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# GLOBAL ATTRACTORS FOR NONLOCAL PARABOLIC EQUATIONS WITH A NEW CLASS OF NONLINEARITIES

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Abstract. In this paper we consider a class of nonlocal parabolic equations in bounded domains with Dirichlet boundary conditions and a new class of nonlinearities. We first prove the existence and uniqueness of weak solutions by using the compactness method. Then we study the existence and fractal dimension estimates of the global attractor for the continuous semigroup generated by the problem. We also prove the existence of stationary solutions and give a sufficient condition for the uniqueness and global exponential stability of the stationary solution. The main novelty of the obtained results is that no restriction is imposed on the upper growth of the nonlinearities.

## 1. Introduction

In this paper we consider the following initial boundary value problem for a nonlinear parabolic equation of nonlocal type

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(l(u))\Delta u + f(u) = g(x), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N (N \geq 1)$, $l : L^2(\Omega) \to \mathbb{R}$ is a continuous functional, $f(u)$ is the nonlinear term, $g$ is the external force, which satisfy some certain conditions specified later. The problem studied here is nonlocal in view of the structure of the diffusion coefficient which is determined by a global quantity. This leads to a number of mathematical difficulties which make the analysis of the problem particularly interesting. This kind of nonlocal problems arises in various situations and has attracted the attention of many authors in recent years because the nonlocal terms allow giving more accurate results, for instance, in population dynamics, the diffusion coefficient $a$ is then supposed to depend upon the entire population in the domain rather than on

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a local density. For more details and motivation in physics, engineering and population dynamics of nonlinear nonlocal parabolic equations of type (1.1), see [11,12] and references therein. We also refer the interested reader to [1,2,13,29] for other kinds of the nonlocal term.

In recent years, the existence, uniqueness and long-time behavior of solutions to nonlinear parabolic equations with nonlocal terms have been extensively studied and there are two main types of the nonlinearities which are usually considered. The first one is the class of nonlinearities which is Lipschitz continuous or more general sublinear [11,20,22], and the second one is the class of nonlinearities which satisfies a polynomial growth

$$\begin{aligned} c_1|u|^p - c_0 \le f(u)u &\le c_2|u|^p + c_0, \\ f'(u) &\ge -\alpha, \end{aligned}$$

for some $p \ge 2$, see for instance [3,10,24]. We also refer the interested reader to [5–8,17–19,23,27,28,30] for results on semilinear non-degenerate/degenerate parabolic equations with (more general) nonlinearities of Sobolev type or polynomial type. Note that all above classes of nonlinearities require some restriction on the upper growth of the nonlinearities, in particular, the exponential nonlinearity, for example, $f(u) = e^u$, does not hold.

In this paper, we try to relax this restriction on the nonlinear term $f(u)$. In particular, we are able to prove the existence of weak solutions and the global attractor for a very large class of nonlinearities that particular covers both the above classes and even the exponential nonlinearities.

To study problem (1.1), we suppose that the nonlinearity $f$, the external force $g$ and the diffusion coefficient $a$ satisfy the following conditions:

**(H1)** $a \in C(\mathbb{R}, \mathbb{R}_+)$ is Lipschitz continuous, i.e., there exists a positive constant $L$ such that

$$|a(t) - a(s)| \le L|t - s|, \ \forall t, s \in \mathbb{R}, \tag{1.2}$$

and $a(\cdot)$ is bounded, i.e., there are two positive constants $m$, $M$ such that

$$0 < m \le a(t) \le M, \ \forall t \in \mathbb{R}. \tag{1.3}$$

Moreover, suppose that $a$ depends upon a continuous linear functional $l(u)$ on $L^2(\Omega)$, i.e.,

$$a = a(l(u)),$$

with $l : L^2(\Omega) \to \mathbb{R}$ is defined by

$$l(u) = \int_\Omega \phi(x)u(x)dx, \tag{1.4}$$

where $\phi(\cdot)$ is a given function in $L^2(\Omega)$.

**(H2)** $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying

$$f(u)u \ge -\mu u^2 - c_1, \tag{1.5}$$

$$f'(u) \geq -\alpha, \tag{1.6}$$

where $c_1, \alpha$ are two positive constants, $0 < \mu < m\lambda_1$ with $\lambda_1 > 0$ is the first eigenvalue of the operator $(-\Delta, H_0^1(\Omega))$.

**(H3)** $g \in L^2(\Omega)$.

The structure of the paper is as follows. In Section 2, we prove the existence and uniqueness of weak solutions by combining the compactness method and the weak convergence techniques in Orlicz spaces introduced in [16], which has been exploited later in [15, 21]. In Section 3, we prove the existence of a global attractor for the semigroup generated by the problem in various spaces. In Section 4, we first show that the boundedness of the global attractor $\mathcal{A}$ in $L^\infty(\Omega)$ under some additional conditions of $f$ and $g$, and then we show the finiteness of fractal dimension of the global attractor by using the Ladyzhenskaya method. The last section is devoted to proving the existence of weak stationary solutions to (1.1) and we give a sufficient condition for the uniqueness and exponential stability of the stationary solution. The main novelty of the paper is that the nonlinearity can grow arbitrarily fast. In particular, the results obtained here improve and extend all previous results for nonlocal parabolic equations in [4, 11, 24].

Before to start, let us introduce some notations that will be used in the sequel. As usual, the inner product in $L^2(\Omega)$ will be denoted by $(\cdot,\cdot)$ and by $|\cdot|_2$ its associated norm. The inner product in $H_0^1(\Omega)$ is presented by $((\cdot,\cdot))$ and by $\|\cdot\|$ its associated norm. By $\langle \cdot,\cdot\rangle$, we represent the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. We identify $L^2(\Omega)$ with its dual, and so, we have a chain of compact and dense embeddings $H_0^1(\Omega) \subset\subset L^2(\Omega) \subset H^{-1}(\Omega)$. This allows us to make an abuse of the notation considering $l \in L^2(\Omega)$ and denoting $(l, u)$ like $l(u)$. We also use $\langle \cdot,\cdot\rangle$ for the duality pairing between $H^{-1}(\Omega) + L^1(\Omega)$ and $H_0^1(\Omega) \cap L^\infty(\Omega)$.

## 2. Existence and uniqueness of weak solutions

In this section, we will prove the existence and uniqueness of weak solutions to problem (1.1). First, we give the definition of weak solutions.

**Definition 2.1.** A weak solution to (1.1) on the interval $(0, T)$ is a function $u \in L^2(0, T; H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega))$ such that $f(u) \in L^1(\Omega_T)$, $u(0) = u_0$, $\frac{du}{dt} \in L^2(0, T; H^{-1}(\Omega)) + L^1(\Omega_T)$ and

$$\langle \frac{d}{dt}u(t), v\rangle + a(l(u(t)))((u(t), v)) + \langle f(u), v\rangle = (g, v) \tag{2.1}$$

for all $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and for a.e. $t \in (0, T)$, where $\Omega_T = \Omega \times (0, T)$.

We now prove the following theorem.

