

Construction of bivariate asymmetric copulas

Saikat Mukherjee^a, Youngsaeng Lee^{1,b}, Jong-Min Kim^c, Jun Jang^d, Jeong-Soo Park^a

^aDepartment of Mathematics, National Institute of Technology, India;

^bDepartment of Statistics, Chonnam National University, Korea;

^cDivision of Science and Mathematics, University of Minnesota-Morris, USA;

^dCenter for Information Analysis, Chungnam National University, Korea

Abstract

Copulas are a tool for constructing multivariate distributions and formalizing the dependence structure between random variables. From copula literature review, there are a few asymmetric copulas available so far while data collected from the real world often exhibit asymmetric nature. This necessitates developing asymmetric copulas. In this study, we discuss a method to construct a new class of bivariate asymmetric copulas based on products of symmetric (sometimes asymmetric) copulas with powered arguments in order to determine if the proposed construction can offer an added value for modeling asymmetric bivariate data. With these newly constructed copulas, we investigate dependence properties and measure of association between random variables. In addition, the test of symmetry of data and the estimation of hyper-parameters by the maximum likelihood method are discussed. With two real example such as car rental data and economic indicators data, we perform the goodness-of-fit test of our proposed asymmetric copulas. For these data, some of the proposed models turned out to be successful whereas the existing copulas were mostly unsuccessful. The method of presented here can be useful in fields such as finance, climate and social science.

Keywords: Cramér-von Mises statistics, empirical copula, Fourier copula, maximum pseudo-likelihood estimation, parametric bootstrap, pseudo-observations

1. Introduction

Copulas offer a useful tool in modeling the dependence among random variables. For example, Busabodhin and Amphanthong (2016) applied copula in the multivariate statistical process control and Kim (2014) used copula-GARCH for the modeling of dependence structure of Korea financial markets. In the literature, most of the existing copulas, however, are symmetric while data collected from the real world may exhibit asymmetric nature. This necessitates developing asymmetric copulas. Many researchers proposed some methods to construct asymmetric copulas; Rodríguez-Lallena and Úbeda-Flores (2004) introduced a class of bivariate copulas that generalizes some known families. Kim *et al.* (2011) and Mesiar and Najjari (2014) extended the method of Rodríguez-Lallena and Úbeda-Flores (2004) to construct new families of symmetric and asymmetric copulas. Alfonsi and Brigo (2005) described a new construction method for asymmetric copulas based on periodic functions. Liebscher (2008) introduced two methods to construct asymmetric multivariate copulas, which is close to what Khoudraji (1995) proposed earlier (Quessy and Kortbi, 2016). The first is connected

¹ Corresponding author: Department of Statistics, Chonnam National University, 77 Yongbong-ro, Buk-gu, Gwangju 61186, Korea. E-mail: yslee82@ejnu.net

with products of copulas while the second one is a generalization of the Archimedean copulas family (Di Bernardino and Rullière, 2015). Durante (2009) suggested a method to construct asymmetric copulas based on products of copulas with powered arguments. Wu (2014) proposed a new method of constructing asymmetric copulas using a mixture of basic copulas and a convex combination of asymmetric copulas that can exhibit different tail dependence along different directions. Di Bernardino and Rullière (2015) constructed multivariate family of copulas by generalizing some known families by using a distortion matrix Σ .

In this study, we discuss a method to construct a new class of bivariate asymmetric copulas based on products of symmetric (sometimes asymmetric) copulas with powered arguments. Then we would like to determine if the proposed construction can offer an added value for modeling asymmetric data. This construction is based on the result of Durante (2009). Our proposal is actually an extension of Durante (2009) for a wide range of copulas which includes some newly constructed copulas in addition to all copula families available in the current literature. With these newly constructed copulas, we investigate dependence properties and measure of association between random variables. We consider the result of Mukherjee *et al.* (2015) in which they obtained meaningful results of the two non-parametric measures of association between two random variables, Spearman's rho (ρ) and Kendall's tau (τ), with the asymmetric copula family. In addition, to test the symmetry of data for using bivariate copulas, we use Cramér-von Mises criterion suggested by Genest *et al.* (2012). Moreover, the estimation of hyper-parameters by the maximum likelihood method are discussed.

This paper is organized as follows. Section 2 contains some basic concepts of copulas and the dependence structure by calculating Spearman's rho and Kendall's tau using asymmetric copulas. In Section 3, we introduce Fourier copula and new class of bivariate asymmetric copulas. Goodness-of-fit of the proposed asymmetric copulas is introduced in Section 4. Test of symmetry for bivariate case and the maximum likelihood estimation of hyper-parameters for the constructed copulas are discussed in Sections 5 and 6, respectively. Section 7 shows the illustrative data analysis for the proposed asymmetric copula models with two real data. Finally, the discussion and conclusion are presented in Section 8.

2. Definition and preliminary

In this section we recall some definitions and results that are necessary to understand a (bivariate) copula. A copula is a multivariate distribution function defined on \mathbb{I}^n , where $\mathbb{I} := [0, 1]$, with uniformly distributed marginals. In this paper, we focus on bivariate copulas.

Definition 1. A bivariate copula is a function $C : \mathbb{I}^2 \rightarrow \mathbb{I}$, which satisfies the following properties:

$$(P1) \quad C(0, v) = C(u, 0) = 0, \quad \forall u, v \in \mathbb{I}$$

$$(P2) \quad C(1, u) = C(u, 1) = u, \quad \forall u \in \mathbb{I}$$

$$(P3) \quad C \text{ is 2-increasing, i.e., } \forall u_1, u_2, v_1, v_2 \in \mathbb{I} \text{ with } u_1 \leq u_2, v_1 \leq v_2,$$

$$C(u_2, v_2) + C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) \geq 0.$$

The importance of copulas has been growing because of their applications in several fields of research. Their relevance primarily comes from Sklar's Theorem (Sklar, 1959): *If X and Y are two continuous random variables with joint distribution function H and marginal distribution functions F and G , respectively, then there exists a unique copula C such that $H(x, y) = C(F(x), G(y))$ for all*

$(x, y) \in \mathbb{R}^2$ and conversely, given a copula C and two univariate distribution functions F and G , the function H defined above is a joint distribution function with margins F and G . Sklar's Theorem clarifies the role that copulas play in the relationship between multivariate distribution functions and their univariate margins. A proof of this theorem can be found in Schweizer and Sklar (1983).

Definition 2. Suppose X and Y are two random variables with marginal distribution functions F and G , respectively. Then Spearman's rho is the ordinary (Pearson) correlation coefficient of the transformed random variables $F(X)$ and $G(Y)$, while Kendall's tau is the difference between the probability of concordance $\Pr[(X_1 - X_2)(Y_1 - Y_2) > 0]$ and the probability of discordance $\Pr[(X_1 - X_2)(Y_1 - Y_2) < 0]$ for two independent pairs (X_1, Y_1) and (X_2, Y_2) of observations drawn from the distribution.

