J. Korean Math. Soc. **55** (2018), No. 3, pp. 605–622
https://doi.org/10.4134/JKMS.j170347
pISSN: 0304-9914 / eISSN: 2234-3008

MIXED MULTIPLICITIES OF MAXIMAL DEGREES

Truong Thi Hong Thanh and Duong Quoc Viet

Abstract. The original mixed multiplicity theory considered the class of mixed multiplicities concerning the terms of highest total degree in the Hilbert polynomial. This paper defines a broader class of mixed multiplicities that concern the maximal terms in this polynomial, and gives many results, which are not only general but also more natural than many results in the original mixed multiplicity theory.

1. Introduction

Let $(A, \mathfrak{m})$ be an Artinian local ring with maximal ideal $\mathfrak{m}$ and infinite residue field $A/\mathfrak{m}$. Let $S = \bigoplus_{\mathbf{n}\in\mathbb{N}^d} S_{\mathbf{n}}$ be a finitely generated standard $\mathbb{N}^d$-graded algebra over A (i.e., S is generated over A by elements of total degree 1) and let $M = \bigoplus_{\mathbf{n}\in\mathbb{N}^d} M_{\mathbf{n}}$ be a finitely generated $\mathbb{N}^d$-graded S-module. Denote by $P_M(\mathbf{n})$ the Hilbert polynomial of the Hilbert function $\ell_A[M_{\mathbf{n}}]$ and by $\mathrm{Proj}S$ the set of the homogeneous prime ideals of S which do not contain $S_{++} = \bigoplus_{\mathbf{n}>\mathbf{0}} S_{\mathbf{n}}$. Put $\mathrm{Supp}_{++}M = \{P \in \mathrm{Proj}S \mid M_P \neq 0\}$ and $\dim \mathrm{Supp}_{++}M = s$. Then by [5, Theorem 4.1], $\deg P_M(\mathbf{n}) = s$. Since $P_M(\mathbf{n})$ is a numerical polynomial, one can write $P_M(\mathbf{n}) = \sum_{\mathbf{k}\in\mathbb{N}^d} e(M;\mathbf{k})\binom{\mathbf{n}+\mathbf{k}}{\mathbf{k}}$, $e(M;\mathbf{k}) \in \mathbb{Z}$ and $\binom{\mathbf{n}+\mathbf{k}}{\mathbf{k}} = \binom{n_1+k_1}{k_1}\cdots\binom{n_d+k_d}{k_d}$ for all $\mathbf{k} = (k_1,\ldots,k_d)$ and $\mathbf{n} = (n_1,\ldots,n_d) \in \mathbb{N}^d$.

In past years, one studied the mixed multiplicities concerning the coefficients of the terms of highest total degree in the Hilbert polynomial $P_M(\mathbf{n})$, i.e., the mixed multiplicities $e(M;\mathbf{k})$ of M of the type $\mathbf{k}$ with $|\mathbf{k}| = k_1 + \cdots + k_d = s$. These mixed multiplicities are briefly called the *original mixed multiplicities* (or the *mixed multiplicities of highest degree*). And the original mixed multiplicity theory has attracted much attention and has been continually developed (see e.g. [2-12; 14-31]).

In this paper, we consider $e(M;\mathbf{k})$ such that $e(M;\mathbf{h}) = 0$ for all $\mathbf{h} > \mathbf{k}$ which concerns the coefficient of the maximal term of degree $\mathbf{k}$ in the Hilbert

Received May 23, 2017; Accepted October 25, 2017.

2010 *Mathematics Subject Classification.* Primary 13H15; Secondary 13A02, 13E05, 13E10, 14C17.

Key words and phrases. mixed multiplicity, Euler-Poincare characteristic, Koszul complex.

This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.04.2015.01.

polynomial of M. Then $e(M;\mathbf{k})$ is called the *mixed multiplicity* (*of maximal degree*) *of M of the type* $\mathbf{k}$ (Definition 2.2). Proposition 2.4 proves that $e(M;\mathbf{k})$ are non-negative integers. Note that the presence of the mixed multiplicity $e(M;\mathbf{k})$ with $|\mathbf{k}| < \deg P_M(\mathbf{n})$ often arises from the process of transforming the mixed multiplicities of highest degree (see [31]). Moreover, the natural appearance of these mixed multiplicities is also expressed via the relationship between their existence and the existence of other familiar objects. This fact is shown by Proposition 2.9 which characterizes the existence of the mixed multiplicity of the type $\mathbf{k}$ for a given $\mathbf{k} \in \mathbb{N}^d$. In addition, for the persuasiveness, Example 2.12 showed the presence of all these mixed multiplicities.

The paper first answers to the question when mixed multiplicities of maximal degrees are positive and characterizes these mixed multiplicities in terms of the length of modules via filter-regular sequences. Recall that a homogeneous element $a \in S$ is called an *S_{++}-filter-regular element with respect to M* if $(0_M : a)_{\mathbf{n}} = 0$ for all large $\mathbf{n}$. And a sequence $x_1, \ldots, x_t$ in S is called an *S_{++}-filter-regular sequence with respect to M* if x_i is an S_{++}-filter-regular element with respect to $M/(x_1, \ldots, x_{i-1})M$ for all $1 \leq i \leq t$. Set $\mathbf{e}_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{N}^d$ and $S_i = S_{\mathbf{e}_i}$ for $1 \leq i \leq d$. A sequence of elements in $\bigcup_{j=1}^d S_j$ consisting of k_1 elements of $S_1, \ldots, k_d$ elements of S_d is called a sequence of the *type* $\mathbf{k} = (k_1, \ldots, k_d)$. Then we have the following result.

Theorem 1.1 (Theorem 2.7)**.** *Let $e(M;\mathbf{k})$ be the mixed multiplicity of maximal degree of the type $\mathbf{k}$ of M. Assume that $\mathbf{x}$ is an S_{++}-filter-regular sequence of the type $\mathbf{k}$ of M. Then we have $e(M;\mathbf{k}) = \ell_A\big[\big(M/\mathbf{x}M\big)_{\mathbf{n}}\big]$ for all large $\mathbf{n}$. And $e(M;\mathbf{k}) \neq 0$ if and only if $\dim \operatorname{Supp}_{++}\big(\frac{M}{\mathbf{x}M}\big) = 0$.*

From this theorem we obtain Proposition 2.9; Remark 2.10; Corollary 2.11; Example 2.12 on the existence of mixed multiplicities of maximal degrees.

To consider the relationship between mixed multiplicities of maximal degrees and other invariants, we turn now to the notion of mixed multiplicity systems and related invariants. Recall that a sequence $\mathbf{y}$ of the type $\mathbf{k} \in \mathbb{N}^d$ is called a *mixed multiplicity system of M of the type* $\mathbf{k}$ if $\dim \operatorname{Supp}_{++}\big(\frac{M}{\mathbf{y}M}\big) \leq 0$. Let $\mathbf{x} = x_1, \ldots, x_n$ be a mixed multiplicity system of M of the type $\mathbf{k}$. Denote by $H_i(\mathbf{x}, M)$ the ith Koszul homology module of M with respect to $\mathbf{x}$. Then one can define that is called the *Euler-Poincare characteristic* $\chi(\mathbf{x}, M) = \sum_{i=0}^n (-1)^i \ell_A[H_i(\mathbf{x}, M)_{\mathbf{n}}]$ (a constant for all $\mathbf{n} \gg \mathbf{0}$). And one also define the *mixed multiplicity symbol* $\widetilde{e}(\mathbf{x}, M)$ as follows. If $n = 0$, then $\ell_A[M_{\mathbf{n}}] = c$ (const) for all $\mathbf{n} \gg \mathbf{0}$ and set $\widetilde{e}(\mathbf{x}, M) = \widetilde{e}(\emptyset, M) = c$. If $n > 0$, set $\widetilde{e}(\mathbf{x}, M) = \widetilde{e}(\mathbf{x}', M/x_1M) - \widetilde{e}(\mathbf{x}', 0_M : x_1)$, here $\mathbf{x}' = x_2, \ldots, x_n$ (see [31]).

With the above notations, the main theorem of this paper is stated as follows.

Theorem 1.2 (Theorem 2.13)**.** *The mixed multiplicity of maximal degree of M of the type $\mathbf{k}$ is defined if and only if there exists a mixed multiplicity system $\mathbf{x}$ of M of the type $\mathbf{k}$. In this case, we have*

$$\chi(\mathbf{x}, M) = \widetilde{e}(\mathbf{x}, M) = e(M;\mathbf{k}).$$

As applications of this theorem, we obtain Corollary 2.16 which characterizes the positivity of mixed multiplicities; Corollary 2.17 on the additivity of mixed multiplicities of maximal degrees; and the following formula which transforms mixed multiplicities of maximal degrees via mixed multiplicity systems.

Corollary 1.3 (Corollary 2.14). *Let* $\mathbf{x} = x_1, \ldots, x_s$ *be a mixed multiplicity system of* M *of the type* $\mathbf{k}$. *Denote by* $\mathbf{h}_i = (h_{i1}, \ldots, h_{id})$ *the type of a subsequence* $x_1, \ldots, x_i$ *of* $\mathbf{x}$ *for each* $1 \leq i \leq s$. *Then for all large* $\mathbf{n}$, *we have*

$$e(M;\mathbf{k}) = \ell_A\big[\big(M/\mathbf{x}M\big)_{\mathbf{n}}\big] - \sum_{i=1}^{s} e\bigg(\frac{(x_1,\ldots,x_{i-1})M : x_i}{(x_1,\ldots,x_{i-1})M}; \mathbf{k}-\mathbf{h}_i\bigg).$$

And by Theorem 1.1 and Corollary 1.3, we immediately get the following result.