**Theorem 2.1.** *Let $u_0 \in L^2(\Omega)$ and $T > 0$ be given. Assume* **(H1), (H2)** *and* **(H3)** *hold. Then problem* (1.1) *has a unique weak solution $u$ on the interval*

$(0,T)$. *Moreover, the mapping* $u_0 \mapsto u(t)$ *is continuous on* $L^2(\Omega)$, *that is, the solution depends continuously on the initial data.*

*Proof.* **i) Existence.** Due to the theory of ordinary differential equations, we can find, for each integer $n \geq 1$, a Galerkin approximate solution $u_n$ in the form

$$u_n(t) = \sum_{j=1}^{n} u_{nj}(t) e_j,$$

where $\{e_j\}_{j=1}^{\infty} \subset H_0^1(\Omega) \cap L^\infty(\Omega)$ is an orthonormal Hilbert basis in $L^2(\Omega)$, and $u_{nj}(t)$ are gotten from solving the following problem

$$\begin{cases} \dfrac{d}{dt}(u_n(t), e_j) + a(l(u_n(t)))((u_n(t), e_j)) + \langle f(u_n(t)), e_j \rangle = (g, e_j), \\ (u_n(0), e_j) = (u_0, e_j). \end{cases} \tag{2.2}$$

Multiplying by $u_{nj}(t)$ in (2.2) and summing from $j = 1$ to $n$, we obtain

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|_2^2 + a(l(u_n(t)))\|u_n(t)\|^2 + \int_\Omega f(u_n(t))u_n(t)dx = \int_\Omega g u_n(t) dx. \tag{2.3}$$

Using (1.5) and the Cauchy inequality yields

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|_2^2 + a(l(u_n(t)))\|u_n(t)\|^2 - \mu|u_n(t)|_2^2 - c_1|\Omega| \leq \frac{|g|_2^2}{2\varepsilon\lambda_1} + \frac{\varepsilon\lambda_1}{2}|u_n(t)|_2^2. \tag{2.4}$$

Therefore, using (1.3) leads to

$$\frac{d}{dt}|u_n(t)|_2^2 + \varepsilon\|u_n(t)\|^2 + 2(m\lambda_1 - \mu - \varepsilon\lambda_1)|u_n(t)|_2^2 \leq \frac{|g|_2^2}{\varepsilon\lambda_1} + 2c_1|\Omega|, \tag{2.5}$$

with $\varepsilon > 0$ small enough so that $m\lambda_1 - \mu - \varepsilon\lambda_1 > 0$. Now, integrating (2.5) from 0 to $t \in (0,T)$ we get

$$\begin{aligned} &|u_n(t)|_2^2 + \varepsilon \int_0^t \|u_n(s)\|^2 ds + 2(m\lambda_1 - \mu - \varepsilon\lambda_1)\int_0^t |u_n(s)|_2^2 ds \\ &\leq \frac{|g|_2^2 T}{\varepsilon\lambda_1} + 2c_1|\Omega|T + |u_0|_2^2. \end{aligned} \tag{2.6}$$

This inequality yields that $\{u_n\}$ is bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$. Therefore, there exist $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, and a subsequence of $u_n$ (relabeled the same) such that

$$\begin{aligned} u_n &\rightharpoonup^* u \quad \text{in } L^\infty(0,T;L^2(\Omega)), \\ u_n &\rightharpoonup u \quad \text{in } L^2(0,T;H_0^1(\Omega)), \\ -\Delta u_n &\rightharpoonup -\Delta u \quad \text{in } L^2(0,T;H^{-1}(\Omega)). \end{aligned} \tag{2.7}$$

By the Cauchy inequality and (1.3), it follows from (2.3) that

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|_2^2 + m\lambda_1|u_n(t)|_2^2 + \int_\Omega f(u_n(t))u_n(t)dx \leq \frac{1}{2m\lambda_1}|g|_2^2 + \frac{m\lambda_1}{2}|u_n(t)|_2^2,$$

hence in particular

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|_2^2 + \int_\Omega f(u_n(t))u_n(t)dx \leq \frac{1}{2m\lambda_1}|g|_2^2.$$

Integrating from 0 to $T$ we obtain

$$\frac{1}{2}|u_n(T)|_2^2 + \int_{\Omega_T} f(u_n(t))u_n(t)dxdt \leq \frac{1}{2m\lambda_1}|g|_2^2 T + \frac{1}{2}|u_0|_2^2.$$

Hence

$$\int_{\Omega_T} f(u_n)u_n dxdt \leq \frac{1}{2m\lambda_1}|g|_2^2 T + \frac{1}{2}|u_0|_2^2. \tag{2.8}$$

We now prove that $\{f(u_n)\}$ is bounded in $L^1(\Omega_T)$. To do this, we put $h(u_n) = f(u_n) - f(0) + \nu u_n$ with $\nu > \alpha$. By using (1.6) we see that $h(s)s \geq 0$ for all $s \in \mathbb{R}$. So, using (2.8) and the boundedness of $\{u_n\}$ in $L^\infty(0,T;L^2(\Omega))$, we have

$$\begin{aligned}
\int_{\Omega_T} |h(u_n)|dxdt &\leq \int_{\Omega_T\cap\{|u_n|>1\}} |h(u_n)u_n|dxdt + \int_{\Omega_T\cap\{|u_n|\leq 1\}} |h(u_n)|dxdt \\
&\leq \int_{\Omega_T} h(u_n)u_n dxdt + \sup_{|s|\leq 1}|h(s)||\Omega_T| \\
&= \int_{\Omega_T} f(u_n)u_n dxdt + \nu\int_{\Omega_T}|u_n|^2 dxdt + |f(0)|\int_{\Omega_T}|u_n|dxdt \\
&\quad + \sup_{|s|\leq 1}|h(s)||\Omega_T| \\
&\leq C.
\end{aligned}$$

This means that $\{h(u_n)\}$ is bounded in $L^1(\Omega_T)$, and so is $\{f(u_n)\}$. Because of (1.3) and the boundedness of $\{u_n\}$ in $L^2(0,T;H_0^1(\Omega))$, we can check that $\{-a(l(u_n))\Delta u_n\}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$. Since

$$\frac{du_n}{dt} = a(l(u_n))\Delta u_n - f(u_n) + g,$$

we deduce that $\{\frac{du_n}{dt}\}$ is bounded in $L^2(0,T;H^{-1}(\Omega)) + L^1(\Omega_T)$, and therefore in $L^1(0,T;H^{-1}(\Omega) + L^1(\Omega))$. By the Aubin-Lions-Simon compactness lemma (see e.g. [9]), we have that $\{u_n\}$ is compact in $L^2(0,T;L^2(\Omega))$. So, up to a subsequence,

$$u_n \to u \text{ in } L^2(0,T;L^2(\Omega)).$$

Therefore,

$$l(u_n) \to l(u) \text{ in } L^2(0,T).$$

Since $a$ is continuous, we have

$$a(l(u_n)) \to a(l(u)) \text{ in } L^2(0,T). \tag{2.9}$$

Combining (2.7) and (2.9), we deduce that for all $\varphi \in C_0^\infty([0,T]; H_0^1(\Omega) \cap L^\infty(\Omega))$,

$$\int_0^T a(l(u_n)) \int_\Omega \nabla u_n \cdot \nabla\varphi dxdt \to \int_0^T a(l(u)) \int_\Omega \nabla u \cdot \nabla\varphi dxdt.$$

We now pass to the limits in the nonlinear term. From (1.6) and (2.8) we see that $h(\cdot)$ is a strictly increasing function and

$$\int_{\Omega_T} h(u_n(t))u_n(t)dxdt \leq \frac{1}{2m\lambda_1}|g|_2^2 T + \frac{1}{2}|u_0|_2^2.$$