In terms of dependence properties, Spearman's rho is a measure of average quadrant dependence, while Kendall's tau is a measure of average likelihood ratio dependence (see Nelsen (2006) for details). If X and Y are two continuous random variables with copula C , then Spearman's rho and Kendall's tau of X and Y are given by,

$$\rho = 12 \iint_{\mathbb{I}^2} C(u, v) du dv - 3, \quad (2.1)$$

$$\tau = 4 \iint_{\mathbb{I}^2} C(u, v) dC(u, v) - 1. \quad (2.2)$$

Definition 3. A copula C is called absolutely continuous if, when considered as a joint distribution function, $C(u, v)$ has a joint density function given by $c(u, v) := \partial^2 C / (\partial u \partial v)$ and in that case $dC(u, v) = \partial^2 C / (\partial u \partial v) du dv$.

Denoting $c(u, v) - 1$ as $h(u, v)$, the following theorem gives a characterization of absolutely continuous copulas (De la Peña *et al.*, 2006).

Theorem 1. A function $C : \mathbb{I}^2 \rightarrow \mathbb{I}$ is an absolutely bivariate copula only if there exists a function $h : \mathbb{I}^2 \rightarrow \mathbb{I}$, satisfying the following conditions,

1. Integrability: $\iint_{\mathbb{I}^2} |h(x, y)| dx dy < \infty$,
2. Degeneracy: $\int_{\mathbb{I}} h(x, \xi) d\xi = \int_{\mathbb{I}} h(\xi, y) d\xi = 0 \forall x, y \in \mathbb{I}$,
3. Positive Definiteness: $h(x, y) \geq -1 \forall (x, y) \in \mathbb{I}^2$,

and such that $C(u, v) = \int_0^v \int_0^u 1 + h(x, y) dx dy$.

A copula C is called symmetric if $C(u, v) = C(v, u)$ for all $u, v \in \mathbb{I}$, otherwise asymmetric. Let us denote the independent copula as $\Pi(u, v) := uv$. In addition, the new asymmetric copulas satisfying all the hypothesis of Theorem 1 were proposed in Mukherjee *et al.* (2015):

$$\begin{aligned} C_{\max}^{\varepsilon}(u, v) &= \Pi(u, v) \\ &\quad + \frac{1}{4} \left(\sqrt{1 + 4\varepsilon^2} - \sqrt{(1 - 2u)^2 + 4\varepsilon^2} \right) \left(\sqrt{1 + 4\varepsilon^2} - \sqrt{(1 - 2v)^2 + 4\varepsilon^2} \right), \\ C_{\min}^{\varepsilon}(u, v) &= \Pi(u, v) \\ &\quad - \frac{1}{4} \left(\sqrt{1 + 4\varepsilon^2} - \sqrt{(1 - 2u)^2 + 4\varepsilon^2} \right) \left(\sqrt{1 + 4\varepsilon^2} - \sqrt{(1 - 2v)^2 + 4\varepsilon^2} \right). \end{aligned}$$

Then corresponding Spearman's rho and Kendall's tau are given by, respectively,

$$\begin{aligned}\rho_{\max}^{\varepsilon} &= \frac{3}{4} \left(\sqrt{1+4\varepsilon^2} - 4\varepsilon^2 \coth^{-1} \left(\sqrt{1+4\varepsilon^2} \right) \right)^2, \\ \rho_{\min}^{\varepsilon} &= -\frac{3}{4} \left(\sqrt{1+4\varepsilon^2} - 4\varepsilon^2 \coth^{-1} \left(\sqrt{1+4\varepsilon^2} \right) \right)^2, \\ \tau_{\max}^{\varepsilon} &= \frac{1}{2} \left[1 + 4\varepsilon^2 + 4\varepsilon^2 \left(\sqrt{1+4\varepsilon^2} - 2\varepsilon^2 \coth^{-1} \left(\sqrt{1+4\varepsilon^2} \right) \right) \ln \left(\frac{1+2\varepsilon^2 - \sqrt{1+4\varepsilon^2}}{2\varepsilon^2} \right) \right], \\ \tau_{\min}^{\varepsilon} &= -\frac{1}{2} \left[1 + 4\varepsilon^2 + 4\varepsilon^2 \left(\sqrt{1+4\varepsilon^2} - 2\varepsilon^2 \coth^{-1} \left(\sqrt{1+4\varepsilon^2} \right) \right) \ln \left(\frac{1+2\varepsilon^2 - \sqrt{1+4\varepsilon^2}}{2\varepsilon^2} \right) \right].\end{aligned}$$

The optimal values of ρ and corresponding τ are obtained by letting $\varepsilon \rightarrow 0$. Mukherjee *et al.* (2015) showed how the values of ρ approach the optimal values as $\varepsilon \rightarrow 0$ and it is clear that $-0.75 \leq \rho \leq 0.75$ and $-0.5 \leq \tau \leq 0.5$.

3. Construction of asymmetric copulas

In this section we will first define Fourier copulas (Lowin, 2010) and then construct asymmetric (in general) copulas using the following theorem (see Durante (2009) for details).

Theorem 2. For all $\alpha, \beta \in (0, 1)$, and for all copulas A and B , the function $C_{\alpha, \beta} : \mathbb{I}^2 \rightarrow \mathbb{I}$, defined by

$$C_{\alpha, \beta}(u, v) = A(u^{\alpha}, v^{\beta}) B(u^{\bar{\alpha}}, v^{\bar{\beta}})$$

is a copula, where $\bar{\alpha} = 1 - \alpha$ and $\bar{\beta} = 1 - \beta$.

3.1. Fourier copula

It is natural to write the function h in Theorem 1 as a Fourier series as follows

$$h(x, y) = \sum_{m, n \in \mathbb{Z}^0} \gamma_m^n \exp(2\pi i(nx + my)), \quad \forall (x, y) \in \mathbb{I}^2,$$

where $\mathbb{Z}^0 = \mathbb{Z} \setminus \{0\}$ and $\sum_{m, n \in \mathbb{Z}^0} |\gamma_m^n| < \infty$ with $\gamma_{-m}^n = \overline{\gamma_m^{-n}}$, $\forall n, m \in \mathbb{Z}^0$. The latter condition guarantees that h is real valued. Then the integrability and degeneracy of h are clear. For positive definiteness, suppose γ_m^n are chosen so that $h(u, v) \geq -1$ for all $u, v \in \mathbb{I}$, then the copula generated by h , defined by

$$\begin{aligned}C_F(u, v) &= \Pi(u, v) + \int_{t=0}^v \int_{s=0}^u h(s, t) ds dt \\ &= \Pi(u, v) + \int_{t=0}^v \int_{s=0}^u \sum_{m, n \in \mathbb{Z}^0} \gamma_m^n \exp(2\pi i(nx + my)) ds dt \\ &= \Pi(u, v) - \frac{1}{4\pi^2} \sum_{m, n \in \mathbb{Z}^0} \frac{\gamma_m^n}{mn} (e^{2\pi i nu} - 1)(e^{2\pi i mv} - 1)\end{aligned}$$