Corollary 1.4 (Corollary 2.15). *Let* $\mathbf{x}$ *be a mixed multiplicity system of* M *of the type* $\mathbf{k}$. *Then we have* $e(M;\mathbf{k}) \leq \ell_A\big[\big(M/\mathbf{x}M\big)_{\mathbf{n}}\big]$ *for all large* $\mathbf{n}$, *and equality holds if* $\mathbf{x}$ *is an* S_{++}*-filter-regular sequence.*

Moreover, applying Theorem 1.2 for mixed multiplicities of maximal degrees of ideals, we get several corollaries (see Theorem 3.5, Corollary 3.8, Corollary 3.9, Corollary 3.10, Corollary 3.11, Corollary 3.12, Theorem 3.13) in Section 3.

The results of this paper show that many important properties of the mixed multiplicities of highest degree not only are still true but also are more natural in the broader class of the mixed multiplicities of maximal degrees. Further, we see that because many constrained conditions in the original mixed multiplicity theory can be eliminated, statements and proofs sometimes become more convenient in the class of these new objects. Moreover, we hope that these objects and results on them not only are a pure extension, but also will bring a certain geometrical significance.

The paper is divided into three sections. Section 2 is devoted to the discussion of mixed multiplicities of multi-graded modules. Section 3 gives applications of Section 2 to mixed multiplicities of ideals.

2. Mixed multiplicities of graded modules

This section studies a class of mixed multiplicities which concern the coefficients of the maximal terms in the Hilbert polynomial of multi-graded modules.

Let d be a positive integer. Put $\mathbf{e}_i = (0, \ldots, \underset{(i)}{1}, \ldots, 0) \in \mathbb{N}^d$ for each $1 \leq i \leq d$ and $\mathbf{k}! = k_1! \cdots k_d!$; $\mid \mathbf{k} \mid = k_1 + \cdots + k_d$ for any $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$. Moreover, set $\mathbf{0} = (0, \ldots, 0) \in \mathbb{N}^d$; $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^d$ and $\mathbf{n}^{\mathbf{k}} = n_1^{k_1} \cdots n_d^{k_d}$ for each $\mathbf{n}, \mathbf{k} \in \mathbb{N}^d$ and $\mathbf{n} \geq \mathbf{1}$. Let $S = \bigoplus_{\mathbf{n} \in \mathbb{N}^d} S_{\mathbf{n}}$ be a finitely generated standard $\mathbb{N}^d$-graded algebra over A (i.e., S is generated over A by elements of total degree 1) and let $M = \bigoplus_{\mathbf{n} \geq \mathbf{0}} M_{\mathbf{n}}$ be a finitely generated $\mathbb{N}^d$-graded S-module. Set $S_{++} = \bigoplus_{\mathbf{n} \geq \mathbf{1}} S_{\mathbf{n}}$ and $S_i = S_{\mathbf{e}_i}$ for any $1 \leq i \leq d$. Denote by

Proj$S$ the set of the homogeneous prime ideals of S which do not contain S_{++}. Put

$$\mathrm{Supp}_{++}M = \{P \in \mathrm{Proj}S \mid M_P \neq 0\}.$$

By [5, Theorem 4.1], $\ell_A[M_{\mathbf{n}}]$ is a polynomial for all large $\mathbf{n}$. Denote by $P_M(\mathbf{n})$ the Hilbert polynomial of the Hilbert function $\ell_A[M_{\mathbf{n}}]$.

Remark 2.1. If we assign $\dim \mathrm{Supp}_{++}M = -\infty$ to the case that $\mathrm{Supp}_{++}M = \emptyset$ and the degree $-\infty$ to the zero polynomial, then by [5, Theorem 4.1] and [27, Proposition 2.7], we always have $\deg P_M(\mathbf{n}) = \dim \mathrm{Supp}_{++}M$.

Recall that in particular, if $\dim \mathrm{Supp}_{++}M = s \geq 0$ and the terms of total degree s in the polynomial $P_M(\mathbf{n})$ have the form $\sum_{|\mathbf{k}|\,=\,s} e(M;\mathbf{k})\frac{\mathbf{n}^{\mathbf{k}}}{\mathbf{k}!}$, then $e(M;\mathbf{k})$ are non-negative integers not all zero, called the *mixed multiplicity of* M *of the type* $\mathbf{k}$ [5]. Obviously, these mixed multiplicities only concern the coefficients of the terms of highest total degree in the Hilbert polynomial $\deg P_M(\mathbf{n})$. And from now on, these mixed multiplicities are called the *original mixed multiplicities* (or the *mixed multiplicities of highest degree*).

Now, we give a broader class than the class of the original mixed multiplicities as in the above introduction.

Since $P_M(\mathbf{n})$ is a numerical polynomial, it is well known that we can write

$$P_M(\mathbf{n}) = \sum_{\mathbf{k}\in\mathbb{N}^d} e(M;\mathbf{k})\binom{\mathbf{n}+\mathbf{k}}{\mathbf{k}}.$$

Then $e(M;\mathbf{k}) \in \mathbb{Z}$ for all $\mathbf{k} \in \mathbb{N}^d$. And we would like to select the following objects.

Definition 2.2. We say that $e(M;\mathbf{k})$ is the *mixed multiplicity of* M *of the type* $\mathbf{k}$ if $e(M;\mathbf{h}) = 0$ for all $\mathbf{h} > \mathbf{k}$. These mixed multiplicities are called the *mixed multiplicities of maximal degrees.*

Remark 2.3. Note that in the above definition, the mixed multiplicity of M of the type $\mathbf{k}$ does not depend on $\dim \mathrm{Supp}_{++}M$. And if all the mixed multiplicities of highest degree of M are positive, then the set of the mixed multiplicities of maximal degrees of M and the set of the original mixed multiplicities of M are the same.

Denote by $\triangle^{\mathbf{k}} f(\mathbf{n})$ the $\mathbf{k}$-difference of the function $f(\mathbf{n})$ for each $\mathbf{k} \in \mathbb{N}^d$.

If $e(M;\mathbf{k})$ is the mixed multiplicity of M of the type $\mathbf{k}$, then it can be verified that $\triangle^{\mathbf{k}}[e(M;\mathbf{h})\binom{\mathbf{n}+\mathbf{h}}{\mathbf{h}}] = 0$ for all $\mathbf{h} \neq \mathbf{k}$. Hence

$$\triangle^{\mathbf{k}} P_M(\mathbf{n}) = \triangle^{\mathbf{k}}\big[e(M;\mathbf{k})\binom{\mathbf{n}+\mathbf{k}}{\mathbf{k}}\big] = e(M;\mathbf{k}).$$

Moreover, in this case, since $P_M(\mathbf{n})$ takes positive values for all large $\mathbf{n}$, it follows that $\triangle^{\mathbf{k}} P_M(\mathbf{n}) \geqslant 0$, and so $e(M;\mathbf{k}) \geqslant 0$. Hence we obtain a result as follows.

Proposition 2.4. *Let $e(M;\mathbf{k})$ be the mixed multiplicity of M of the type $\mathbf{k}$. Then*

(i) *$e(M;\mathbf{k})$ is a non-negative integer.*
(ii) *$\triangle^{\mathbf{k}} P_M(\mathbf{n}) = e(M;\mathbf{k})$.*

Note that one can also get (i) as an immediate consequence of Theorem 2.7.

Next, we discuss filter-regular sequences as one of tools in the paper. The notion of filter-regular sequences was introduced by Stuckrad and Vogel in [13](see [1]). The theory of these sequences became an important tool to study some classes of singular rings and has been continually developed (see e.g. [1, 17, 27, 29, 32]).

Definition 2.5. Let $a \in S$ be a homogeneous element. Then a is called an *S_{++}-filter-regular element with respect to M* if $(0_M : a)_{\mathbf{n}} = 0$ for all large $\mathbf{n}$. Let $x_1, \ldots, x_t$ be homogeneous elements in S. We call that $x_1, \ldots, x_t$ is an *S_{++}-filter-regular sequence with respect to M* if x_i is an S_{++}-filter-regular element with respect to $M/(x_1, \ldots, x_{i-1})M$ for all $1 \le i \le t$.

Remark 2.6. We need to emphasize the following notes for filter-regular sequences:

(i) By [27, Proposition 2.2 and Note (ii)], for each $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ there exists an S_{++}-filter-regular sequence $\mathbf{x}$ in $\bigcup_{i=1}^{d} S_i$ with respect to M consisting of k_1 elements of $S_1, \ldots, k_d$ elements of S_d. In this case, $\mathbf{x}$ is called an S_{++}-filter-regular sequence of the type $\mathbf{k}$.
(ii) If $a \in S_i$ is an S_{++}-filter-regular element, then by [27, Remark 2.6] we obtain $\ell_A[(M/aM)_{\mathbf{n}}] = \ell_A[M_{\mathbf{n}}] - \ell_A[M_{\mathbf{n}-\mathbf{e}_i}]$ for large $\mathbf{n}$. Hence $\triangle^{\mathbf{e}_i} P_M(\mathbf{n}) = P_{M/aM}(\mathbf{n})$. From this it follows that for any S_{++}-filter-regular sequence $\mathbf{x}$ of the type $\mathbf{k}$, we get $\triangle^{\mathbf{k}} P_M(\mathbf{n}) = P_{M/\mathbf{x}M}(\mathbf{n})$.

In a recently appeared paper [27], by using S_{++}-filter-regular sequences, Manh and Viet answered to the question when original mixed multiplicities are positive and characterized these mixed multiplicities in terms of lengths of modules (see [27, Theorem 3.4]). This theorem is developed to a broader class as the following.

Theorem 2.7. *Let S be a finitely generated standard $\mathbb{N}^d$-graded algebra over an Artinian local ring A and let M be a finitely generated standard $\mathbb{N}^d$-graded S-module. Let $e(M;\mathbf{k})$ be the mixed multiplicity of maximal degree of the type $\mathbf{k}$ of M. Assume that $\mathbf{x}$ is an S_{++}-filter-regular sequence of the type $\mathbf{k}$ of M. Then we have*

$$e(M;\mathbf{k}) = \ell_A\big[\big(M/\mathbf{x}M\big)_{\mathbf{n}}\big]$$

for all large $\mathbf{n}$. And $e(M;\mathbf{k}) \neq 0$ if and only if $\dim \operatorname{Supp}_{++}\big(\frac{M}{\mathbf{x}M}\big) = 0$.