Since $u_n \to u$ in $L^2(\Omega_T)$, then up to a subsequence we have $u_n \to u$ a.e. in $\Omega_T$. Applying Lemma 6.1 in [15], we obtain that $h(u) \in L^1(\Omega_T)$ and for all test functions $\varphi \in C_0^\infty([0,T]; H_0^1(\Omega) \cap L^\infty(\Omega))$,

$$\int_{\Omega_T} h(u_n)\varphi dxdt \to \int_{\Omega_T} h(u)\varphi dxdt \quad \text{as } n \to \infty.$$

Hence, $f(u) \in L^1(\Omega_T)$ and for all $\varphi \in C_0^\infty([0,T]; H_0^1(\Omega) \cap L^\infty(\Omega))$,

$$\int_{\Omega_T} f(u_n)\varphi dxdt \to \int_{\Omega_T} f(u)\varphi dxdt \quad \text{as } n \to \infty.$$

Passing to the limits, we deduce that $u$ satisfies (2.1). Since $u \in L^2(0,T; H_0^1(\Omega))$ and $\frac{du}{dt} \in L^2(0,T; H^{-1}(\Omega)) + L^1(\Omega_T)$, we get $u \in C([0,T]; L^2(\Omega))$. Moreover, repeating the arguments in [7], we get $u(0) = u_0$ and this implies that $u$ is a weak solution to problem (1.1).

**ii) Uniqueness and continuous dependence on the initial data.** Let $u_1$ and $u_2$ be two weak solutions of (1.1) with initial data $u_{01}$, $u_{02} \in L^2(\Omega)$, respectively. Then $w = u_1 - u_2$ satisfies

(2.10)
$$\begin{cases} w_t - a(l(u_1))\Delta w + \widehat{f}(u_1) - \widehat{f}(u_2) = -\left(a(l(u_2)) - a(l(u_1))\right)\Delta u_2 + \alpha w, \\ w(0) = u_{01} - u_{02}, \end{cases}$$

where $\widehat{f}(s) = f(s) + \alpha s$ and $w = u_1 - u_2$. Here, we cannot choose $w(t)$ as a test function since $w(t)$ does not belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$. Therefore, the proof is more involved.

We use some ideas in [16]. Let $B_k : \mathbb{R} \to \mathbb{R}$ be the truncated function defined by

$$B_k(s) = \begin{cases} k & \text{if} \quad s > k, \\ s & \text{if} \quad |s| \leq k, \\ -k & \text{if} \quad s < -k. \end{cases}$$

We construct the following Nemytskii mapping

$$\begin{aligned} \widehat{B}_k : H_0^1(\Omega) \cap L^\infty(\Omega) &\to H_0^1(\Omega) \cap L^\infty(\Omega) \\ w &\mapsto \widehat{B}_k(w)(x) = B_k(w(x)). \end{aligned}$$

It follows from Lemma 2.3 in [16] that $\|\widehat{B}_k(w)-w\|_{H_0^1(\Omega)\cap L^\infty(\Omega)} \to 0$ as $k\to\infty$. Multiplying the first equation in (2.10) by $\widehat{B}_k(w)$, then integrating from $\delta$ to $t$, we obtain

$$\begin{aligned}
&\int_\delta^t\int_\Omega \frac{d}{ds}(w(s)\widehat{B}_k(w)(s))dxds - \int_\delta^t\int_\Omega w\frac{d}{ds}(\widehat{B}_k(w)(s))dxds\\
&+\int_\delta^t a(l(u_1))\int_{\{x\in\Omega:|w(x,s)|\le k\}}|\nabla w|^2dxds\\
&+\int_\delta^t\int_\Omega\left(\widehat{f}(u_1)-\widehat{f}(u_2)\right)\widehat{B}_k(w)dxds\\
\le&\int_\delta^t|a(l(u_2))-a(l(u_1))|\,|\int_{\{x\in\Omega:|w(x,s)|\le k\}}\nabla u_2\cdot\nabla wdx|ds\\
&+\alpha\int_\delta^t\int_\Omega w\widehat{B}_k(w)dxds.
\end{aligned}\tag{2.11}$$

The following estimate is deduced by using (1.2), (1.4), the Hölder and the Cauchy inequalities

$$\begin{aligned}
&\int_\delta^t|a(l(u_2))-a(l(u_1))|\,|\int_{\{x\in\Omega:|w(x,s)|\le k\}}\nabla u_2\cdot\nabla wdx|ds\\
\le\,& L\int_\delta^t(\int_\Omega|\phi|\,|w|dx)\,(\int_{\{x\in\Omega:|w(x,s)|\le k\}}|\nabla u_2\cdot\nabla w|dx)ds\\
\le\,& L\int_\delta^t|\phi|_2\,|w(s)|_2\,\|u_2(s)\|\,\|\widehat{B}_k(w)(s)\|ds\\
\le\,&\frac{m}{2}\int_\delta^t\|\widehat{B}_k(w)(s)\|^2ds+\frac{L^2|\phi|_2^2}{2m}\int_\delta^t\|u_2(s)\|^2|w(s)|_2^2ds.
\end{aligned}\tag{2.12}$$

Since $w\frac{d}{dt}\widehat{B}_k(w)=\frac{1}{2}\frac{d}{dt}(\widehat{B}_k(w))^2$, we obtain from (1.3), (2.11) and (2.12) that

$$\begin{aligned}
&\int_\Omega w(t)\widehat{B}_k(w)(t)dx-\frac{1}{2}|\widehat{B}_k(w)(t)|_2^2\\
&+m\int_\delta^t\|\widehat{B}_k(w)(s)\|^2ds+\int_\delta^t\int_\Omega\widehat{f}'(\xi)w\widehat{B}_k(w)dxds\\
\le&\int_\Omega w(\delta)\widehat{B}_k(w)(\delta)dx-\frac{1}{2}|\widehat{B}_k(w)(\delta)|_2^2+\frac{m}{2}\int_\delta^t\|\widehat{B}_k(w)(s)\|^2ds\\
&+\frac{L^2|\phi|_2^2}{2m}\int_\delta^t\|u_2(s)\|^2|w(s)|_2^2ds+\alpha\int_\delta^t\int_\Omega w\widehat{B}_k(w)dxds.
\end{aligned}$$

Since $\widehat{f}'(s)\ge 0$ and $s\widehat{B}_k(s)\ge 0$ for all $s\in\mathbb{R}$, letting $\delta\to 0$ and $k\to\infty$ in the above inequality we obtain

$$|w(t)|_2^2\le|w(0)|_2^2+\int_0^t\left(\frac{L^2|\phi|_2^2}{m}\|u_2(s)\|^2+2\alpha\right)|w(s)|_2^2ds.\tag{2.13}$$

By (2.6), we have for some fixed $\varepsilon > 0$,

$$\int_0^t \|u_2(s)\|^2 ds \leq \frac{|g|_2^2 T}{\varepsilon^2 \lambda_1} + \frac{1}{\varepsilon}\left(2c_1|\Omega|T + |u_{02}|_2^2\right).$$

Applying the Gronwall inequality of integral form for (2.13) we get

$$\begin{aligned} |w(t)|_2^2 &\leq |w(0)|_2^2 \exp\left(\int_0^t \left(\frac{L^2|\phi|_2^2}{m}\|u_2(s)\|^2 + 2\alpha\right) ds\right) \\ &\leq |w(0)|_2^2 \exp\left(\frac{L^2|\phi|_2^2}{m}\left[\frac{|g|_2^2 T}{\varepsilon^2\lambda_1} + \frac{1}{\varepsilon}\left(2c_1|\Omega|T + |u_{02}|_2^2\right)\right] + 2\alpha t\right). \end{aligned}$$

This implies the desired result. □

## 3. Existence of a global attractor

Thanks to Theorem 2.1, we can define a continuous (nonlinear) semigroup $S(t) : L^2(\Omega) \to L^2(\Omega)$ associated to problem (1.1) as follows

$$S(t)u_0 := u(t),$$

where $u(\cdot)$ is the unique global weak solution of (1.1) with the initial datum $u_0$. We will prove that the semigroup $S(t)$ has a compact global attractor $\mathcal{A}$. For the general theory of global attractors, we refer the reader to [14, 23, 26].