is called a Fourier copula, which was apparently introduced by Ibragimov (2009). It is sufficient that if

$$\sum_{n, m \in \mathbb{N}} |\gamma_m^n| + |\gamma_m^{-n}| \leq \frac{1}{2}, \quad (3.1)$$

then h is positive definite and will generate a Fourier copula C_F as mentioned above. Using (2.1) and (2.2), Spearman's rho and Kendall's tau of a Fourier copula C_F are given by $\rho = -(3/\pi^2) \sum_{m,n \in \mathbb{Z}^0} (\gamma_m^n / mn)$ and $\tau = -(1/\pi^2) \sum_{m,n \in \mathbb{Z}^0} \{(2\gamma_m^n + |\gamma_m^n|^2)/mn\}$. The last equality follows from the assumption that $\gamma_{-m}^n = \overline{\gamma_m^{-n}}$, $\forall n, m \in \mathbb{Z}^0$. One can show that

$$-\frac{6}{\pi^2} \sum_{m,n \in \mathbb{N}} (|\gamma_m^n| + |\gamma_m^{-n}|) \leq \rho \leq \frac{6}{\pi^2} \sum_{m,n \in \mathbb{N}} (|\gamma_m^n| + |\gamma_m^{-n}|),$$

$$-\frac{2}{\pi^2} \sum_{m,n \in \mathbb{N}} (2|\gamma_m^n| + 3|\gamma_m^{-n}|) < \tau < \frac{2}{\pi^2} \sum_{m,n \in \mathbb{N}} (3|\gamma_m^n| + 2|\gamma_m^{-n}|)$$

and hence using (3.1) we have, $|\rho| \leq 3/\pi^2 \approx 0.304$ and $|\tau| < 3/\pi^2 \approx 0.304$. Even though Fourier copulas are in general asymmetric in nature, the above results show its applications are quite limited. In the following subsection we will construct asymmetric copulas by utilizing the existing copulas including Fourier with mind of convenient application.

3.2. New class of bivariate asymmetric copulas

In this subsection we use Theorem 2 to construct a class of asymmetric copulas and will find corresponding Spearman's rho and Kendall's tau for these new copulas to have a qualitative idea of which asymmetric copula has a better range of ρ and τ values. In Durante (2009), the author mentions that Theorem 2 will generate an asymmetric copula for all $\alpha, \beta \in (0, 1)$ with $\alpha \neq 1/2$ or $\beta \neq 1/2$. But if the copulas A and B in Theorem 2 are symmetric then we have,

$$C_{\alpha,\beta}(v, u) = A(v^\alpha, u^\beta) B(v^{\bar{\alpha}}, u^{\bar{\beta}}) = A(u^\beta, v^\alpha) B(u^{\bar{\beta}}, v^{\bar{\alpha}}) = C_{\beta,\alpha}(u, v).$$

Therefore we would like to mention here that $C_{\alpha,\beta}$ in Theorem 2 can be symmetric if $\alpha = \beta$ and hence in our case we will choose $\alpha \neq \beta$. The following lemma will give an interesting symmetric behavior of ρ and τ .

Lemma 1. *If A and B are two symmetric copulas and $\alpha, \beta \in (0, 1)$, then*

$$\rho(C_{\alpha,\beta}) = \rho(C_{\beta,\alpha}) \quad \text{and} \quad \tau(C_{\alpha,\beta}) = \tau(C_{\beta,\alpha}),$$

where $\rho(C_{\alpha,\beta})$, $\tau(C_{\alpha,\beta})$ are Spearman's rho, Kendall's tau of $C_{\alpha,\beta}$, respectively, and

$$C_{\alpha,\beta}(u, v) = A(u^\alpha, v^\beta) B(u^{\bar{\alpha}}, v^{\bar{\beta}}).$$

Proof: The symmetry of ρ follows from the fact that $C_{\alpha,\beta}(v, u) = C_{\beta,\alpha}(u, v)$. To show that $\tau(C_{\alpha,\beta}) = \tau(C_{\beta,\alpha})$, first recall that

$$\tau(C) = 1 - 4 \iint_{\mathbb{I}^2} \frac{\partial C}{\partial u} \frac{\partial C}{\partial v} du dv$$

is equivalent to (2.2). Secondly notice that

$$\begin{aligned} \frac{\partial}{\partial u} C_{\alpha,\beta}(u, v) &= \frac{\partial}{\partial u} A(u^\alpha, v^\beta) B(u^{\bar{\alpha}}, v^{\bar{\beta}}) \\ &= A(u^\alpha, v^\beta) \frac{\partial}{\partial u} B(u^{\bar{\alpha}}, v^{\bar{\beta}}) + B(u^{\bar{\alpha}}, v^{\bar{\beta}}) \frac{\partial}{\partial u} A(u^\alpha, v^\beta) \\ &= A(v^\beta, u^\alpha) \frac{\partial}{\partial v} B(v^{\bar{\beta}}, u^{\bar{\alpha}}) + B(v^{\bar{\beta}}, u^{\bar{\alpha}}) \frac{\partial}{\partial v} A(v^\beta, u^\alpha) \end{aligned}$$

Table 1: Nine basic Copula Functions used in this study to construct bivariate asymmetric copulas

Copula name	Copula function
Fourier	$C1(u, v) := C_F(u, v)$
Max	$C2(u, v) := C_{\max}^{\varepsilon_1}(u, v), \quad \varepsilon_1 > 0$
Min	$C3(u, v) := C_{\min}^{\varepsilon_2}(u, v), \quad \varepsilon_2 > 0$
Independent	$C4(u, v) := \Pi(u, v) = uv$
FGM	$C5(u, v) := \text{FGM}(u, v, \theta_1) = uv + \theta_1 uv(1-u)(1-v), \quad \theta_1 \in (-1, 1]$
Clayton	$C6(u, v) := \text{Clayton}(u, v, \theta_2) = (u^{-\theta_2} + v^{-\theta_2} - 1)^{-1/\theta_2}, \quad \theta_2 \in (0, \infty)$
Frank	$C7(u, v) := \text{Frank}(u, v, \theta_3) = -\frac{1}{\theta_3} \log \left[1 + \frac{(e^{-\theta_3 u} - 1)(e^{-\theta_3 v} - 1)}{e^{-\theta_3} - 1} \right], \quad \theta_3 \in \mathbb{R} \setminus \{0\}$
Gumbel	$C8(u, v) := \text{Gumbel}(u, v, \theta_4) = \exp \left[-\left((-\log u)^{\theta_4} + (-\log v)^{\theta_4} \right)^{1/\theta_4} \right], \quad \theta_4 \geq 1$
AMH	$C9(u, v) := \text{AMH}(u, v, \theta_5) = \frac{uv}{1 - \theta_5(1-u)(1-v)}, \quad \theta_5 \in (-1, 1]$