Proof. First, we have $\triangle^{\mathbf{k}} P_M(\mathbf{n}) = P_{M/\mathbf{x}M}(\mathbf{n})$ by Remark 2.6(ii). Note that

$$P_{M/\mathbf{x}M}(\mathbf{n}) = \ell_A\big[\big(M/\mathbf{x}M\big)_{\mathbf{n}}\big]$$

for all $\mathbf{n} \gg \mathbf{0}$ and $\triangle^{\mathbf{k}} P_M(\mathbf{n}) = e(M;\mathbf{k})$ by Proposition 2.4(ii). So for all $\mathbf{n} \gg \mathbf{0}$, we get $e(M;\mathbf{k}) = \ell_A\big[\big(M/\mathbf{x}M\big)_{\mathbf{n}}\big]$. From this it follows that $e(M,\mathbf{k}) \neq 0$ if and only if $\deg P_{M/\mathbf{x}M}(\mathbf{n}) = 0$. Remember that $\dim \mathrm{Supp}_{++}\big(M/\mathbf{x}M\big) = \deg P_{M/\mathbf{x}M}(\mathbf{n})$ by Remark 2.1. Thus,

$$e(M,\mathbf{k}) \neq 0 \text{ if and only if } \dim \mathrm{Supp}_{++}\big(M/\mathbf{x}M\big) = 0. \qquad \square$$

The following important notions will be used in the next parts of the paper.

Definition 2.8 ([31])**.** Let $\mathbf{y} = y_1, \ldots, y_n$ be a sequence of elements in $\bigcup_{j=1}^{d} S_j$ consisting of m_1 elements of $S_1, \ldots, m_d$ elements of S_d. Then $\mathbf{y}$ is called a *mixed multiplicity system of M of the type* $\mathbf{m} = (m_1, \ldots, m_d)$ if

$$\dim \mathrm{Supp}_{++}\big(M/\mathbf{y}M\big) \leq 0.$$

Let $\mathbf{x} = x_1, \ldots, x_n$ be a mixed multiplicity system of M of the type $\mathbf{k}$. Then

(i) If $n = 0$, then $\ell_A[M_{\mathbf{n}}] = c$ (const) for all $\mathbf{n} \gg \mathbf{0}$ and set $\widetilde{e}(\mathbf{x}, M) = \widetilde{e}(\emptyset, M) = c$. If $n > 0$, set $\widetilde{e}(\mathbf{x}, M) = \widetilde{e}(\mathbf{x}', M/x_1M) - \widetilde{e}(\mathbf{x}', 0_M : x_1)$, here $\mathbf{x}' = x_2, \ldots, x_n$. Then $\widetilde{e}(\mathbf{x}, M)$ is called the *mixed multiplicity symbol of M with respect to* $\mathbf{x}$ *of the type* $\mathbf{k}$.

(ii) From the Koszul complex of M with respect to $\mathbf{x}$

$$0 \longrightarrow K_n(\mathbf{x}, M) \longrightarrow K_{n-1}(\mathbf{x}, M) \longrightarrow \cdots \longrightarrow K_1(\mathbf{x}, M) \longrightarrow K_0(\mathbf{x}, M) \longrightarrow 0,$$

one obtain the sequence of the homology modules

$$\ldots, H_0(\mathbf{x}, M), H_1(\mathbf{x}, M), \ldots, H_n(\mathbf{x}, M), \ldots.$$

Then $\chi(\mathbf{x}, M) = \sum_{i=0}^{n} (-1)^i \ell_A[H_i(\mathbf{x}, M)_{\mathbf{n}}]$(const) for all $\mathbf{n} \gg \mathbf{0}$ is called the *Euler-Poincare characteristic of M with respect to* $\mathbf{x}$ *of the type* $\mathbf{k}$.

The compatibility of mixed multiplicities with other familiar objects is shown by the following proposition which characterizes the existence of mixed multiplicities of maximal degrees in different terms.

Proposition 2.9. *Let S be a finitely generated standard $\mathbb{N}^d$-graded algebra over an Artinian local ring A and let M be a finitely generated standard $\mathbb{N}^d$-graded S-module. Then the following are equivalent*:

(i) *There exists an S_{++}-filter-regular sequence* $\mathbf{x}$ *of the type* $\mathbf{k}$ *such that* $\mathbf{x}$ *is a mixed multiplicity system of M.*

(ii) *There exists a mixed multiplicity system of M of the type* $\mathbf{k}$.

(iii) $\triangle^{\mathbf{k}} P_M(\mathbf{n})$ *is a constant.*

(iv) *The mixed multiplicity of M of the type* $\mathbf{k}$ *is defined.*

Proof. (i)$\Rightarrow$ (ii) is clear. (ii)$\Rightarrow$ (iii): Let $\mathbf{x}$ be a mixed multiplicity system of M of the type $\mathbf{k}$. By [31, Theorem 3.7], we obtain

$$\chi(\mathbf{x}, M) = \widetilde{e}(\mathbf{x}, M) = \triangle^{\mathbf{k}} P_M(\mathbf{n}).$$

Hence $\triangle^{\mathbf{k}}P_M(\mathbf{n})$ is a constant. (iii)⇒ (iv): Since $\triangle^{\mathbf{k}}P_M(\mathbf{n})$ is a constant, it follows that $\triangle^{\mathbf{h}}P_M(\mathbf{n}) = 0$ for all $\mathbf{h} > \mathbf{k}$. Therefore, $e(M; \mathbf{h}) = 0$ for all $\mathbf{h} > \mathbf{k}$. So the mixed multiplicity of M of the type $\mathbf{k}$ is defined. (iv)⇒ (i): By Remark 2.6(i), there exists an S_{++}-filter-regular sequence $\mathbf{x}$ of the type $\mathbf{k}$. By Theorem 2.7, we get

$$e(M; \mathbf{k}) = \ell_A\big[\big(M/\mathbf{x}M\big)_{\mathbf{n}}\big]$$

for all large $\mathbf{n}$. Hence $P_{M/\mathbf{x}M}(\mathbf{n})$ is a constant. So $\dim \operatorname{Supp}_{++}\big(M/\mathbf{x}M\big) \leq 0$ by Remark 2.1. Therefore, $\mathbf{x}$ is a mixed multiplicity system of M. □

From Proposition 2.9 and the proof of Proposition 2.9, we obtain some following effective comments for the existence of mixed multiplicities.

Remark 2.10. Assume that the mixed multiplicity of M of the type $\mathbf{k}$ is defined. Then there exists a mixed multiplicity system $\mathbf{x} = x_1, \ldots, x_s$ of M of the type $\mathbf{k}$ by Proposition 2.9. Now let $x_1, \ldots, x_i$ be a subsequence of $\mathbf{x}$ of the type $\mathbf{h} = (h_1, \ldots, h_d)$ for each $1 \leq i \leq s$. It is easily seen that $x_{i+1}, \ldots, x_s$ is a mixed multiplicity system of $M/(x_1, \ldots, x_i)M$ of the type $\mathbf{k} - \mathbf{h}$. Hence the mixed multiplicity of $M/(x_1, \ldots, x_i)M$ of the type $\mathbf{k} - \mathbf{h}$ is also defined by Proposition 2.9. Moreover, from the proof of Proposition 2.9, it follows that any S_{++}-filter-regular sequence $\mathbf{x}$ of the type $\mathbf{k}$ of M is a mixed multiplicity system of the type $\mathbf{k}$ of M. Hence if $a \in S_i$ is an S_{++}-filter-regular element and $k_i > 0$, then $e(M/aM; \mathbf{k} - \mathbf{e}_i)$ is defined.

Let $a \in S_i$ be an S_{++}-filter-regular element. Now if $e(M; \mathbf{k})$ is defined and $k_i > 0$, then $e(M/aM; \mathbf{k} - \mathbf{e}_i)$ is defined by Remark 2.10. Since $\triangle^{\mathbf{e}_i}P_M(\mathbf{n}) = P_{M/aM}(\mathbf{n})$ by Remark 2.6(ii), it implies that

$$\triangle^{\mathbf{k}-\mathbf{e}_i}P_{M/aM}(\mathbf{n}) = \triangle^{\mathbf{k}-\mathbf{e}_i}[\triangle^{\mathbf{e}_i}P_M(\mathbf{n})] = \triangle^{\mathbf{k}}P_M(\mathbf{n}).$$

So $e(M; \mathbf{k}) = e(M/aM; \mathbf{k} - \mathbf{e}_i)$ by Proposition 2.4. From this it follows that for any S_{++}-filter-regular sequence $\mathbf{y}$ of the type $\mathbf{h}$ with $\mathbf{h} \leq \mathbf{k}$, $e(M/\mathbf{y}M; \mathbf{k} - \mathbf{h})$ is defined and $e(M; \mathbf{k}) = e(M/\mathbf{y}M; \mathbf{k} - \mathbf{h})$. In particular, if $\mathbf{x}$ is an S_{++}-filter-regular sequence of the type $\mathbf{k}$, then $e(M/\mathbf{x}M; \mathbf{0})$ is defined and $e(M; \mathbf{k}) = e(M/\mathbf{x}M; \mathbf{0})$.