For the sake of brevity, in the following lemmas we only give some formal calculations, the rigorous proof is done by use of Galerkin approximations and Lemma 11.2 in [23].

**Lemma 3.1.** *The semigroup $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $L^2(\Omega)$.*

*Proof.* Multiplying the first equation in (1.1) by $u$ we have

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + a(l(u))\|u\|^2 + \int_\Omega f(u)u dx = (g, u). \tag{3.1}$$

Thanks to (1.3) and (1.5), we obtain the following estimate by using the Cauchy inequality and the Poincaré inequality

$$\frac{d}{dt}|u|_2^2 + (m\lambda_1 - \mu)|u|_2^2 \leq 2c_1|\Omega| + \frac{|g|_2^2}{m\lambda_1 - \mu}.$$

Applying the Gronwall inequality we infer that

$$|u(t)|_2^2 \leq |u_0|_2^2 e^{-(m\lambda_1-\mu)t} + R_1,$$

where

$$R_1 = \frac{2c_1|\Omega|(m\lambda_1 - \mu) + |g|_2^2}{(m\lambda_1 - \mu)^2}.$$

Therefore, if choosing $\rho_1 = 2R_1$, we can assert that

$$|u(t)|_2^2 \leq \rho_1 \tag{3.2}$$

for all $t \geq T_1 = T_1(\lambda_1, \mu, m, |u_0|_2)$, and so the proof is complete. □

**Lemma 3.2.** *The semigroup $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $H_0^1(\Omega)$.*

*Proof.* Multiplying the first equation in (1.1) by $-\Delta u$ and integrating by parts, then using (1.6) and the Cauchy inequality we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2 + a(l(u))|\Delta u|_2^2 &= -\int_\Omega f'(u)|\nabla u|^2dx - \int_\Omega g\Delta u dx \\ &\le \alpha\|u\|^2 + \frac{1}{2m}|g|_2^2 + \frac{m}{2}|\Delta u|_2^2.\end{aligned}$$

Combining this with (1.3) leads to

$$\frac{d}{dt}\|u(t)\|^2 \le 2\alpha\|u(t)\|^2 + \frac{1}{m}|g|_2^2. \tag{3.3}$$

On the other hand, integrating (3.1) from $t$ to $t+1$ and taking (1.3), (1.5) and (3.2) into account, we get

$$\begin{aligned}\int_t^{t+1}\|u(s)\|^2ds &\le \frac{1}{2m}|u(t)|_2^2 + \frac{c_1}{m}|\Omega| + \frac{\mu+1}{m}\int_t^{t+1}|u(s)|_2^2ds + \frac{1}{4m}|g|_2^2 \\ &\le \frac{c_1}{m}|\Omega| + \frac{2\mu+3}{2m}\rho_1 + \frac{1}{4m}|g|_2^2\end{aligned} \tag{3.4}$$

for all $t \ge T_1 = T_1(\lambda_1, \mu, m, |u_0|_2)$. By the uniform Gronwall inequality, from (3.3) and (3.4) we deduce that

$$\|u(t)\|^2 \le \rho_2 := \left(\frac{c_1}{m}|\Omega| + \frac{2\mu+3}{2m}\rho_1 + \frac{5}{4m}|g|_2^2\right)e^{2\alpha} \tag{3.5}$$

for all $t \ge T_2 = T_1 + 1$. The proof is complete. $\square$

As a direct consequence of Lemma 3.2 and the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, we get the following result.

**Theorem 3.1.** *Suppose that the hypotheses* **(H1), (H2)** *and* **(H3)** *hold. Then the semigroup $S(t)$ generated by problem* (1.1) *has a connected compact global attractor $\mathcal{A}$ in $L^2(\Omega)$.*

It is possible to show that the global attractor will be more regular when $a$ is more regular. To do this, we suppose that

**(H1bis)** $a \in C(\mathbb{R}, \mathbb{R}_+)$ is continuously differentiable and satisfies condition **(H1)**.

**Lemma 3.3.** *Under the hypotheses* **(H1bis), (H2)** *and* **(H3)**, *the semigroup $\{S(t)\}_{t\ge 0}$ has a bounded absorbing set in $H^2(\Omega)\cap H_0^1(\Omega)$.*

*Proof.* Differentiating the first equation in (1.1) with respect to $t$, then taking the duality with $u_t$ yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|u_t|_2^2 + a(l(u))|\nabla u_t|_2^2 + \int_\Omega f'(u)|u_t|^2dx \\ &= -a'(l(u))\int_\Omega \phi(x)u_tdx\int_\Omega \nabla u\cdot\nabla u_tdx,\end{aligned}$$

and performing the following estimate by using the assumptions (1.3), (1.6) and the Hölder inequality

$$\frac{d}{dt}|u_t|_2^2 + 2m\|u_t\|^2 - 2\alpha|u_t|_2^2 \leq 2|a'(l(u))|\,|\phi|_2\,|u_t|_2\,\|u\|\,\|u_t\|. \tag{3.6}$$

In addition, by using the Hölder inequality and (3.2), we have

$$|l(u)| \leq |\phi|_2\,|u|_2 \leq |\phi|_2\sqrt{\rho_1}, \quad \forall t \geq T_1.$$

We denote

$$\gamma = \sqrt{\rho_2}\sup_{|s|\leq|\phi|_2\sqrt{\rho_1}}|a'(s)|\,|\phi|_2. \tag{3.7}$$

In view of (3.6) and (3.7), we obtain the following estimate by using the Cauchy inequality

$$\frac{d}{dt}|u_t|_2^2 + 2m\|u_t\|^2 - 2\alpha|u_t|_2^2 \leq 2\gamma|u_t|_2\,\|u_t\| \leq \frac{\gamma^2}{2m}|u_t|_2^2 + 2m\|u_t\|^2.$$

Hence

$$\frac{d}{dt}|u_t|_2^2 \leq \left(\frac{\gamma^2}{2m} + 2\alpha\right)|u_t|_2^2. \tag{3.8}$$

Multiplying the first equation in (1.1) by $u_t$ and integrating by parts we obtain

$$a(l(u))\int_\Omega \nabla u\cdot\nabla u_t dx + \int_\Omega f(u)u_t dx - \int_\Omega g u_t dx = -|u_t|_2^2. \tag{3.9}$$