FGM = Farlie-Gumbel-Morgenstern family; AMH = Ali-Mikhail-Haq family.

$$\begin{aligned}
 &= \frac{\partial}{\partial v} A(v^\beta, u^\alpha) B(v^{\bar{\beta}}, u^{\bar{\alpha}}) \\
 &= \frac{\partial}{\partial v} C_{\beta, \alpha}(v, u).
 \end{aligned}$$

Hence we have

$$\begin{aligned}
 \tau(C_{\alpha, \beta}) &= 1 - 4 \iint_{\mathbb{I}^2} \frac{\partial C_{\alpha, \beta}}{\partial u} \frac{\partial C_{\alpha, \beta}}{\partial v} du dv \\
 &= 1 - 4 \iint_{\mathbb{I}^2} \frac{\partial C_{\beta, \alpha}}{\partial v} \frac{\partial C_{\beta, \alpha}}{\partial u} du dv \\
 &= \tau(C_{\beta, \alpha}).
 \end{aligned}$$

□

For convenience we adopt the following notations, for $j = 1, 2, \dots$, copulas C_j are defined in Table 1. The list of copulas in Table 1 is considered in this study. We define the set of parameters ψ and the copulas that arise from Theorem 2 as,

$$\begin{aligned}
 \psi &:= \{\gamma_m^n, \varepsilon_1, \varepsilon_2, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5\} \\
 C_j k_{\alpha, \beta}(u, v) &:= C_j(u^\alpha, v^\beta) C_k(u^{\bar{\alpha}}, v^{\bar{\beta}}), \quad \text{for } j, k = 1, 2, \dots, 9.
 \end{aligned}$$

Notice that $C_j k_{\alpha, \beta} \equiv C_k j_{\bar{\alpha}, \bar{\beta}}$.

For $\psi = \{0.5\delta_m^n (\delta_m^1 + \delta_n^{-1}), 0.01, 0.01, 1, 20, 30, 20, 1\}$, where δ_m^n is the Kronecker delta, we have calculated (Mathematica code and the results of many other different cases can be found at <http://goo.gl/plkJ7>). Spearman’s rho and Kendall’s tau of the copulas $C_j k_{\alpha, \beta}$, for $j, k = 1, 2, \dots, 9; j < k$ with different α, β values. In general, our results show that ρ, τ values stay away from zero if $(\alpha, \beta) \approx (0, 0)$ or/and $(1, 1)$. For instance, we would like to mention ρ, τ values for two copulas $C_{12_{\alpha, \beta}}$ and $C_{17_{\alpha, \beta}}$ (Table 2 and Table 3).

Figures 1 and 2 clearly show that the contour plots of $C_{12_{\alpha, \beta}}$ and $C_{17_{\alpha, \beta}}$ are asymmetric. In this article, the authors just showed the contour plots of two asymmetric copulas, but readers can download the *Mathematica* code from the linked website and reproduce the contour plots of the other remaining asymmetric copulas. So depending on the readers’ provided data, readers can choose one of

Table 2: ρ, τ values (correct up to 2 decimal places) for $C12_{\alpha,\beta}$

α		β							
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.2	ρ	0.62							
	τ	0.41							
0.3	ρ	0.56	0.51						
	τ	0.37	0.34						
0.4	ρ	0.49	0.44	0.38					
	τ	0.32	0.29	0.25					
0.5	ρ	0.41	0.36	0.30	0.23				
	τ	0.27	0.24	0.20	0.15				
0.6	ρ	0.33	0.28	0.21	0.15	0.08			
	τ	0.22	0.18	0.14	0.10	0.05			
0.7	ρ	0.24	0.19	0.13	0.07	0.01	-0.05		
	τ	0.16	0.13	0.09	0.04	0.00	-0.04		
0.8	ρ	0.15	0.10	0.05	-0.00	-0.06	-0.11	-0.15	
	τ	0.10	0.07	0.03	-0.01	-0.05	-0.08	-0.12	
0.9	ρ	0.06	0.02	-0.02	-0.07	-0.12	-0.16	-0.20	-0.23
	τ	0.04	0.01	-0.02	-0.05	-0.09	-0.12	-0.15	-0.18

Table 3: ρ, τ values (correct up to 2 decimal places) for $C17_{\alpha,\beta}$

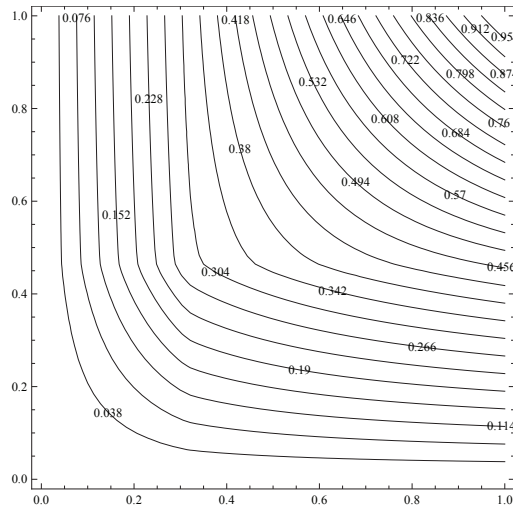
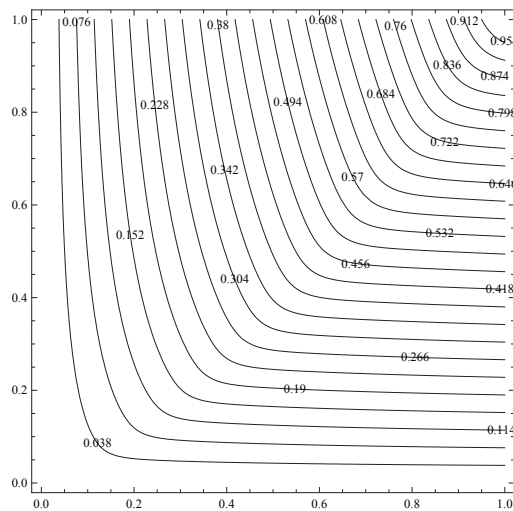
α		β							
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.2	ρ	0.81							
	τ	0.67							
0.3	ρ	0.73	0.68						
	τ	0.59	0.54						
0.4	ρ	0.65	0.60	0.54					
	τ	0.51	0.47	0.41					
0.5	ρ	0.57	0.52	0.46	0.38				
	τ	0.43	0.39	0.34	0.28				
0.6	ρ	0.47	0.42	0.36	0.30	0.23			
	τ	0.35	0.31	0.26	0.21	0.16			
0.7	ρ	0.36	0.32	0.27	0.21	0.15	0.08		
	τ	0.26	0.23	0.19	0.14	0.09	0.05		
0.8	ρ	0.25	0.21	0.16	0.11	0.06	0.00	-0.05	
	τ	0.17	0.14	0.11	0.07	0.03	-0.01	-0.05	
0.9	ρ	0.12	0.09	0.04	0.00	-0.05	-0.09	-0.13	-0.17
	τ	0.08	0.06	0.02	-0.01	-0.05	-0.08	-0.11	-0.15

the proposed asymmetric copula by looking at the contour plots of all proposed asymmetric copulas. Figure 3 is scatter plots of random numbers generated from the nine basic copulas. Figure 4 is scatter plots of random numbers generated from some of the constructed asymmetric copulas. These figures may be helpful to choose which copula will be appropriate to fit the given data well.