The above facts yield:

Corollary 2.11. *Let $e(M; \mathbf{k})$ be the mixed multiplicity of the type $\mathbf{k}$ of M. Assume that $\mathbf{y}$ is an S_{++}-filter-regular sequence of the type $\mathbf{h}$ of M with $\mathbf{h} \leq \mathbf{k}$, and $\mathbf{x}$ is an S_{++}-filter-regular sequence of the type $\mathbf{k}$ of M. Then $e(M/\mathbf{y}M; \mathbf{k} - \mathbf{h})$ and $e(M/\mathbf{x}M; \mathbf{0})$ are defined. Moreover, we have*

$$e(M; \mathbf{k}) = e(M/\mathbf{y}M; \mathbf{k} - \mathbf{h}) = e(M/\mathbf{x}M; \mathbf{0}).$$

The following example will build a finitely generated standard $\mathbb{N}^3$-graded algebra containing full mixed multiplicity kinds.

Example 2.12. Let k be an infinite field and let $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3$ be indeterminates. Let

$$B = k[x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3]$$

be a finitely generated standard $\mathbb{N}^3$-graded algebra over k with

$$\deg x_i = (1, 0, 0),\ \deg y_i = (0, 1, 0),\ \deg z_i = (0, 0, 1), \quad i = 1, 2, 3.$$

Set

$$I = (x_1, y_1, z_1) \cap (x_1, x_2) \cap (y_1, y_2) \cap (z_1, z_2)$$

and $S = B/I$. Then S is a finitely generated standard $\mathbb{N}^3$-graded algebra over k and $\dim S = 7$. Denote by $P_S(n_1, n_2, n_3)$ the Hilbert polynomial of S. For $x \in B$, denote by $\bar{x}$ the image of x in S. Then [27, Example 3.7] showed that $\deg P_S(n_1, n_2, n_3) = 4$ and

$$\begin{aligned} e(S; 2, 2, 0) &= e(S; 2, 0, 2) = e(S; 0, 2, 2) = 1; \\ e(S; 3, 1, 0) &= e(S; 1, 3, 0) = e(S; 3, 0, 1) = e(S; 1, 0, 3) \\ &= e(S; 0, 3, 1) = e(S; 0, 1, 3) = e(S; 4, 0, 0) \\ &= e(S; 0, 4, 0) = e(S; 0, 0, 4) = e(S; 2, 1, 1) \\ &= e(S; 1, 2, 1) = e(S; 1, 1, 2) = 0. \end{aligned}$$

Hence the mixed multiplicities $e(S; 3, 0, 0)$; $e(S; 0, 3, 0)$; $e(S; 0, 0, 3)$; $e(S; 1, 1, 1)$ are defined. By the symmetry, we get

$$e(S; 3, 0, 0) = e(S; 0, 3, 0) = e(S; 0, 0, 3).$$

By [27, Example 3.7], $\bar{x}_3, \bar{x}_2, \bar{x}_1$ is an S_{++}-filter-regular sequence consisting of 3 elements in S_1 and $\frac{S}{(\bar{x}_3, \bar{x}_2, \bar{x}_1) : S_{++}^{\infty}} = 0$. So $\dim \operatorname{Supp}_{++} \frac{S}{(\bar{x}_3, \bar{x}_2, \bar{x}_1)} < 0$. From this it follows that

$$\ell_k\Big[\Big(\frac{S}{(\bar{x}_3, \bar{x}_2, \bar{x}_1)}\Big)_{(n_1, n_2, n_3)}\Big] = 0$$

for all $n_1, n_2, n_3 \gg 0$. Hence by Theorem 2.7, we obtain $e(S; 3, 0, 0) = 0$. Thus $e(S; 3, 0, 0) = e(S; 0, 3, 0) = e(S; 0, 0, 3) = 0$.

Upon simple computation, we show that $\bar{x}_3, \bar{y}_3, \bar{z}_3$ is an S_{++}-filter-regular sequence of S consisting of 1 element of S_1, 1 element of S_2 and 1 element of S_3, and

$$\begin{aligned} S/(\bar{x}_3, \bar{y}_3, \bar{z}_3) : S_{++}^{\infty} \cong B/(I, x_3, y_3, z_3) : B_{++}^{\infty} &= B/(x_1, y_1, z_1, x_3, y_3, z_3) \\ &\cong k[x_2, y_2, z_2]. \end{aligned}$$

By [27, Remark 2.4], we have

$$\Big[\frac{(\bar{x}_3, \bar{y}_3, \bar{z}_3) : S_{++}^{\infty}}{(\bar{x}_3, \bar{y}_3, \bar{z}_3)}\Big]_{(n_1, n_2, n_3)} = 0$$

for all $n_1, n_2, n_3 \gg 0$. Therefore, the mixed multiplicities of $S/[(\bar{x}_3, \bar{y}_3, \bar{z}_3) : S_{++}^{\infty}]$ and the mixed multiplicities of $S/(\bar{x}_3, \bar{y}_3, \bar{z}_3)$ are the same. Moreover, note that by Corollary 2.11, we get $e(S;1,1,1) = e(S/(\bar{x}_3, \bar{y}_3, \bar{z}_3); 0,0,0)$. So we obtain

$$e(S;1,1,1) = e(S/[(\bar{x}_3, \bar{y}_3, \bar{z}_3) : S_{++}^{\infty}]; 0,0,0) = e(k[x_2, y_2, z_2]; 0,0,0).$$

Now, since $\deg x_2 = (1,0,0), \deg y_2 = (0,1,0), \deg z_2 = (0,0,1)$ in the standard $\mathbb{N}^3$-graded algebra $k[x_2, y_2, z_2]$ over k, it follows that $e(k[x_2, y_2, z_2]; 0,0,0) = 1$. Thus, $e(S;1,1,1) = 1$. Consequently, it can be verified that all mixed multiplicities of maximal degrees of this finitely generated standard $\mathbb{N}^3$-graded algebra S are indicated.

The following useful result proves that mixed multiplicity of the type $\mathbf{k}$; the Euler-Poincare characteristic and the mixed multiplicity symbol of any mixed multiplicity system of the type $\mathbf{k}$ are the same.

Theorem 2.13. *The mixed multiplicity of maximal degree of M of the type $\mathbf{k}$ is defined if and only if there exists a mixed multiplicity system $\mathbf{x}$ of M of the type $\mathbf{k}$. In this case, we have*

$$\chi(\mathbf{x}, M) = \widetilde{e}(\mathbf{x}, M) = e(M; \mathbf{k}).$$

Proof. The mixed multiplicity of maximal degree of M of the type $\mathbf{k}$ is defined if and only if there exists a mixed multiplicity system $\mathbf{x}$ of M of the type $\mathbf{k}$ by Proposition 2.9. In this case, $\triangle^{\mathbf{k}} P_M(\mathbf{n}) = e(M; \mathbf{k})$ by Proposition 2.4. Moreover,

$$\chi(\mathbf{x}, M) = \widetilde{e}(\mathbf{x}, M) = \triangle^{\mathbf{k}} P_M(\mathbf{n})$$

by [31, Theorem 3.7]. Hence we obtain $\chi(\mathbf{x}, M) = \widetilde{e}(\mathbf{x}, M) = e(M; \mathbf{k})$. □

We would like to comment here that Theorem 2.13 not only covers, but also is more natural than [31, Theorem 3.9 and Theorem 3.10].

As an immediate consequence of Theorem 2.13, we also obtain a more natural result than [31, Corollary 3.11(ii)] in the original mixed multiplicity theory.

Corollary 2.14. *Let $\mathbf{x} = x_1, \ldots, x_s$ be a mixed multiplicity system of M of the type $\mathbf{k}$. Denote by $\mathbf{h}_i = (h_{i1}, \ldots, h_{id})$ the type of a subsequence $x_1, \ldots, x_i$ of $\mathbf{x}$ for each $1 \le i \le s$. Then for all large $\mathbf{n}$, we have*

$$e(M; \mathbf{k}) = \ell_A\big[\big(M/\mathbf{x}M\big)_{\mathbf{n}}\big] - \sum_{i=1}^{s} e\bigg(\frac{(x_1, \ldots, x_{i-1})M : x_i}{(x_1, \ldots, x_{i-1})M}; \mathbf{k} - \mathbf{h}_i\bigg).$$

Proof. By Definition 2.8(i), it follows that

$$\widetilde{e}(\mathbf{x}, M) = \widetilde{e}\bigg(\emptyset, \frac{M}{\mathbf{x}M}\bigg) - \sum_{i=1}^{s} \widetilde{e}\bigg(x_{i+1}, \ldots, x_s, \frac{(x_1, \ldots, x_{i-1})M : x_i}{(x_1, \ldots, x_{i-1})M}\bigg)$$

and $\widetilde{e}\big(\emptyset, \frac{M}{\mathbf{x}M}\big) = \ell_A\big[\big(\frac{M}{\mathbf{x}M}\big)_{\mathbf{n}}\big]$ for all large $\mathbf{n}$. Note that $x_{i+1}, \ldots, x_s$ is a mixed multiplicity system of $\frac{(x_1, \ldots, x_{i-1})M : x_i}{(x_1, \ldots, x_{i-1})M}$ of the type $\mathbf{k} - \mathbf{h}_i$. Hence by Theorem 2.13, we get $e(M; \mathbf{k}) = \ell_A\big[\big(M/\mathbf{x}M\big)_{\mathbf{n}}\big] - \sum_{i=1}^{s} e\bigg(\frac{(x_1, \ldots, x_{i-1})M : x_i}{(x_1, \ldots, x_{i-1})M}; \mathbf{k} - \mathbf{h}_i\bigg)$ for all large $\mathbf{n}$. □

By Theorem 2.7 and Corollary 2.14, we immediately obtain the following.