Putting

$$F(u) = \int_0^u f(s)ds. \tag{3.10}$$

It follows from (1.6) that

$$F(u) \leq f(u)u + \alpha\frac{u^2}{2}, \quad \forall u \in \mathbb{R}, \tag{3.11}$$

and

$$F(u) \geq -\frac{\alpha+1}{2}u^2 - \frac{|f(0)|^2}{2}, \quad \forall u \in \mathbb{R}. \tag{3.12}$$

We now have from (3.9) and (3.10) that

$$\frac{d}{dt}\left(\frac{a(l(u))}{2}\|u\|^2 + \int_\Omega F(u)dx - \int_\Omega g u dx\right) = \frac{1}{2}\|u\|^2 a'(l(u))\int_\Omega \phi(x)u_t dx - |u_t|_2^2. \tag{3.13}$$

By the Hölder inequality and the Cauchy inequality, it follows from (3.13) that

$$\begin{aligned}\frac{d}{dt}\left(\frac{a(l(u))}{2}\|u\|^2 + \int_\Omega F(u)dx - \int_\Omega g u dx\right) &\leq \frac{1}{2}\|u\|^2|a'(l(u))|\,|\phi|_2\,|u_t|_2 - |u_t|_2^2\\ &\leq \frac{1}{8}\|u\|^4|a'(l(u))|^2|\phi|_2^2 - \frac{1}{2}|u_t|_2^2.\end{aligned}$$

Taking (3.5) and (3.7) into account, we have

$$\frac{1}{2}|u_t|_2^2 + \frac{d}{dt}\left(\frac{a(l(u))}{2}\|u\|^2 + \int_\Omega F(u)dx - \int_\Omega gudx\right) \leq \frac{\gamma^2\rho_2}{8}. \tag{3.14}$$

On the other hand, integrating (3.1) from $t$ to $t+1$, and using (3.2), we have

$$\int_t^{t+1} \Big[a(l(u))\|u\|^2 + \int_\Omega f(u)udx - \int_\Omega gudx\Big]ds \leq \rho_1. \tag{3.15}$$

From (3.15), (3.11) and (3.2), we get

$$\int_t^{t+1} \Big[\frac{a(l(u))}{2}\|u\|^2 + \int_\Omega F(u)dx - \int_\Omega gudx\Big]ds \leq (1+\frac{\alpha}{2})\rho_1 \tag{3.16}$$

for all $t \geq T_1$. Therefore, from (3.14) and (3.16), by using the uniform Gronwall inequality, we obtain

$$\frac{a(l(u))}{2}\|u\|^2 + \int_\Omega F(u)dx - \int_\Omega gudx \leq (1+\frac{\alpha}{2})\rho_1 + \frac{\gamma^2\rho_2}{8} \tag{3.17}$$

for all $t \geq T_2 = T_1 + 1$. Using the Cauchy inequality and (1.3), we have

$$\frac{a(l(u))}{2}\|u\|^2 + \int_\Omega F(u)dx - \int_\Omega gudx \geq \frac{m}{2}\|u\|^2 + \int_\Omega F(u)dx - \frac{1}{2m\lambda_1}|g|_2^2 - \frac{m\lambda_1}{2}|u|_2^2.$$

By the Poincaré inequality and (3.12), we get

$$\frac{a(l(u))}{2}\|u\|^2 + \int_\Omega F(u)dx - \int_\Omega gudx \geq -\frac{\alpha+1}{2}|u|_2^2 - \frac{|f(0)|^2}{2}|\Omega| - \frac{1}{2m\lambda_1}|g|_2^2. \tag{3.18}$$

Now, integrating (3.14) from $t$ to $t+1$ and using (3.17), (3.18), we have

$$\frac{1}{2}\int_t^{t+1} |u_t(s)|_2^2 ds \leq \frac{2\alpha+3}{2}\rho_1 + \frac{\gamma^2\rho_2}{4} + \frac{|f(0)|^2}{2}|\Omega| + \frac{1}{2m\lambda_1}|g|_2^2 \tag{3.19}$$

for all $t \geq T_2$. Here we have used (3.2).

Combining (3.8) with (3.19) and using the uniform Gronwall inequality, we get

$$|u_t|_2^2 \leq \rho_3 \text{ for all } t \geq T_3 = T_2 + 1, \tag{3.20}$$

where

$$\rho_3 = \left((2\alpha+3)\rho_1 + \frac{\gamma^2\rho_2}{2} + |f(0)|^2|\Omega| + \frac{1}{m\lambda_1}|g|_2^2\right)\exp\left(\frac{\gamma^2}{2m} + 2\alpha\right).$$

Multiplying the first equation in (1.1) by $-\Delta u$ and using (1.3) and (1.5), we obtain

$$m|\Delta u|_2^2 \leq \alpha\|u\|^2 + |u_t|_2|\Delta u|_2 + |g|_2|\Delta u|_2.$$

By the Cauchy inequality, we have

$$m|\Delta u|_2^2 \leq \alpha\|u\|^2 + \frac{1}{m}|u_t|_2^2 + \frac{1}{m}|g|_2^2 + \frac{m}{2}|\Delta u|_2^2.$$

Hence
$$|\Delta u|_2^2 \leq \frac{2\alpha}{m}\|u\|^2 + \frac{2}{m^2}|u_t|_2^2 + \frac{2}{m^2}|g|_2^2.$$
Using the estimates (3.5) and (3.20), we deduce that
$$|\Delta u|_2^2 \leq \rho_4 := \frac{2\alpha}{m}\rho_2 + \frac{2}{m^2}\rho_3 + \frac{2}{m^2}|g|_2^2$$
for all $t \geq T_3$. This completes the proof. □

Due to the compactness of the embedding $H^2(\Omega) \hookrightarrow H_0^1(\Omega)$, we get the following theorem.

**Theorem 3.2.** *Suppose that the hypotheses* **(H1bis)**, **(H2)** *and* **(H3)** *hold. Then the semigroup* $S(t)$ *generated by problem* (1.1) *has a connected compact global attractor* $\mathcal{A}$ *in* $H_0^1(\Omega)$.

## 4. Fractal dimension estimates of the global attractor

In this section we will study the finiteness of fractal dimension of the global attractor for the semigroup generated by problem (1.1). To do this, we assume that the nonlinearity $f$ and the external force $g$ satisfy stronger conditions:

**(H2bis)** $f$ satisfies the condition **(H2)** and there exists $s_0 > 0$ such that
$$f(s) \geq \|g\|_{L^\infty(\Omega)} \text{ if } s \geq s_0, \quad f(s) \leq \|g\|_{L^\infty(\Omega)} \text{ if } s \leq -s_0.$$

**(H3bis)** $g \in L^\infty(\Omega)$.

**Lemma 4.1.** *Assume that* **(H1)**, **(H2bis)**, *and* **(H3bis)** *hold. Then the global attractor* $\mathcal{A}$ *of problem* (1.1) *is bounded in* $L^\infty(\Omega)$.