4. Estimation and goodness-of-fit

4.1. Fitting copulas to data

We assume that we have a random sample $\mathbf{X}_1, \dots, \mathbf{X}_n$ from a d-dimensional cumulative distribution function (CDF) F with continuous marginal CDFs F_1, \dots, F_d . Hence, F has the unique representation, $F(x_1, \dots, x_d) = C[F_1(x_1), \dots, F_d(x_d)]$, by Sklar's Theorem. Let $\hat{F}_1, \dots, \hat{F}_d$ denote the rescaled empirical CDFs computed from the data as that for every $j \in \{1, \dots, d\}$, $\hat{F}_j(x) = \{1/(n + 1)\} \sum_{i=1}^n \mathbf{1}(X_{i,j} \leq x)$

Figure 1: Contour plot of $C12_{\alpha,\beta}$ with $\alpha = 0.4, \beta = 0.1$.Figure 2: Contour plot of $C17_{\alpha,\beta}$ with $\alpha = 0.1, \beta = 0.4$.

$\leq x$). The rescaled empirical CDFs differ from the usual empirical CDF by the use of denominator $n + 1$ rather than n . This guarantees that the pseudo-observations lie strictly in the interior of $[0, 1]^d$.

The maximum pseudo-likelihood estimate (MPLE) of θ is obtained by maximizing the log pseudo-likelihood with respect to θ ;

$$\log L(\theta; \hat{\mathbf{U}}_1, \dots, \hat{\mathbf{U}}_n) = \sum_{i=1}^n \log c_{\theta}(\hat{\mathbf{U}}_i), \quad (4.1)$$

where c_{θ} denotes the copula density (Kojadinovic, 2013), and

$$\hat{\mathbf{U}}_i = (\hat{U}_{i,1}, \dots, \hat{U}_{i,d}) = (\hat{F}_1(X_{i,1}), \dots, \hat{F}_d(X_{i,d})) \quad (4.2)$$

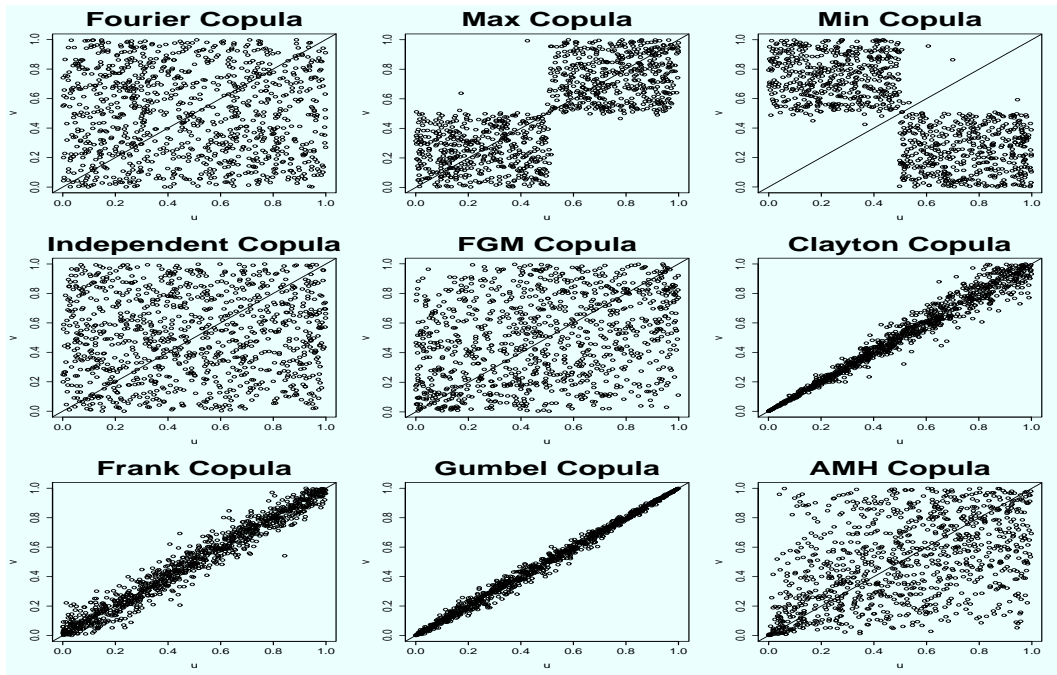


Figure 3: Scatter plots of random numbers generated from the nine basic copulas.

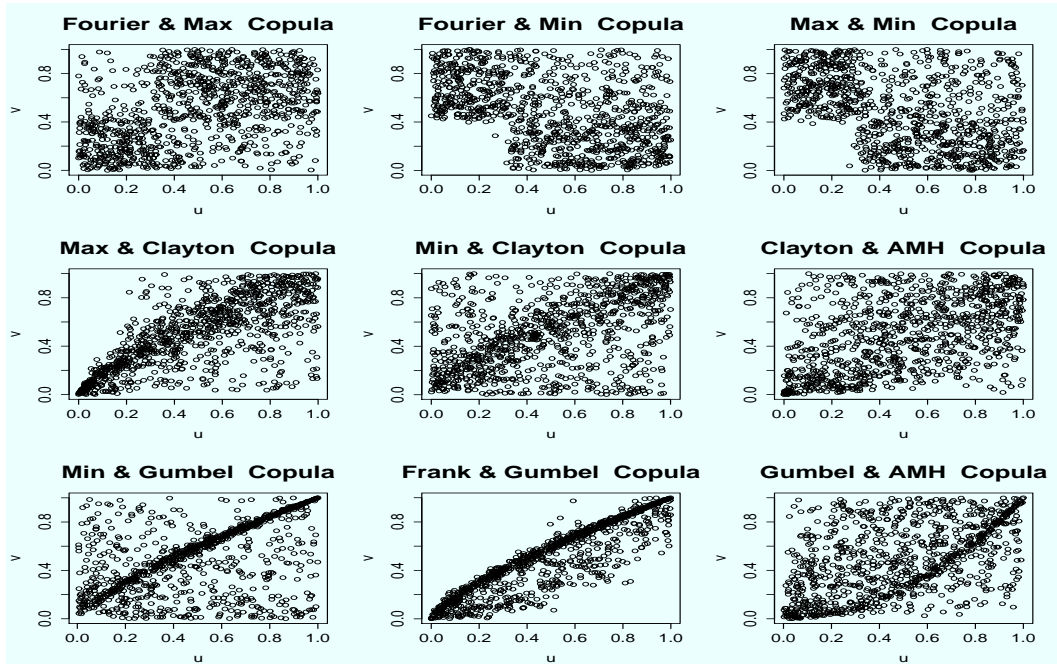


Figure 4: Scatter plots of random numbers generated from the proposed asymmetric copulas. Only nine copula cases are shown here as a sample.

are the pseudo-observations. The estimate is generally found by numerical maximization of (4.1). For the computation in this paper, we used a R package ‘copula’ developed by Kojadinovic and Yan (2010) for basic symmetric copulas and our own R program for constructed asymmetry copulas.