Corollary 2.15. *Let* $\mathbf{x}$ *be a mixed multiplicity system of* M *of the type* $\mathbf{k}$. *Then we have* $e(M;\mathbf{k}) \leq \ell_A\big[\big(M/\mathbf{x}M\big)_{\mathbf{n}}\big]$ *for all large* $\mathbf{n}$, *and equality holds if* $\mathbf{x}$ *is an* S_{++}*-filter-regular sequence.*

Combining Remark 2.6 and Theorem 2.7 with Corollary 2.14, we get:

Corollary 2.16. *Let* $e(M;\mathbf{k})$ *be the mixed multiplicity of* M *of the type* $\mathbf{k}$. *Then the following are equivalent*:

(i) $e(M;\mathbf{k}) > 0$.
(ii) $\dim \operatorname{Supp}_{++}\big(M/\mathbf{x}M\big) = 0$ *for any mixed multiplicity system* $\mathbf{x}$ *of the type* $\mathbf{k}$.
(iii) $\dim \operatorname{Supp}_{++}\big(M/\mathbf{x}M\big) = 0$ *for any* S_{++}*-filter-regular sequence* $\mathbf{x}$ *of the type* $\mathbf{k}$.
(iv) *There exists an* S_{++}*-filter-regular sequence* $\mathbf{x}$ *of* M *of the type* $\mathbf{k}$ *such that*
$$\dim \operatorname{Supp}_{++}\big(M/\mathbf{x}M\big) = 0.$$

Proof. (i)⇒ (ii): Let $\mathbf{x}$ be a mixed multiplicity system of M of the type $\mathbf{k}$. Then by Corollary 2.15, we have $\ell_A\big[\big(M/\mathbf{x}M\big)_{\mathbf{n}}\big] > 0$ for all large $\mathbf{n}$ since $e(M;\mathbf{k}) > 0$. So $\deg P_{M/\mathbf{x}M}(\mathbf{n}) = 0$. By Remark 2.1, we get
$$\dim \operatorname{Supp}_{++}\big(M/\mathbf{x}M\big) = \deg P_{M/\mathbf{x}M}(\mathbf{n}).$$
Hence $\dim \operatorname{Supp}_{++}\big(M/\mathbf{x}M\big) = 0$.

(ii)⇒ (iii): By Remark 2.10, any S_{++}-filter-regular sequence $\mathbf{x}$ of the type $\mathbf{k}$ of M is a mixed multiplicity system of the type $\mathbf{k}$ of M. Hence by (ii), we obtain $\dim \operatorname{Supp}_{++}\big(M/\mathbf{x}M\big) = 0$ for any S_{++}-filter-regular sequence $\mathbf{x}$ of the type $\mathbf{k}$.

(iii)⇒ (iv): By Remark 2.6(i), there exists an S_{++}-filter-regular sequence $\mathbf{x}$ of the type $\mathbf{k}$ of M. And by (iii), $\dim \operatorname{Supp}_{++}\big(M/\mathbf{x}M\big) = 0$. Hence there exists an S_{++}-filter-regular sequence $\mathbf{x}$ of the type $\mathbf{k}$ of M such that
$$\dim \operatorname{Supp}_{++}\big(M/\mathbf{x}M\big) = 0.$$

(iv)⇒ (i): Assume that there exists an S_{++}-filter-regular sequence $\mathbf{x}$ of M of the type $\mathbf{k}$ such that
$$\dim \operatorname{Supp}_{++}\big(M/\mathbf{x}M\big) = 0.$$
Then by Theorem 2.7, we get $e(M;\mathbf{k}) > 0$. □

We obtain the following result which shows that mixed multiplicities of maximal degrees are additive on short exact sequences.

Corollary 2.17. *Let* $0 \longrightarrow M' \longrightarrow M \longrightarrow M" \longrightarrow 0$ *be a short exact sequence of* $\mathbb{N}^d$*-graded* S*-modules. Then the following statements hold.*

(i) $e(M;\mathbf{k})$ *is defined if and only if both* $e(M';\mathbf{k})$ *and* $e(M";\mathbf{k})$ *are defined.*

(ii) *Assume that $e(M;\mathbf{k})$ is defined. Then $e(M;\mathbf{k}) = e(M';\mathbf{k}) + e(M";\mathbf{k})$, i.e., the mixed multiplicities are additive on short exact sequences.*

Proof. Let $\mathbf{x}$ be a sequence of elements in $\bigcup_{j=1}^{d} S_j$ of the type $\mathbf{k}$. Then $\mathbf{x}$ is a mixed multiplicity system of M if and only if $\mathbf{x}$ is a mixed multiplicity system of both M' and $M"$ by [31, Lemma 2.7]. Hence $e(M;\mathbf{k})$ is defined if and only if both $e(M';\mathbf{k})$ and $e(M";\mathbf{k})$ are defined by Proposition 2.9. We get (i). The proof of (ii): Since $e(M;\mathbf{k})$ is defined, it follows that there exists a mixed multiplicity system $\mathbf{y}$ of M of the type $\mathbf{k}$ by Proposition 2.9. Then we have $\chi(\mathbf{y}, M) = \chi(\mathbf{y}, M') + \chi(\mathbf{y}, M")$ by [31, Lemma 3.2(i)]. So $e(M;\mathbf{k}) = e(M';\mathbf{k}) + e(M";\mathbf{k})$ by Theorem 2.13. □

3. Mixed multiplicities of ideals

In this section, we will give some applications of Section 2 to mixed multiplicities of modules over local rings with respect to ideals.

Let $(R, \mathfrak{n})$ be a Noetherian local ring with maximal ideal $\mathfrak{n}$ and infinite residue field $R/\mathfrak{n}$. Let N be a finitely generated R-module. Let $J, I_1, \ldots, I_d$ be ideals of R with J being $\mathfrak{n}$-primary. For any $\mathbf{k} = (k_1, \ldots, k_d); \mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$ and $\mathbf{I} = I_1, \ldots, I_d$, set $\mathbf{I}^{[\mathbf{k}]} = I_1^{[k_1]}, \ldots, I_d^{[k_d]}$ and $\mathbb{I}^{\mathbf{n}} = I_1^{n_1} \cdots I_d^{n_d}$. We get an $\mathbb{N}^{(d+1)}$-graded algebra and an $\mathbb{N}^{(d+1)}$-graded module:

$$T = \bigoplus_{n \geq 0,\ \mathbf{n} \geq \mathbf{0}} \frac{J^n \mathbb{I}^{\mathbf{n}}}{J^{n+1} \mathbb{I}^{\mathbf{n}}} \ \text{ and } \ \mathcal{N} = \bigoplus_{n \geq 0,\ \mathbf{n} \geq \mathbf{0}} \frac{J^n \mathbb{I}^{\mathbf{n}} N}{J^{n+1} \mathbb{I}^{\mathbf{n}} N}.$$

Then T is a finitely generated standard $\mathbb{N}^{(d+1)}$-graded algebra over an Artinian local ring R/J and $\mathcal{N}$ is a finitely generated standard $\mathbb{N}^{(d+1)}$-graded T-module. The mixed multiplicity of $\mathcal{N}$ of the type $(k_0, \mathbf{k})$ is denoted by $e\big(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N\big)$, i.e., $e\big(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N\big) := e(\mathcal{N}; k_0, \mathbf{k})$ and which is called the *mixed multiplicity of N with respect to ideals $J, \mathbf{I}$ of the type* $(k_0 + 1, \mathbf{k})$. The mixed multiplicity of N with respect to ideals $J, \mathbf{I}$ of the type $(k_0+1, \mathbf{k})$ with $k_0 + \mid \mathbf{k} \mid = \dim \frac{N}{0_N : I^\infty} - 1$ (see e.g. [5,10,19]) is called the *original mixed multiplicity of N with respect to ideals $J, \mathbf{I}$ of the type* $(k_0 + 1, \mathbf{k})$. Set $I = JI_1 \cdots I_d$; $I_0 = J$ and $T_i = I_i/JI_i$ for all $0 \leq i \leq d$.

Remark 3.1. Assign $\dim \frac{N}{0_N : I^\infty} = -\infty$ to the case that $\frac{N}{0_N : I^\infty} = 0$. Then we always have $\dim \operatorname{Supp}_{++}\mathcal{N} = \dim \frac{N}{0_N : I^\infty} - 1$ by [31, Remark 4.1].

Definition 3.2 ([31, Definition 4.2])**.** An element $a \in R$ is called a *Rees superficial element* of N with respect to $\mathbf{I}$ if there exists $i \in \{1, \ldots, d\}$ such that $a \in I_i$ and

$$aN \bigcap \mathbb{I}^{\mathbf{n}} I_i N = a \mathbb{I}^{\mathbf{n}} N$$

for all $\mathbf{n} \gg \mathbf{0}$. A sequence $x_1, \ldots, x_t$ in R is called a *Rees superficial sequence* of N with respect to $\mathbf{I}$ if x_{j+1} is a Rees superficial element of $N/(x_1, \ldots, x_j)N$ with respect to $\mathbf{I}$ for all $j = 0, 1, \ldots, t-1$. A Rees superficial sequence of N

consisting of k_1 elements of $I_1, \ldots, k_d$ elements of I_d is called a Rees superficial sequence of N of the type $\mathbf{k} = (k_1, \ldots, k_d)$.

Definition 3.3 ([31, Definition 4.4])**.** Let $\mathbf{x}$ be a Rees superficial sequence of N with respect to ideals $J, \mathbf{I}$ of the type $(k_0, \mathbf{k}) \in \mathbb{N}^{d+1}$. Then $\mathbf{x}$ is called a *mixed multiplicity system of N with respect to ideals $J, \mathbf{I}$ of the type* $(k_0, \mathbf{k})$ if $\dim \frac{N}{\mathbf{x}N:I^\infty} \leq 1$.