*Proof.* To prove Lemma 4.1, we will use a bounded version of function $u(x)-M$ for some appropriate constant $M$. We define
$$u_+(x) = \begin{cases} u(x), & \text{if } u(x) > 0, \\ 0, & \text{otherwise,} \end{cases}$$
and
$$u_-(x) = \begin{cases} u(x), & \text{if } u(x) < 0, \\ 0, & \text{otherwise.} \end{cases}$$
Suppose that $u(t) \in \mathcal{A}$. We multiply (1.1) by $(u-M)_+$ and integrate over $\Omega$ to obtain
$$\frac{1}{2}\frac{d}{dt}\int_{\Omega(u\geq M)} |(u-M)_+|^2 dx + a(l(u))\int_{\Omega(u\geq M)} |\nabla(u-M)_+|^2 dx$$
$$+ \int_{\Omega(u\geq M)} f(u)(u-M)_+ dx = \int_{\Omega(u\geq M)} g(u-M)_+ dx,$$
where $\Omega(u \geq M) = \{x \in \Omega : u(x) - M \geq 0\}$. Using the fact that
$$\int_{\Omega(u\geq M)} |\nabla(u-M)_+|^2 dx \geq \lambda_{\Omega(u\geq M)} \int_{\Omega(u\geq M)} |(u-M)_+|^2 dx$$

and (1.3), it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\Omega(u\geq M)}|(u-M)_+|^2dx+m\lambda_{\Omega(u\geq M)}\int_{\Omega(u\geq M)}|(u-M)_+|^2dx\\ &\leq \int_{\Omega(u\geq M)}(g-f(u))(u-M)_+dx.\end{aligned}$$

By virtue of the condition **(H2bis)**, we can choose $M$ large enough such that $f(u) \geq \|g\|_{L^\infty(\Omega)}$ whenever $u \geq M$. Then

$$\frac{d}{dt}\int_{\Omega(u\geq M)}|(u-M)_+|^2dx+2m\lambda_{\Omega(u\geq M)}\int_{\Omega(u\geq M)}|(u-M)_+|^2dx\leq 0.$$

Using the Gronwall inequality, we have

$$\int_{\Omega(u\geq M)}|(u-M)_+|^2dx\leq e^{-2m\lambda_{\Omega(u\geq M)}t}\int_{\Omega(u\geq M)}|(u_0-M)_+|^2dx\to 0 \text{ as } t\to\infty.$$

Since $\mathcal{A}$ is invariant, we get

$$\int_{\Omega(u\geq M)}|(u-M)_+|^2dx=0. \tag{4.1}$$

Repeating the arguments in the same way for $(u+M)_-$ instead of $(u-M)_+$, we obtain

$$\int_{\Omega(u\leq -M)}|(u+M)_-|^2dx=0. \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$\|u\|_{L^\infty(\Omega)}\leq M \quad \text{for all } u\in\mathcal{A}.$$

□

To prove the finiteness of fractal dimension of the global attractor, we will use the following abstract result.

**Theorem 4.1** ([14])**.** *Assume that $E$ is a compact set in a Hilbert space $H$ with the norm $\|\cdot\|$. Let $V$ be a continuous mapping in $H$ such that $E\subset V(E)$. Assume that there exists a finite dimensional projector $P$ in the space $H$ such that*

$$\|P(Vv_1-Vv_2)\|\leq\kappa\|v_1-v_2\| \quad \text{for all } v_1,v_2\in E,$$

*and*

$$\|(1-P)(Vv_1-Vv_2)\|\leq\delta\|v_1-v_2\| \quad \text{for all } v_1,v_2\in E,$$

*where $\delta<1$. We also assume that $\kappa>1-\delta$. Then the compact set $E$ possesses a finite fractal dimension and*

$$\dim_f E\leq \dim P \ln\left(\frac{9\kappa}{1-\delta}\right)\left[\ln\left(\frac{2}{1+\delta}\right)\right]^{-1}.$$

The following theorem is the main result of this section.

**Theorem 4.2.** *Assume that* **(H1), (H2bis)**, *and* **(H3bis)** *hold. Then the global attractor $\mathcal{A}$ of problem* (1.1) *has a finite fractal dimension in $L^2(\Omega)$, namely,*

$$\dim_f \mathcal{A} \le q \ln\left(\frac{9e^C}{1-\delta}\right)\left[\ln\left(\frac{2}{1+\delta}\right)\right]^{-1}.$$

*Proof.* Suppose that $u_{01}$, $u_{02} \in \mathcal{A}$ are given, and $u_1(t) = S(t)u_{01}$ and $u_2(t) = S(t)u_{02}$ are solutions of problem (1.1) with the initial values $u_{01}$, $u_{02}$, respectively. We define $w(t) = u_1(t) - u_2(t)$. Assume that $w(t) = w_1(t) + w_2(t)$, where $w_1(t)$ is the projection onto $P_qL^2(\Omega) = \text{span}\{e_1, e_2, \ldots, e_q\}$, where $\{e_j\}_{j=1}^{\infty} \subset H_0^1(\Omega) \cap L^\infty(\Omega)$ is an orthonormal Hilbert basis in $L^2(\Omega)$.

As we know that $S(t)\mathcal{A} = \mathcal{A}$ for all $t \ge 0$, so $u_i(t) \in H_0^1(\Omega)$ for any $t \ge 0$ and

$$\|u_i(t)\|^2 \le \rho_2, \quad i = 1, 2, \quad t \ge 0. \tag{4.3}$$

Moreover, by Lemma 4.1, we have $w(t) \in L^\infty(\Omega)$ for all $t \ge 0$, and then $w_1(t), w_2(t) \in L^\infty(\Omega)$ for all $t \ge 0$.

Performing the same arguments in the proof of Theorem 2.1 we get

$$w_t - a(l(u_1))\Delta w + f(u_1) - f(u_2) = -(a(l(u_2)) - a(l(u_1)))\Delta u_2. \tag{4.4}$$

It follows that

$$|w(t)|_2^2 \le e^{2Ct}\, |w(0)|_2^2. \tag{4.5}$$

Therefore,

$$|w_1(t)|_2^2 \le e^{2Ct}\, |w(0)|_2^2. \tag{4.6}$$

Multiplying (4.4) by $w_2(t)$, integrating over $\Omega$ and using (1.2), (1.3), (1.4), (1.6) and the Cauchy inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w_2(t)|_2^2 + m\|w_2(t)\|^2 &\le L|\phi|_2\, |w|_2\, \|u_2\|\, \|w_2\| + \alpha|w_2|_2^2 \\ &\le \frac{m}{2}\|w_2(t)\|^2 + \frac{L^2|\phi|_2^2}{2m}\|u_2\|^2|w|_2^2 + \alpha|w_2|_2^2.\end{aligned}$$

Hence, using the Poincaré inequality and (4.3) lead to

$$\frac{d}{dt}|w_2(t)|_2^2 + (m\lambda_q - 2\alpha)|w_2(t)|_2^2 \le \frac{L^2|\phi|_2^2}{m}\rho_2|w|_2^2.$$

By the Gronwall inequality, we have

$$|w_2(t)|_2^2 \le e^{-(m\lambda_q-2\alpha)t}|w_2(0)|_2^2 + \frac{L^2|\phi|_2^2\rho_2}{m}e^{-(m\lambda_q-2\alpha)t}\int_0^t e^{(m\lambda_q-2\alpha)s}|w(s)|_2^2 ds. \tag{4.7}$$