4.2. Goodness-of-fit test

A rigorous approach to compare the fit of different copulas to the same data consists of using goodness-of-fit tests. The issue is whether the unknown copula C actually belongs to the chosen parametric copula family $C_0 = \{C_\theta\}$ or not. Formally, one wants to test $H_0 : C \in C_0$ vs. $H_1 : C \notin C_0$. A relatively large number of testing procedures have been proposed in the literature as can be concluded from the review of Genest *et al.* (2009). One approach that appears to perform particularly well according to recent large scale simulations is based on the empirical copula of the data $\mathbf{X}_1, \dots, \mathbf{X}_n$, which is defined by

$$C_n(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^n l(\hat{\mathbf{U}}_i < \mathbf{u}), \quad \mathbf{u} \in [0, 1]^d, \quad (4.3)$$

where the random vectors $\hat{\mathbf{U}}_i$ are the pseudo-observations as in (4.2). The empirical copula C_n is a consistent estimator of the unknown copula C whether H_0 is true or not. Hence, as suggested by several authors, a natural goodness-of-fit test consists of comparing C_n with an estimation C_{θ_n} of C obtained assuming that $C \in C_0$ holds Kojadinovic (2013), where θ_n is an estimator of θ computed from the pseudo-observations $\hat{\mathbf{U}}_1, \dots, \hat{\mathbf{U}}_n$. Precisely, it was proposed to base a test of goodness-of-fit on the empirical process

$$\mathbb{C}_n(\mathbf{u}) = \sqrt{n} \{C_n(\mathbf{u}) - C_{\theta_n}(\mathbf{u})\}, \quad \mathbf{u} \in [0, 1]^d. \quad (4.4)$$

According to the large scale simulations carried out in Genest *et al.* (2009), the most powerful version of this procedure is based on the following Cramér-von Mises statistic:

$$S_n = \int_{[0,1]^d} \mathbb{C}_n(\mathbf{u})^2 dC_n(\mathbf{u}) = \sum_{i=1}^n \{C_n(\hat{\mathbf{U}}_i) - C_{\theta_n}(\hat{\mathbf{U}}_i)\}^2. \quad (4.5)$$

An approximate p -value for S_n can be obtained by means of the parametric bootstrap-based procedure (see Genest *et al.* (2009) for the details omitted here). This procedure is computationally very intensive. Thus, as n reaches 300, the extensive Monte Carlo experiments carried for $d = 2, 3, 4$ in Kojadinovic *et al.* (2011) indicate that one can safely use the fast multiplier approach as an alternative.

5. Test of symmetry for bivariate data

This section briefly deals with methods to test the symmetry of bivariate data. For that, it is reasonable to compare values of $\hat{C}_n(u, v)$ and $\hat{C}_n(v, u)$. Base on this idea, for the test the hypothesis of exchangeability data, Genest *et al.* (2012) suggested three measures as:

$$R_n = \int_0^1 \int_0^1 \{\hat{C}_n(u, v) - \hat{C}_n(v, u)\}^2 dv du, \quad (5.1)$$

$$S_n^* = \int_0^1 \int_0^1 \{\hat{C}_n(u, v) - \hat{C}_n(v, u)\}^2 d\hat{C}_n(v, u), \quad (5.2)$$

$$T_n = \sup_{(u,v) \in [0,1]^2} |\hat{C}_n(u, v) - \hat{C}_n(v, u)|. \quad (5.3)$$

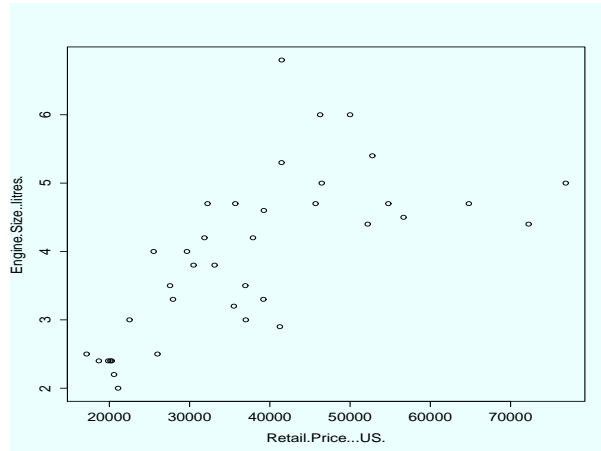


Figure 5: Scatter plot of car rental data.

See also Bouzebda and Cherfi (2012) and Quessy and Bahraoui (2013) for other test procedures for the symmetry of copulas. Nelsen (2007) considered another measure of asymmetry. In this study, we use a Cramér-von Mises statistic S_n^* as a measure to check asymmetry of bivariate data for the computational convenience. The 'exchTest' function of the 'copula' package in R program was used for the calculation of the S_n^* .

6. Estimation of hyper-parameters in constructed asymmetric copula

In the previous section, we estimated the parameters of copulas with the powered hyper-parameters α and β . In this section, we explain how to estimate simultaneously the parameter in copulas as well as the hyper-parameters of constructed asymmetric copula.

By the equation (4.1), the log pseudo-likelihood of the constructed copula by copulas C_1 and C_2 is as:

$$\log L(\theta_1, \theta_2, \alpha, \beta; (\hat{u}_1, \hat{v}_1), \dots, (\hat{u}_n, \hat{v}_n)) = \sum_{i=1}^n \log C_1(u_1^\alpha, v_1^\beta) C_2(u_1^{\bar{\alpha}}, v_1^{\bar{\beta}}), \quad (6.1)$$

where (\hat{u}_i, \hat{v}_i) is i^{th} pseudo-observation and θ_1 and θ_2 are the parameters of copulas C_1 and C_2 , respectively. We estimated the parameters $\theta_1, \theta_2, \alpha, \beta$ by maximizing (6.1), simultaneously. For this optimization computation, we used a quasi-Newton algorithm with numerical differentiation in a 'L-BFGS-B' method in R function 'optim'.

7. Real data example

7.1. Car rental data

We consider two datasets to illustrate the usefulness of our proposed asymmetric copulas. The first dataset is car rental data of American new cars and trucks data for sport utility vehicle (SUV) with four wheel drive which is available at Nayland College. Engine size and retail price variables with sample size $n = 38$ are considered for this study. Figure 5 is a scatter plot of two variables, engine size and retail price. For the symmetry test on this data, we have $S_n^* = 0.056$ with p -value = 0.004 as described in Section 5, which means the data is not symmetric.