Remark 3.4. Recall that $e(\mathcal{N}; k_0, \mathbf{k}) = e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N)$. Then we have:

(i) Let $a \in I_i$ be a Rees superficial element of N with respect to $J, \mathbf{I}$, a^* the image of a in T_i, $e(0_{\mathcal{N}} : a^*; h_0, \mathbf{h})$; $e(\mathcal{N}/a^*\mathcal{N}; h_0, \mathbf{h})$ be defined. By [31, (4.1)], $\big(\mathcal{N}/a^*\mathcal{N}\big)_{(m,\ \mathbf{m})} \cong \left[\bigoplus_{n\geq 0,\ \mathbf{n}\geq \mathbf{0}} \frac{J^n\mathbb{I}^{\mathbf{n}}(N/aN)}{J^{n+1}\mathbb{I}^{\mathbf{n}}(N/aN)}\right]_{(m,\ \mathbf{m})}$ for $m \gg 0$; $\mathbf{m} \gg \mathbf{0}$. Hence $e(\mathcal{N}/a^*\mathcal{N}; h_0, \mathbf{h}) = e(J^{[h_0+1]}, \mathbf{I}^{[\mathbf{h}]}; N/aN)$.

Recall that there exists $u \gg 0$ such that $(0_{\mathcal{N}} : a^*)_{(n+u,\ \mathbf{n}+u\mathbf{1})} = \frac{WJ^n\mathbb{I}^{\mathbf{n}}}{WJ^{n+1}\mathbb{I}^{\mathbf{n}}}$ for all $n \geq 0$; $\mathbf{n} \geq \mathbf{0}$ by [31, (4.3)], here $W = (0_N : a) \bigcap I^u N$. Therefore, we get $e(0_{\mathcal{N}} : a^*; h_0, \mathbf{h}) = e(J^{[h_0+1]}, \mathbf{I}^{[\mathbf{h}]}; W)$. Let Y be a submodule of N. We put $\mathfrak{R} = \bigoplus_{n\geq 0,\ \mathbf{n}\geq \mathbf{0}} J^n\mathbb{I}^{\mathbf{n}}$; $\mathcal{Y} = \bigoplus_{n\geq 0,\ \mathbf{n}\geq \mathbf{0}} J^n\mathbb{I}^{\mathbf{n}}Y$ and $\mathcal{X} = \mathcal{Y} : \mathfrak{R}_{++}^\infty$. Then by [27, Lemma 2.3], there exists $v \gg 0$ such that $\mathcal{X}_{(n+v,\mathbf{n}+v\mathbf{1})} = \mathfrak{R}_{(n,\ \mathbf{n})}\mathcal{X}_{(v,\ v\mathbf{1})}$ for all $n \geq 0$; $\mathbf{n} \geq \mathbf{0}$. Note that for all $n \gg 0$; $\mathbf{n} \gg \mathbf{0}$, $\mathfrak{R}_{(n,\ \mathbf{n})}\mathcal{X}_{(v,\ v\mathbf{1})} \subset \mathcal{Y}_{(n+v,\mathbf{n}+v\mathbf{1})}$. Consequently $\mathcal{X}_{(n+v,\mathbf{n}+v\mathbf{1})} = \mathcal{Y}_{(n+v,\mathbf{n}+v\mathbf{1})}$ for all $n \gg 0$; $\mathbf{n} \gg \mathbf{0}$. From this it follows that $(Y : I^\infty)J^n\mathbb{I}^{\mathbf{n}} = YJ^n\mathbb{I}^{\mathbf{n}}$ for all $n \gg 0$; $\mathbf{n} \gg \mathbf{0}$. Therefore, we have $e(J^{[h_0+1]}, \mathbf{I}^{[\mathbf{h}]}; Y) = e(J^{[h_0+1]}, \mathbf{I}^{[\mathbf{h}]}; Y : I^\infty)$. Next set $U = 0_N : a$. Then since $U : I^\infty = W : I^\infty$, we obtain $e(J^{[h_0+1]}, \mathbf{I}^{[\mathbf{h}]}; U) = e(J^{[h_0+1]}, \mathbf{I}^{[\mathbf{h}]}; W)$. Thus $e(0_{\mathcal{N}} : a^*; h_0, \mathbf{h}) = e(J^{[h_0+1]}, \mathbf{I}^{[\mathbf{h}]}; 0_N : a)$.

(ii) We always have $\dim \frac{N}{0_N:I^\infty} \neq 0$ since $(0_N : I^\infty) : I = 0_N : I^\infty$.

(iii) If $\dim \frac{N}{0_N:I^\infty} = 1$, then $e(J^{[1]}, \mathbf{I}^{[\mathbf{0}]}; N) = e(J; \frac{N}{0_N:I^\infty})$ by [20, Proposition 3.2]. Moreover, if $\dim \frac{N}{0_N:I^\infty} < 0$, then $\frac{N}{0_N:I^\infty} = 0$. In this case, $e(J; \frac{N}{0_N:I^\infty}) = 0$, and $e(J^{[1]}, \mathbf{I}^{[\mathbf{0}]}; N) = 0$ since $\dim \operatorname{Supp}_{++}\mathcal{N} < 0$ by Remark 3.1. Thus, if $\dim \frac{N}{0_N:I^\infty} \leq 1$, then $e(J^{[1]}, \mathbf{I}^{[\mathbf{0}]}; N) = e(J; \frac{N}{0_N:I^\infty})$ by (ii).

(iv) Let $\mathbf{x}$ be a Rees superficial sequence of N with respect to $J, \mathbf{I}$ of the type $(k_0, \mathbf{k})$ and let $\mathbf{x}^*$ be the image of $\mathbf{x}$ in $\bigcup_{i=0}^d T_i$. Then $\mathbf{x}$ is a mixed multiplicity system of N with respect to $J, \mathbf{I}$ of the type $(k_0, \mathbf{k})$ if and only if $\mathbf{x}^*$ is a mixed multiplicity system of $\mathcal{N}$ of the type $(k_0, \mathbf{k})$ by [31, Remark 4.5].

By Corollary 2.14 and Remark 3.4, we obtain the following result.

Theorem 3.5. *Let $\mathbf{x} = x_1, \ldots, x_s$ be a mixed multiplicity system of N with respect to ideals $J, \mathbf{I}$ of the type $(k_0, \mathbf{k})$. Denote by $(m_i, \mathbf{h}_i) = (m_i, h_{i1}, \ldots, h_{id})$ the type of a subsequence $x_1, \ldots, x_i$ of $\mathbf{x}$ for each $1 \leq i \leq s$. For $1 \leq i \leq s$, set*

$N_i = \frac{(x_1,\ldots,x_{i-1})N:x_i}{(x_1,\ldots,x_{i-1})N}$. *Then we have*

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e\big(J; \frac{N}{\mathbf{x}N : I^\infty}\big) - \sum_{i=1}^{s} e\big(J^{[k_0-m_i+1]}, \mathbf{I}^{[\mathbf{k}-\mathbf{h}_i]}; N_i\big).$$

Proof. Since $\mathbf{x}$ is a mixed multiplicity system of N with respect to ideals $J, \mathbf{I}$ of the type $(k_0, \mathbf{k})$, it follows that $\dim \frac{N}{\mathbf{x}N:I^\infty} \le 1$, and $\mathbf{x}^*$ is a mixed multiplicity system of the type $(k_0, \mathbf{k})$ of $\mathcal{N}$ by Remark 3.4(iv). Recall that $e(\mathcal{N}; k_0, \mathbf{k}) = e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N)$. It can be verified (see [31, (4.1)]) that

$$(1)\quad \big[\mathcal{N}/(x_1^*, \ldots, x_i^*)\mathcal{N}\big]_{(m,\,\mathbf{m})} \cong \Big[\bigoplus_{n\ge 0,\,\mathbf{n}\ge\mathbf{0}} \frac{J^n\mathbb{I}^{\mathbf{n}}(N/(x_1,\ldots,x_i)N)}{J^{n+1}\mathbb{I}^{\mathbf{n}}(N/(x_1,\ldots,x_i)N)}\Big]_{(m,\,\mathbf{m})}$$

for all $m \gg 0$; $\mathbf{m} \gg \mathbf{0}$ and $1 \le i \le s$. Hence it is easily seen by Remark 3.4(i) that

$$e(\mathcal{N}/\mathbf{x}^*\mathcal{N}; 0, \mathbf{0}) = e(J^{[1]}, \mathbf{I}^{[\mathbf{0}]}; N/\mathbf{x}N)$$

and

$$e\Big(\frac{(x_1^*, \ldots, x_{i-1}^*)\mathcal{N} : x_i^*}{(x_1^*, \ldots, x_{i-1}^*)\mathcal{N}}; k_0 - m_i, \mathbf{k} - \mathbf{h}_i\Big)$$
$$= e\Big(J^{[k_0-m_i+1]}, \mathbf{I}^{[\mathbf{k}-\mathbf{h}_i]}; \frac{(x_1, \ldots, x_{i-1})N : x_i}{(x_1, \ldots, x_{i-1})N}\Big)$$

for each $1 \le i \le s$. Moreover, $e(J^{[1]}, \mathbf{I}^{[\mathbf{0}]}; N/\mathbf{x}N) = e(J; \frac{N}{\mathbf{x}N:I^\infty})$ by Remark 3.4(iii). Consequently, by Corollary 2.14 we get

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e\big(J; \frac{N}{\mathbf{x}N : I^\infty}\big) - \sum_{i=1}^{s} e\big(J^{[k_0-m_i+1]}, \mathbf{I}^{[\mathbf{k}-\mathbf{h}_i]}; N_i\big),$$

which finishes the proof. □

One expressed original mixed multiplicities of ideals in terms of the Hilbert-Samuel multiplicity by using different sequences. First, in the case of $\mathfrak{n}$-primary ideals, Risler-Teissier [15] in 1973 showed that each original mixed multiplicity is the multiplicity of an ideal generated by a superficial sequence and Rees [11] in 1984 proved that original mixed multiplicities are multiplicities of ideals generated by joint reductions. For the case of arbitrary ideals, Viet [20] in 2000 characterized original mixed multiplicities as the Hilbert-Samuel multiplicity via (FC)-sequences.