Assume that $q$ is large enough such that $m\lambda_q - 2\alpha > 0$. It follows from (4.3), (4.5) and (4.7) that

$$\begin{aligned}|w_2(t)|_2^2 &\le e^{-(m\lambda_q-2\alpha)t}|w_2(0)|_2^2\\ &\quad + \frac{L^2|\phi|_2^2\rho_2}{m}e^{-(m\lambda_q-2\alpha)t}\int_0^t e^{(m\lambda_q-2\alpha+C)s}|w(0)|_2^2 ds\\ &\le e^{-(m\lambda_q-2\alpha)t}|w_2(0)|_2^2 + \frac{L^2|\phi|_2^2\rho_2 e^{2Ct}}{m(m\lambda_q-2\alpha+2C)}|w(0)|_2^2.\end{aligned}$$

Therefore,

$$|w_2(t)|_2^2 \le \left(e^{-(m\lambda_q-2\alpha)t} + \frac{L^2|\phi|_2^2\rho_2 e^{2Ct}}{m(m\lambda_q-2\alpha+2C)}\right)|w(0)|_2^2. \tag{4.8}$$

From (4.6) and (4.8), we have

$$|w_1(1)|_2^2 \le e^{2C}\ |w(0)|_2^2,\ \ |w_2(1)|_2^2 \le \delta^2\ |w(0)|_2^2,$$

where $\delta^2 = e^{-(m\lambda_q-2\alpha)} + \frac{L^2|\phi|_2^2\rho_2 e^{2C}}{m(m\lambda_q-2\alpha+2C)} < 1$ whenever $q$ is large enough. Using Theorem 4.1, one obtains the bound given in the statement of the theorem. □

## 5. Existence and exponential stability of stationary solutions

A weak stationary solution to problem (1.1) is an element $u^* \in H_0^1(\Omega)$ such that

$$a(l(u^*))\int_\Omega \nabla u^*\cdot\nabla v dx + \int_\Omega f(u^*)v dx = \int_\Omega g v dx$$

for all test functions $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$.

**Theorem 5.1.** *Under hypotheses* **(H1)**, **(H2)** *and* **(H3)**, *problem* (1.1) *has at least one weak stationary solution $u^*$ satisfying*

$$\|u^*\|^2 \le \ell(\tau_0), \tag{5.1}$$

*where*

$$\ell(\tau) = \frac{\lambda_1(4k_1\tau + k_2)}{4\tau(k_3-\tau)},$$

*with*

$$\tau_0 = \frac{\sqrt{k_2^2+4k_1k_2k_3}-k_2}{4k_1}\ \textit{for}\ k_1 = c_1|\Omega|,\ k_2 = |g|_2^2,\ k_3 = m\lambda_1-\mu.$$

*Moreover, if the following condition holds*

$$m\lambda_1 > \alpha + \sqrt{L^2\ |\phi|_2^2\ \ell(\tau_0)\ \lambda_1}, \tag{5.2}$$

*then for any weak solution $u$ of* (1.1), *we have*

$$|u(t)-u^*|_2^2 \le |u(0)-u^*|_2^2 e^{-2\Gamma_0 t}\ \textit{for all}\ t>0, \tag{5.3}$$

*where $\Gamma_0 = m\lambda_1 - \alpha - \sqrt{L^2|\phi|_2^2\ \ell(\tau_0)\lambda_1} > 0$. That is, the weak stationary solution of* (1.1) *is unique and exponentially stable.*

*Proof.* **i) Existence.** Assume that $\{e_j\}_{j=1}^{\infty} \subset H_0^1(\Omega) \cap L^\infty(\Omega)$ is an orthonormal Hilbert basis in $L^2(\Omega)$. For each $n \geq 1$, we denote $V_n = \mathrm{span}\{e_1, e_2, \dots, e_n\}$. We will find an approximate stationary solution $u_n$ by

$$u_n = \sum_{j=1}^{n} \beta_{nj} e_j,$$

such that

$$a(l(u_n)) \int_\Omega \nabla u_n \cdot \nabla v dx + \int_\Omega f(u_n) v dx = \int_\Omega g v dx \tag{5.4}$$

for all test functions $v \in V_n$. To do this, we define the following operators $R_n : V_n \to V_n$ by

$$[R_n u, v] = a(l(u)) \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega f(u) v dx - \int_\Omega g v dx$$

for all $u, v \in V_n$. Using the Cauchy inequality, (1.3) and (1.5), it follows that

$$\begin{aligned}
[R_n u, u] &= a(l(u)) \int_\Omega |\nabla u|^2 dx + \int_\Omega f(u) u dx - \int_\Omega g u dx \\
&\geq m\|u\|^2 - \mu|u|_2^2 - c_1|\Omega| - \int_\Omega g u dx \\
&\geq m\|u\|^2 - (\mu+\tau)|u|_2^2 - \left(c_1|\Omega| + \frac{|g|_2^2}{4\tau}\right) \\
&\geq \left(m - \frac{\mu+\tau}{\lambda_1}\right)\|u\|^2 - \left(c_1|\Omega| + \frac{|g|_2^2}{4\tau}\right) \\
&= \left(m - \frac{\mu+\tau}{\lambda_1}\right)\left(\|u\|^2 - \frac{\lambda_1(4\tau c_1|\Omega| + |g|_2^2)}{4\tau(m\lambda_1 - \mu - \tau)}\right)
\end{aligned}$$

for all $0 < \tau < m\lambda_1 - \mu$. We consider the following function

$$\ell(\tau) = \frac{\lambda_1(4k_1\tau + k_2)}{4\tau(k_3 - \tau)}, \ 0 < \tau < k_3,$$

with $k_1 = c_1|\Omega|$, $k_2 = |g|_2^2$, $k_3 = m\lambda_1 - \mu$. It attains the minimum value at

$$\tau_0 = \frac{\sqrt{k_2^2 + 4k_1k_2k_3} - k_2}{4k_1} \in (0, k_3).$$

Therefore,

$$[R_n u, u] \geq \left(m - \frac{\mu+\tau_0}{\lambda_1}\right)(\|u\|^2 - \ell(\tau_0)). \tag{5.5}$$

We deduce from (5.5) that $[R_n u, u] \geq 0$ for all $u \in V_n$ satisfying $\|u\| = \sqrt{\ell(\tau_0)}$. Consequently, by a corollary of the Brouwer fixed point theorem (see [25, Chapter 2, Lemma 1.4]), for each $n \geq 1$, there exists $u_n \in V_n$ such that $R_n(u_n) = 0$, and

$$\|u_n\|^2 \leq \ell(\tau_0). \tag{5.6}$$

We deduce from (5.6) that the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$, and by the compactness of injection of $H_0^1(\Omega)$ into $L^2(\Omega)$, we can extract a subsequence of $\{u_n\}$ (relabeled the same) such that

$$\begin{aligned} u_n &\rightharpoonup u^* \text{ in } H_0^1(\Omega), \\ u_n &\to u^* \text{ in } L^2(\Omega). \end{aligned}$$

Notice that $-a(l(u_n))\Delta u_n$ defines an element of $H^{-1}(\Omega)$, given by the duality

$$\langle -a(l(u_n))\Delta u_n, v\rangle = a(l(u_n))\int_\Omega \nabla u_n \cdot \nabla v dx$$

for all $v \in H_0^1(\Omega)$. Using (1.3) and (5.6), one can check that $\{-a(l(u_n))\Delta u_n\}$ is bounded in $H^{-1}(\Omega)$.

