - \mathcal{C}) \subset \frac{1}{2}\left(\varphi(\mathcal{C}') - \varphi(\mathcal{C}')\right) \subset \frac{1}{2}((\lambda\mathcal{C} + X) - (\lambda\mathcal{C} + X)).$$

We simplify and write this as

$$\tilde{\mathcal{C}} \subset \frac{1}{2}\left(\varphi(\mathcal{C}') - \varphi(\mathcal{C}')\right) \subset \frac{\lambda}{2}(\mathcal{C} - \mathcal{C}) = \lambda\tilde{\mathcal{C}}. \tag{14}$$

A typical element in the middle term in (14) is $(\varphi(X') - \varphi(Y'))/2$, $X', Y' \in \mathcal{C}'$. Letting $\varphi(X) = A \cdot X + Z, Z \in \mathcal{X}$, $A \in GL(\mathcal{X})$, we write this as

$$\frac{1}{2}(\varphi(X') - \varphi(Y')) = \frac{1}{2}(A \cdot X' + Z - (A \cdot Y' + Z)) = A \cdot \frac{1}{2}(X' - Y'). \tag{15}$$

This is a typical element in $\varphi_0(\tilde{\mathcal{C}}') = A \cdot \tilde{\mathcal{C}}'$, where $\varphi_0 \in GL(\mathcal{X})$ is the projection of Aff. Putting these together, (14) becomes

$$\tilde{\mathcal{C}} \subset \varphi_0(\tilde{\mathcal{C}}') \subset \lambda\tilde{\mathcal{C}}.$$

This gives

$$d_{BM}(\tilde{\mathcal{C}}, \tilde{\mathcal{C}}') \leq \lambda.$$

Since $d_{BM}(\mathcal{C}, \mathcal{C}') \leq \lambda$, we obtain the first inequality in (7).

The second inequality in (7) is an easy application of (BM3) along with Corollary 2 as follows:

$$d_{BM}(\mathcal{C}, \mathcal{C}') \leq d_{BM}(\mathcal{C}, \tilde{\mathcal{C}}) d_{BM}(\mathcal{C}', \tilde{\mathcal{C}}') d_{BM}(\tilde{\mathcal{C}}, \tilde{\mathcal{C}}') = \mathfrak{m}^*_{\mathcal{C}} \mathfrak{m}^*_{\mathcal{C}'} \cdot d_{BM}(\tilde{\mathcal{C}}, \tilde{\mathcal{C}}'). \quad \square$$

Remark. The first inequality in (7) is sharp, e.g. if $\mathcal{C}$ and $\mathcal{C}'$ are symmetric, then equality holds. Note that, for $\mathcal{C}$ not symmetric and $\mathcal{C}' = \tilde{\mathcal{C}}$, then, by the classical Minkowski estimate, we have

$$1 = d_{BM}(\tilde{\mathcal{C}}, \tilde{\mathcal{C}}') < d_{BM}(\mathcal{C}, \mathcal{C}') = d_{BM}(\mathcal{C}, \tilde{\mathcal{C}}) = \mathfrak{m}^*_{\mathcal{C}}.$$

Proof of Proposition 5. We first claim that the homotopy $\Phi : \mathfrak{B} \times [0, 1] \to \mathfrak{B}$, $\Phi(\mathcal{C}, \lambda) = (1 - \lambda)\mathcal{C} + \lambda\tilde{\mathcal{C}}$, $\mathcal{C} \in \mathfrak{B}$, $\lambda \in [0, 1]$, defined before the statement of Proposition 5 factors through the projection $\mathfrak{B} \to \mathfrak{B}/\mathrm{Aff}$ and gives rise to a homotopy $\mathfrak{B}/\mathrm{Aff} \times [0, 1] \to \mathfrak{B}/\mathrm{Aff}$. Let $\lambda \in [0, 1]$ be fixed. For $\mathcal{C} \in \mathfrak{B}$ and $\varphi \in \mathrm{Aff}$, using (MS4), we calculate

$$\begin{aligned}
\Phi(\varphi(\mathcal{C}), \lambda) &= (1 - \lambda)\varphi(\mathcal{C}) + \lambda\widetilde{\varphi(\mathcal{C})} \\
&= (1 - \lambda)\varphi(\mathcal{C}) + \lambda\varphi_0(\tilde{\mathcal{C}}) \\
&= (1 - \lambda)\, A \cdot \mathcal{C} + (1 - \lambda)\, Z + \lambda\, A \cdot \tilde{\mathcal{C}} \\
&= A \cdot \Big((1 - \lambda)\, \mathcal{C} + \lambda\, \tilde{\mathcal{C}}\Big) + (1 - \lambda)\, Z \\
&= A \cdot \Phi(\mathcal{C}, \lambda) + (1 - \lambda)\, Z,
\end{aligned}$$

where, as usual, $\varphi(X) = A \cdot X + Z$, $A \in GL(\mathcal{X})$, $Z \in \mathcal{X}$.

The last convex body in the computation above is affinely equivalent to the convex body $\Phi(\mathcal{C}, \lambda)$, so that the projection of the homotopy is well-defined. The claim follows.

The rest of the proof of Proposition 5 is straightforward. We have $\Phi(\mathcal{C}, 0) = \mathcal{C}$ and $\Phi(\mathcal{C}, 1) = \tilde{\mathcal{C}}$, $\mathcal{C} \in \mathfrak{B}$. Finally, for $\mathcal{C}' \in \mathfrak{S}$ and $\lambda \in [0, 1]$, we have

$$\Phi(\mathcal{C}', \lambda) = (1 - \lambda)\mathcal{C}' + \lambda\tilde{\mathcal{C}}' = (1 - \lambda)\mathcal{C}' + \lambda\mathcal{C}' = \mathcal{C}'.$$

The proposition follows. $\square$

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