Definition 3.6 ([20])**.** Let $\mathbf{I} = I_1, \ldots, I_d$ be ideals of R. Set $\mathfrak{I} = I_1 \cdots I_d$. An element $a \in R$ is called a *weak-(FC)-element of N with respect to* $\mathbf{I}$ if there exists $i \in \{1, \ldots, d\}$ such that $a \in I_i$ and the following conditions are satisfied:

(i) a is an $\mathfrak{I}$-filter-regular element with respect to N, i.e., $0_N : a \subseteq 0_N : \mathfrak{I}^\infty$.

(ii) a is a Rees superficial element of N with respect to $\mathbf{I}$.

A sequence $x_1, \ldots, x_t$ in R is called a *weak-(FC)-sequence of N with respect to* $\mathbf{I}$ if x_{i+1} is a weak-(FC)-element of $N/(x_1, \ldots, x_i)N$ with respect to $\mathbf{I}$ for all $0 \le i \le t-1$. A weak-(FC)-sequence of N consisting of k_1 elements of $I_1, \ldots, k_d$ elements of I_d is called a weak-(FC)-sequence of N of the *type* $\mathbf{k} = (k_1, \ldots, k_d)$.

Remember that [20] defined weak-(FC)-sequences in the condition

$$\mathfrak{I} \nsubseteq \sqrt{\mathrm{Ann}_R N}$$

(see e.g. [3,10,21–23,25,26,28,29]). In Definition 3.6, we omitted this condition. Moreover, the authors of [4] proved that the superficial sequences in [15,17,18, 27] are weak-(FC)-sequences (see [4, Remark 3.8]).

Remark 3.7. Note that for any $\mathbf{k} \in \mathbb{N}^d$, there exists a weak-(FC)-sequence of N with respect to $\mathbf{I}$ of the type $\mathbf{k}$ by [10, Proposition 2.3]. And if $\mathbf{x}$ is a weak-(FC)-sequence of N with respect to $J, \mathbf{I}$, then $\mathbf{x}^*$ is a T_{++}-filter-regular sequence with respect to $\mathcal{N}$ by [29, Proposition 4.5]. Hence by Remark 2.10, it follows that if $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N)$ is defined, then any weak-(FC)-sequence $\mathbf{x}$ of N with respect to $J, \mathbf{I}$ of the type $(k_0, \mathbf{k})$ is a mixed multiplicity system of N, i.e., $\dim \frac{N}{\mathbf{x}N : I^\infty} \le 1$.

From Proposition 2.9 and Remark 3.4, and Remark 3.7, we have the following.

Corollary 3.8. *The following are equivalent:*

(i) *There exists a weak-(FC)-sequence $\mathbf{x}$ of N with respect to $J, \mathbf{I}$ of the type $(k_0, \mathbf{k})$ such that $\mathbf{x}$ is a mixed multiplicity system of N.*
(ii) *There exists a mixed multiplicity system of N of the type $(k_0, \mathbf{k})$.*
(iii) *The mixed multiplicity of N of the type $(k_0, \mathbf{k})$ is defined.*

Proof. (i)⇒(ii) is clear. (ii)⇒(iii): Since $\mathbf{x}$ is a mixed multiplicity system of N of the type $(k_0, \mathbf{k})$, $\mathbf{x}^*$ is a mixed multiplicity system of $\mathcal{N}$ of the type $(k_0, \mathbf{k})$ by Remark 3.4(iv). Hence $e(\mathcal{N}; k_0, \mathbf{k})$ is defined by Proposition 2.9. So $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N)$ is defined since $e(\mathcal{N}; k_0, \mathbf{k}) = e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N)$. (iii)⇒(i) is evident by Remark 3.7. □

Next, as an immediate application of Corollary 2.11 and Remark 3.4, and Remark 3.7, we obtain:

Corollary 3.9. *Let $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N)$ be the mixed multiplicity of the type $(k_0, \mathbf{k})$. Let $\mathbf{x}$ be a weak-(FC)-sequence of N with respect to $J, \mathbf{I}$ of the type $(h_0, \mathbf{h})$ with $(h_0, \mathbf{h}) \le (k_0, \mathbf{k})$. Then $e(J^{[k_0-h_0+1]}, \mathbf{I}^{[\mathbf{k}-\mathbf{h}]}; N/\mathbf{x}N)$ is defined and*

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e(J^{[k_0-h_0+1]}, \mathbf{I}^{[\mathbf{k}-\mathbf{h}]}; N/\mathbf{x}N).$$

Proof. Since $\mathbf{x}$ is a weak-(FC)-sequence of N with respect to J, $\mathbf{I}$ of the type $(h_0, \mathbf{h})$, it follows that $\mathbf{x}^*$ is a T_{++}-filter-regular sequence with respect to $\mathcal{N}$ of the type $(h_0, \mathbf{h})$ by Remark 3.7. Hence $e(\mathcal{N}/\mathbf{x}^*\mathcal{N}; k_0 - h_0, \mathbf{k} - \mathbf{h})$ is defined and

$$e(\mathcal{N}; k_0, \mathbf{k}) = e(\mathcal{N}/\mathbf{x}^*\mathcal{N}; k_0 - h_0, \mathbf{k} - \mathbf{h})$$

by Corollary 2.11. So $e(J^{[k_0-h_0+1]}, \mathbf{I}^{[\mathbf{k}-\mathbf{h}]}; N/\mathbf{x}N)$ is defined. Moreover,

$$e(\mathcal{N}/\mathbf{x}^*\mathcal{N}; k_0-h_0, \mathbf{k}-\mathbf{h}) = e(J^{[k_0-h_0+1]}, \mathbf{I}^{[\mathbf{k}-\mathbf{h}]}; N/\mathbf{x}N)$$

by Remark 3.4(i) and (1). Thus,

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e(J^{[k_0-h_0+1]}, \mathbf{I}^{[\mathbf{k}-\mathbf{h}]}; N/\mathbf{x}N). \qquad \square$$

By Remark 3.4, Remark 3.7 and Corollary 3.9, we get the following result which can be considered as a corollary of Theorem 3.5.

Corollary 3.10. *Let $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N)$ be the mixed multiplicity of the type $(k_0, \mathbf{k})$. Let $\mathbf{x}$ be a weak-(FC)-sequence of N with respect to J, $\mathbf{I}$ of the type $(k_0, \mathbf{k})$. Then*

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e(J; \frac{N}{\mathbf{x}N : I^\infty}).$$

And $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \neq 0$ if and only if $\dim \frac{N}{\mathbf{x}N:I^\infty} = 1$.

Proof. Since $\mathbf{x}$ is a weak-(FC)-sequence of N with respect to J, $\mathbf{I}$ of the type $(k_0, \mathbf{k})$, it follows that $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e(J^{[1]}, \mathbf{I}^{[\mathbf{0}]}; N/\mathbf{x}N)$ by Corollary 3.9. Note that $\dim \frac{N}{\mathbf{x}N:I^\infty} \leq 1$ by Remark 3.7. Therefore, $e(J^{[1]}, \mathbf{I}^{[\mathbf{0}]}; N/\mathbf{x}N) = e(J; \frac{N}{\mathbf{x}N:I^\infty})$ by Remark 3.4(iii). So $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e(J; \frac{N}{\mathbf{x}N:I^\infty})$. Hence $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \neq 0$ if and only if $e(J; \frac{N}{\mathbf{x}N:I^\infty}) \neq 0$. This is equivalent to $\dim \frac{N}{\mathbf{x}N:I^\infty} = 1$ by Remark 3.4(ii) and Remark 3.4(iii). $\square$

Note that one can prove Corollary 3.10 by using Theorem 3.5 as follows:

Set $N_i = \frac{(x_1,\ldots,x_{i-1})N:x_i}{(x_1,\ldots,x_{i-1})N}$ for each $1 \leq i \leq s$. Since $\mathbf{x}$ is a weak-(FC)-sequence of N with respect to J, $\mathbf{I}$, $\mathbf{x}$ is an I-filter-regular sequence with respect to N. Consequently, $N_i/(0_{N_i} : I^\infty) = 0$ since $(x_1, \ldots, x_{i-1})N : x_i \subseteq (x_1, \ldots, x_{i-1})N : I^\infty$. So $\dim N_i/(0_{N_i} : I^\infty) < 0$. Hence

$$\sum_{i=1}^{s} e\big(J^{[k_0-m_i+1]}, \mathbf{I}^{[\mathbf{k}-\mathbf{h}_i]}; N_i\big) = 0.$$

Then we obtain $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e(J; \frac{N}{\mathbf{x}N:I^\infty})$ by Theorem 3.5. We get the proof of Corollary 3.10.

Recall that in the case that $k_0+ \mid \mathbf{k} \mid= \dim \frac{N}{0_N:I^\infty} - 1$, [20, Theorem 3.4] in 2000 showed (see e.g. [4,10,21,22,25,26,28]) that $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \neq 0$ *if and only if there exists a weak-(FC)-sequence of N of the type $(0, \mathbf{k})$ with respect to $J, \mathbf{I}$ such that $\dim N/\mathbf{x}N : I^\infty = \dim N/0_N : I^\infty - \mid \mathbf{k} \mid$. In this case, we obtain* $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e\big(J; \frac{N}{\mathbf{x}N:I^\infty}\big)$. Hence Corollary 3.10 is a more natural result than [20, Theorem 3.4] in the original mixed multiplicity theory. Note that [4] proved that [20, Theorem 3.4] covers the results of Risler and Teissier [15] in 1973; Trung [17, Theorem 3.4] in 2001; Trung and Verma [18, Theorem 1.4] in 2007 (see [4, Remark 3.8]).

As an immediate consequence of Theorem 3.5 and Corollary 3.10, we get the following interesting corollary.

Corollary 3.11. *Let* $\mathbf{x}$ *be a mixed multiplicity system of* N *with respect to ideals* $J, \mathbf{I}$ *of the type* $(k_0, \mathbf{k})$*. Then we have*

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \leq e\big(J; \frac{N}{\mathbf{x}N : I^\infty}\big),$$

and equality holds if $\mathbf{x}$ *is a weak-*(FC)*-sequence of* N *with respect to* J, $\mathbf{I}$.