Table 4: Result of parameter estimates, values of BIC, AIC, and S_n with approximated p -values for nine basic copulas on the car rental data

Copula name	Parameter	BIC	AIC	S_n	p -value
Frank	7.328	-29.328	-30.966	0.035	0.143
Clayton	1.972	-25.367	-27.005	0.066	0.029
Max	0.113	-20.905	-22.543	0.094	0.007
AMH	1.000	-19.877	-21.515	0.155	0.001
Gumbel	1.959	-18.270	-19.907	0.071	0.042
FGM	1.000	-11.796	-13.433	0.243	0.001
Independent		0.000	0.000	0.505	0.001
Min	45423.710	3.638	2.000	0.505	0.001
Fourier	0.000	3.640	2.003	0.505	0.001

BIC = Bayesian information criterion; AIC = Akaike information criterion; S_n = Cramér-von Mises goodness-of-fit statistics.

Table 5: Result of parameter estimates, values of BIC, AIC, and S_n with approximated p -values for some combined asymmetric copulas on the car rental data

Copula name	par1	par2	α	β	BIC	AIC	S_n	p -value
Max \times Clayton	0.001	7.905	0.448	0.423	-26.315	-32.865	0.055	0.197
Independent \times Frank		7.195	0.001	0.001	-22.024	-26.937	0.036	0.276
Clayton \times Frank	13.119	7.647	0.334	0.410	-21.499	-28.050	0.031	0.356
Independent \times Clayton		4.280	0.186	0.071	-21.144	-26.056	0.049	0.216
Frank \times Gumbel	8.691	21.269	0.804	0.948	-20.853	-27.401	0.480	0.351
Fourier \times Clayton	0.999	4.950	0.228	0.124	-20.202	-26.752	0.049	0.177
Clayton \times Gumbel	3.715	20.611	0.754	0.934	-20.195	-26.745	0.042	0.201
Max \times Frank	3.571	7.360	0.001	0.001	-18.397	-24.947	0.035	0.425
Min \times Frank	3.578	7.359	0.001	0.001	-18.397	-24.947	0.035	0.336
FGM \times Frank	0.020	7.320	0.001	0.001	-18.397	-24.947	0.035	0.311
Fourier \times Frank	0.867	7.245	0.001	0.001	-18.393	-24.944	0.036	0.311
Frank \times AMH	7.586	0.849	0.999	0.999	-18.371	-24.922	0.033	0.391
Min \times Clayton	0.001	4.232	0.200	0.052	-18.197	-24.747	0.054	0.142
Max \times Gumbel	0.001	2.742	0.608	0.462	-17.735	-24.286	0.061	0.067
FGM \times Clayton	-0.999	4.185	0.173	0.058	-17.687	-24.237	0.049	0.187
Clayton \times AMH	4.181	-0.999	0.827	0.943	-17.658	-24.208	0.049	0.192
Max \times Independent	0.113		0.996	0.999	-13.571	-18.484	0.094	0.017
Independent \times AMH		0.999	0.001	0.001	-12.558	-17.471	0.155	0.001
Independent \times Gumbel		1.961	0.000	0.000	-10.988	-15.900	0.071	0.047
Fourier \times Max	0.449	0.113	0.001	0.001	-9.945	-16.495	0.094	0.022
Max \times FGM	0.113	-0.601	0.991	0.999	-9.880	-16.431	0.095	0.022
Max \times AMH	0.113	-0.601	0.991	0.999	-9.880	-16.431	0.095	0.167
Max \times Min	0.113	1.924	0.990	0.999	-9.877	-16.427	0.095	0.017
Gumbel \times AMH	5.596	0.999	0.184	0.238	-9.666	-16.216	0.130	0.012
Min \times AMH	0.925	1.000	0.001	0.000	-8.938	-15.488	0.155	0.001
Fourier \times AMH	0.962	1.000	0.001	0.001	-8.928	-15.478	0.155	0.001
FGM \times AMH	-0.539	0.999	0.001	0.001	-8.903	-15.454	0.156	0.001
Min \times Gumbel	8.174	1.960	0.001	0.001	-7.345	-13.896	0.071	0.027
Fourier \times Gumbel	0.900	1.970	0.001	0.001	-7.336	-13.886	0.070	0.042
FGM \times Gumbel	0.797	1.970	0.001	0.001	-7.336	-13.886	0.070	0.201
Independent \times FGM		0.998	0.001	0.001	-4.495	-9.408	0.244	0.001
Fourier \times FGM	0.606	0.999	0.001	0.001	-0.860	-7.410	0.244	0.001
Min \times FGM	9.931	1.000	0.000	0.003	-0.850	-7.400	0.244	0.001
Fourier \times Independent	0.999		0.182	0.187	9.109	4.196	0.478	0.001
Min \times Independent	8.980		0.067	0.001	10.913	6.000	0.505	0.001
Fourier \times Min	0.999	6.595	0.175	0.192	12.833	6.282	0.480	0.001

BIC = Bayesian information criterion; AIC = Akaike information criterion; S_n = Cramér-von Mises goodness-of-fit statistics.

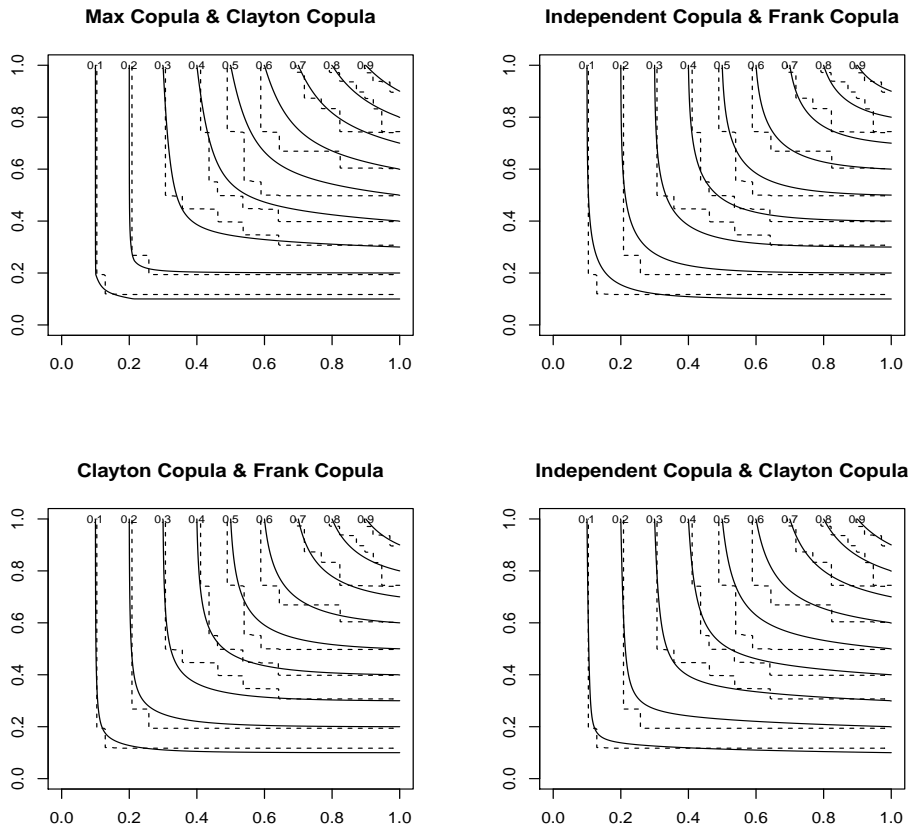


Figure 6: Contour plots of fitted copulas C_{θ_n} (solid line) and empirical copulas C_n (dotted line) for the four constructed asymmetric copulas on the car rental data.