Since the continuity of $l$ and $a$ then $a(l(u_n)) \to a(l(u^*))$. It follows that

$$-a(l(u_n))\Delta u_n \rightharpoonup -a(l(u^*))\Delta u^* \quad \text{in} \quad H^{-1}(\Omega).$$

Taking $v = u_n$ in (5.4), we have

$$a(l(u_n))\|u_n\|^2 + \int_\Omega f(u_n)u_n dx = \int_\Omega g u_n dx.$$

Using (1.3), the Cauchy inequality and the Poincaré inequality, we obtain

$$\int_\Omega f(u_n)u_n dx \leq \frac{|g|_2^2}{2m\lambda_1}.$$

Putting $h(s) = f(s) - f(0) + \nu s$, where $\nu > \alpha$. By (1.6), one sees that $h(s)s \geq 0$ for all $s \in \mathbb{R}$. So, we have

$$\begin{aligned} \int_\Omega |h(u_n)|dx &\leq \int_{\Omega\cap\{|u_n|>1\}} |h(u_n)u_n|dx + \int_{\Omega\cap\{|u_n|\leq 1\}} |h(u_n)|dx \\ &\leq \int_\Omega h(u_n)u_n dx + \sup_{|s|\leq 1} |h(s)||\Omega| \\ &\leq \int_\Omega f(u_n)u_n dx + \nu \int_\Omega |u_n|^2 dx + |f(0)| \int_\Omega |u_n| dx + \sup_{|s|\leq 1} |h(s)||\Omega| \\ &\leq \frac{|g|_2^2}{2m\lambda_1} + (\nu+1)\,\frac{\ell(\tau_0)}{\lambda_1} + \frac{|f(0)|^2}{4}|\Omega| + \sup_{|s|\leq 1} |h(s)||\Omega|. \end{aligned}$$

This means that $\{h(u_n)\}$ is bounded in $L^1(\Omega)$, and so is $\{f(u_n)\}$. Since the strong convergence of $u_n$ to $u^*$ in $L^2(\Omega)$, $u_n \to u^*$ a.e. in $\Omega$. Hence, by Lemma 6.1 in [15], we have $f(u^*) \in L^1(\Omega)$ and

$$\int_\Omega f(u_n)\varphi dx \to \int_\Omega f(u^*)\varphi dx \quad \text{ for all } \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

Taking limits in (5.4), we obtain that $u^*$ is a weak stationary solution to (1.1). The estimate (5.1) directly follows from (5.6) as $n$ tends to infinity.

**ii) Uniqueness and exponential stability.** Denote $w(t) = u(t) - u^*$, we have

$$
\begin{aligned}
&(\frac{d}{dt}w, v) + a(l(u)) \int_\Omega \nabla w \cdot \nabla v dx + \langle \widehat{f}(u) - \widehat{f}(u^*), v\rangle \\
&= (a(l(u^*)) - a(l(u))) \int_\Omega \nabla u^* \cdot \nabla v dx + \alpha(w, v)
\end{aligned}
\tag{5.7}
$$

for all $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$, where $\widehat{f}(s) = f(s) + \alpha s$. Taking $v = \widehat{B}_k(w)$, where $\widehat{B}_k$ is defined as in the proof of Theorem 2.1, we obtain from (5.7) that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega w\widehat{B}_k(w)dx - \frac{1}{2}\frac{d}{dt}\int_\Omega |\widehat{B}_k(w)|^2 dx \\
&+ a(l(u)) \int_{\{x\in\Omega:|w|\le k\}} |\nabla w|^2 dx + \int_\Omega (\widehat{f}(u) - \widehat{f}(u^*))\widehat{B}_k(w)dx \\
&\le |a(l(u^*)) - a(l(u))| \left| \int_{\{x\in\Omega:|w|\le k\}} \nabla u^* \cdot \nabla w dx \right| + \alpha \int_\Omega w\widehat{B}_k(w)dx.
\end{aligned}
\tag{5.8}
$$

Here, we have used the fact that $w\dfrac{d}{dt}(\widehat{B}_k(w)) = \dfrac{1}{2}\dfrac{d}{dt}((\widehat{B}_k(w))^2)$.

Using (1.2) and (1.4) we have

$$
\begin{aligned}
&|a(l(u^*)) - a(l(u))| \left| \int_{\{x\in\Omega:|w|\le k\}} \nabla u^* \cdot \nabla w dx \right| \\
&\le L \int_\Omega |\phi|\ |w| dx \int_{\{x\in\Omega:|w|\le k\}} |\nabla u^* \cdot \nabla w| dx \\
&\le L|\phi|_2 \, \|u^*\| \ |w|_2 \, \|\widehat{B}_k(w)\|.
\end{aligned}
$$

In view of (5.1) and the Cauchy inequality, the last inequality leads to

$$
\begin{aligned}
&|a(l(u^*)) - a(l(u))| \left| \int_{\{x\in\Omega:|w(x,s)|\le k\}} \nabla u^* \cdot \nabla w dx \right| \\
&\le \rho\|\widehat{B}_k(w)\|^2 + \frac{L^2|\phi|_2^2\ \ell(\tau_0)}{4\rho}|w|_2^2
\end{aligned}
\tag{5.9}
$$

for all $\rho > 0$. Using (1.3), we obtain from (5.8) and (5.9) that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega w\widehat{B}_k(w)dx - \frac{1}{2}\frac{d}{dt}\int_\Omega |\widehat{B}_k(w)|^2 dx \\
&+ m \int_{\{x\in\Omega:|w|\le k\}} |\nabla w|^2 dx + \int_\Omega \widehat{f}'(\xi) w\widehat{B}_k(w)dx \\
&\le \rho\|\widehat{B}_k(w)\|^2 + \frac{L^2|\phi|_2^2\ \ell(\tau_0)}{4\rho}|w|_2^2 + \alpha \int_\Omega w\widehat{B}_k(w)dx.
\end{aligned}
$$

Since $\widehat{f'}(s) \geq 0$ and $s\widehat{B}_k(s) \geq 0$ for all $s \in \mathbb{R}$, if we choose $0 < \rho \leq m$, and let $k \to \infty$ in the above inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}|w|_2^2 + (m-\rho)\|w\|^2 \leq \frac{L^2|\phi|_2^2\,\ell(\tau_0)}{4\rho}|w|_2^2 + \alpha|w|_2^2.$$

So, using the Poincaré inequality, we get from the above inequality

$$\frac{d}{dt}|w|_2^2 + 2\left((m-\rho)\lambda_1 - \alpha - \frac{L^2|\phi|_2^2\,\ell(\tau_0)}{4\rho}\right)|w|_2^2 \leq 0. \tag{5.10}$$

Consider the following function on $(0, +\infty)$

$$\Gamma(\rho) = (m-\rho)\lambda_1 - \alpha - \frac{L^2|\phi|_2^2\ell(\tau_0)}{4\rho}.$$

It has the maximum value at

$$\rho_0 = \sqrt{\frac{L^2|\phi|_2^2\,\ell(\tau_0)}{4\lambda_1}},$$

and

$$\Gamma(\rho_0) = m\lambda_1 - \alpha - \sqrt{L^2|\phi|_2^2\,\ell(\tau_0)\lambda_1} =: \Gamma_0.$$

It follows from (5.2) that $\Gamma_0 > 0$ and $\rho_0 \in (0, m)$. So, if we choose $\rho = \rho_0$ then from (5.10) we have

$$\frac{d}{dt}|w|_2^2 + 2\Gamma_0|w|_2^2 \leq 0.$$

Hence, by using the Gronwall inequality, we get (5.3). $\square$

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