By combining Corollary 3.10 and Corollary 3.11 with Remark 3.4 and Remark 3.7, we have the following.

Corollary 3.12. *Let* $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N)$ *be the mixed multiplicity of the type* $(k_0, \mathbf{k})$*. Then the following are equivalent*:

(i) $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) > 0$.

(ii) $\dim \frac{N}{\mathbf{x}N:I^\infty} = 1$ *for any mixed multiplicity system* $\mathbf{x}$ *of* N *of the type* $(k_0, \mathbf{k})$.

(iii) $\dim \frac{N}{\mathbf{x}N:I^\infty} = 1$ *for any weak-*(FC)*-sequence* $\mathbf{x}$ *of* N *of the type* $(k_0, \mathbf{k})$.

(iv) *There exists a weak-*(FC)*-sequence* $\mathbf{x}$ *of* N *of the type* $(k_0, \mathbf{k})$ *such that*

$$\dim \frac{N}{\mathbf{x}N : I^\infty} = 1.$$

Proof. (i) $\Rightarrow$(ii): By Corollary 3.11, we have $e\big(J; \frac{N}{\mathbf{x}N:I^\infty}\big) > 0$. Hence since $\dim \frac{N}{\mathbf{x}N:I^\infty} \leq 1$, it follows that $\dim \frac{N}{\mathbf{x}N:I^\infty} = 1$ by Remark 3.4(ii). (ii)$\Rightarrow$ (iii) is clear by Remark 3.7. (iii)$\Rightarrow$ (iv) is evident by Remark 3.7. (iv)$\Rightarrow$(i) is immediate by Corollary 3.10. The corollary is proved. Note that the proof of this corollary can be based on Corollary 2.16, Remark 3.4 and Remark 3.7. □

Suppose that $\mathbf{x}$ is a mixed multiplicity system of N with respect to J, $\mathbf{I}$ of the type $(k_0, \mathbf{k})$. Then $\mathbf{x}^*$ is a mixed multiplicity system of $\mathcal{N}$ of the type $(k_0, \mathbf{k})$ by Remark 3.4(iv). Hence we obtain $\widetilde{e}(\mathbf{x}^*, \mathcal{N}) = \chi(\mathbf{x}^*, \mathcal{N}) = e(\mathcal{N}; k_0, \mathbf{k})$ by Theorem 2.13. Moreover, we have $e(\mathcal{N}; k_0, \mathbf{k}) = e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N)$. So we get a version of [31, Theorem 4.9] for these mixed multiplicities.

Theorem 3.13. *Let* $\mathbf{x}$ *be a mixed multiplicity system of* N *with respect to ideals* $J, \mathbf{I}$ *of the type* $(k_0, \mathbf{k})$ *and let* $\mathbf{x}^*$ *be the image of* $\mathbf{x}$ *in* $\bigcup_{i=0}^d T_i$*. Then*

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = \chi(\mathbf{x}^*, \mathcal{N}) = \widetilde{e}(\mathbf{x}^*, \mathcal{N}).$$

Finally, we would like to give some following comments.

Remark 3.14. From the results of [31] and this paper, we find that the presence of the mixed multiplicity of M of the type $\mathbf{k}$ with $|\mathbf{k}| < \mathrm{Supp}_{++}M$ also arises from the process of transforming original mixed multiplicities (see e.g. [31, Corollary 3.11, Corollary 4.10, Corollary 4.11] and Corollary 2.11, Corollary 2.14, Theorem 3.5, Corollary 3.9). Moreover, in this broader class, many hypotheses for results in the original mixed multiplicity theory have been removed. It seems that many results of the paper not only cover, but also are more natural than results in the original mixed multiplicity theory. This contributes to the explanation the meaning of mixed multiplicities of maximal degrees.

References

[1] M. P. Brodmann and R. Y. Sharp, *Local Cohomology: An Algebraic Introduction with Geometric Applications*, Cambridge Studies in Advanced Mathematics, **60**, Cambridge University Press, Cambridge, 1998.

[2] R. Callejas-Bedregal and V. H. Jorge Pérez, *Mixed multiplicities and the minimal number of generator of modules*, J. Pure Appl. Algebra **214** (2010), no. 9, 1642–1653.

[3] ———, *Multiplicities for arbitrary modules and reduction*, Rocky Mountain J. Math. **43** (2013), no. 4, 1077–1113.

[4] L. V. Dinh, N. T. Manh, and T. T. H. Thanh, *On some superficial sequences*, Southeast Asian Bull. Math. **38** (2014), no. 6, 803–811.

[5] M. Herrmann, E. Hyry, J. Ribbe, and Z. Tang, *Reduction numbers and multiplicities of multigraded structures*, J. Algebra **197** (1997), no. 2, 311–341.

[6] D. Katz and J. K. Verma, *Extended Rees algebras and mixed multiplicities*, Math. Z. **202** (1989), no. 1, 111–128.

[7] D. Kirby and D. Rees, *Multiplicities in graded rings. I. The general theory*, in Commutative algebra: syzygies, multiplicities, and birational algebra (South Hadley, MA, 1992), 209–267, Contemp. Math., 159, Amer. Math. Soc., Providence, RI, 1994.

[8] ———, *Multiplicities in graded rings. II. Integral equivalence and the Buchsbaum-Rim multiplicity*, Math. Proc. Cambridge Philos. Soc. **119** (1996), no. 3, 425–445.

[9] S. Kleiman and A. Thorup, *Mixed Buchsbaum-Rim multiplicities*, Amer. J. Math. **118** (1996), no. 3, 529–569.

[10] N. T. Manh and D. Q. Viet, *Mixed multiplicities of modules over Noetherian local rings*, Tokyo J. Math. **29** (2006), no. 2, 325–345.

[11] D. Rees, *Generalizations of reductions and mixed multiplicities*, J. London Math. Soc. (2) **29** (1984), no. 3, 397–414.

[12] P. Roberts, *Local Chern classes, multiplicities, and perfect complexes*, Mém. Soc. Math. France (N.S.) No. 38 (1989), 145–161.

[13] J. Stückrad and W. Vogel, *Buchsbaum Rings and Applications*, Mathematische Monographien, **21**, VEB Deutscher Verlag der Wissenschaften, Berlin, 1986.

[14] I. Swanson, *Mixed multiplicities, joint reductions and quasi-unmixed local rings*, J. London Math. Soc. (2) **48** (1993), no. 1, 1–14.

[15] B. Teissier, *Cycles évanescents, sections planes et conditions de Whitney*, in Singularités à Cargèse (Rencontre Singularités Géom. Anal., Inst. Études Sci., Cargèse, 1972), 285–362. Astérisque, Nos. 7 et 8, Soc. Math. France, Paris.

[16] N. V. Trung, *Reduction exponent and degree bound for the defining equations of graded rings*, Proc. Amer. Math. Soc. **101** (1987), no. 2, 229–236.

[17] ———, *Positivity of mixed multiplicities*, Math. Ann. **319** (2001), no. 1, 33–63.

[18] N. V. Trung and J. Verma, *Mixed multiplicities of ideals versus mixed volumes of polytopes*, Trans. Amer. Math. Soc. **359** (2007), no. 10, 4711–4727.

[19] J. K. Verma, *Multigraded Rees algebras and mixed multiplicities*, J. Pure Appl. Algebra **77** (1992), no. 2, 219–228.

[20] D. Q. Viet, *Mixed multiplicities of arbitrary ideals in local rings*, Comm. Algebra **28** (2000), no. 8, 3803–3821.

[21] ———, *On some properties of* (FC)*-sequences of ideals in local rings*, Proc. Amer. Math. Soc. **131** (2003), no. 1, 45–53.

[22] ———, *Sequences determining mixed multiplicities and reductions of ideals*, Comm. Algebra **31** (2003), no. 10, 5047–5069.

[23] ———, *Reductions and mixed multiplicities of ideals*, Comm. Algebra **32** (2004), no. 11, 4159–4178.

[24] ———, *The multiplicity and the Cohen-Macaulayness of extended Rees algebras of equimultiple ideals*, J. Pure Appl. Algebra **205** (2006), no. 3, 498–509.

[25] D. Q. Viet and L. V. Dinh, *On mixed multiplicities of good filtrations*, Algebra Colloq. **22** (2015), no. 3, 421–436.
[26] D. Q. Viet, L. V. Dinh, and T. T. H. Thanh, *A note on joint reductions and mixed multiplicities*, Proc. Amer. Math. Soc. **142** (2014), no. 6, 1861–1873.
[27] D. Q. Viet and N. T. Manh, *Mixed multiplicities of multigraded modules*, Forum Math. **25** (2013), no. 2, 337–361.
[28] D. Q. Viet and T. T. H. Thanh, *On (FC)-sequences and mixed multiplicities of multi-graded algebras*, Tokyo J. Math. **34** (2011), no. 1, 185–202.
[29] ______, *On some multiplicity and mixed multiplicity formulas*, Forum Math. **26** (2014), no. 2, 413–442.
[30] ______, *A note on formulas transmuting mixed multiplicities*, Forum Math. **26** (2014), no. 6, 1837–1851.
[31] ______,, *The Euler-Poincaré characteristic and mixed multiplicities*, Kyushu J. Math. **69** (2015), no. 2, 393–411.
[32] ______, *On filter-regular sequences of multi-graded modules*, Tokyo J. Math. **38** (2015), no. 2, 439–457.

Truong Thi Hong Thanh
Department of Mathematics
Hanoi National University of Education
136 Xuan Thuy Street, Hanoi, Vietnam
Email address: thanhtth@hnue.edu.vn

Duong Quoc Viet
Department of Mathematics
Hanoi National University of Education
136 Xuan Thuy Street, Hanoi, Vietnam
Email address: vduong99@gmail.com