Table 4 shows the result of parameter estimates, values of Bayesian information criterion (BIC), values of Akaike information criterion (AIC), and Cramér-von Mises goodness-of-fit statistics (S_n) with approximated p -values for nine basic copulas on the car rental data. Only one basic copula fits well in the sense of 5% level of Cramér-von Mises test: Frank copula. Table 5 shows the result of analysis for the constructed asymmetric copulas. Here $\text{par1}(\theta_1)$, $\text{par2}(\theta_2)$, α and β are estimated simultaneously by the MPLE as presented in Subsection 4.1 and Section 6. Fifteen combinations show p -values greater than 0.05 in Table 5, which means that asymmetric copulas are appropriate model. Figure 6 is the contour plots of empirical copulas (C_n) and fitted copulas (C_{θ_n}) for four asymmetric copulas: Clayton \times Frank, Fourier \times Frank, Fourier \times Gumbel, and Fourier \times AMH copula.

7.2. Economic indicators data

The second datasets is monthly economic indicators of Korea from Jan. 2011 to Aug. 2013, available at (Statistics Korea). Certificate of deposit (CD) rate and interest rate variables with sample size $n = 44$ are considered for this study. Figure 7 is a scatter plot of two variables, CD rate and interest rate. For the symmetry test on this data, we have $S_n^* = 0.102$ with p -value = 0.008 as mentioned in

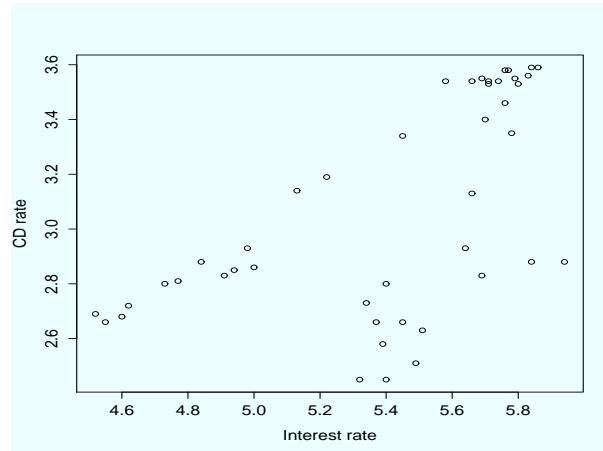


Figure 7: Scatter plot of economic indicators data.

Table 6: Result of parameter estimates, values of BIC, AIC, and S_n with approximated p -values for nine basic copulas on the economic indicators data

Copula name	par	BIC	AIC	S_n	p -value
Max	0.096	-26.055	-27.839	0.089	0.017
Gumbel	1.867	-20.218	-22.002	0.062	0.107
Frank	5.048	-19.045	-20.829	0.073	0.032
FGM	1.000	-11.786	-13.570	0.223	0.001
AMH	0.859	-8.993	-10.777	0.208	0.001
Clayton	0.826	-6.796	-8.580	0.197	0.001
Independent		0.000	0.000	0.462	0.001
Min	87109.604	3.784	2.000	0.462	0.001
Fourier	0.000	3.802	2.017	0.462	0.001

BIC = Bayesian information criterion; AIC = Akaike information criterion; S_n = Cramér-von Mises goodness-of-fit statistics.

Section 5, which means the data is not symmetric.

Table 6 shows the result of parameter estimates, values of BIC, AIC, and S_n with approximated p -values for nine basic copulas on the car rental data. Only one basic copula fits well in the sense of 5% level of Cramér-von Mises test: Gumbel copula. Table 7 shows the result of analysis for the constructed asymmetric copulas. Fifteen combinations show p -values greater than 0.05 in Table 7, which means that asymmetric copulas are appropriate model. Figure 8 is the contour plots of empirical copulas (C_n) and fitted copulas (C_{θ_n}) for four asymmetric copulas: Clayton \times Frank, Fourier \times Frank, Fourier \times Gumbel, and Fourier \times AMH copula.

8. Conclusion and discussion

We discussed a new generalized copula family which includes a class of asymmetric copulas as well as all copula families available in the current literature, including Fourier copula. The construction of new asymmetric family is based on and an extension of the result by Durante (2009). With diverse data such as simulated data, car rental data, and economic indicators, we performed parameter estimation by using the maximum pseudo-likelihood estimation method and Cramér-von Mises type of goodness-of-fit tests for the newly constructed asymmetric copula family. For these data, some of the proposed models turned out to be successful whereas the existing copulas were mostly unsuccessful. We thus argue that the proposed construction can offer an added value to model asymmetric bivariate

Table 7: Result of parameter estimates, values of BIC, AIC, and S_n with approximated p -values for some combined asymmetric copulas on the economic indicators data

Copula name	par1	par2	α	β	BIC	AIC	S_n	p -value
Clayton \times Gumbel	13.478	5.576	0.605	0.172	-27.187	-34.324	0.044	0.236
Frank \times Gumbel	17.132	5.471	0.598	0.166	-26.130	-33.267	0.043	0.221
Independent \times Gumbel		5.037	0.500	0.001	-25.799	-31.151	0.064	0.107
Independent \times Clayton		13.916	0.483	0.080	-25.585	-30.938	0.072	0.082
Fourier \times Clayton	0.999	10.218	0.491	0.088	-23.759	-30.896	0.078	0.057
Clayton \times Frank	11.221	18.584	0.446	0.852	-23.380	-30.516	0.052	0.152
Min \times Clayton	0.001	13.566	0.474	0.067	-23.172	-30.309	0.078	0.052
Fourier \times Gumbel	0.999	4.641	0.543	0.128	-23.139	-30.276	0.089	0.062
Clayton \times AMH	13.112	-0.999	0.531	0.947	-22.091	-29.228	0.071	0.072
FGM \times Gumbel	0.728	4.705	0.492	0.001	-22.090	-29.227	0.063	0.107
Max \times Gumbel	4.443	4.571	0.482	0.001	-22.062	-29.199	0.061	0.201
Min \times Gumbel	4.446	4.569	0.481	0.001	-22.062	-29.199	0.061	0.152
FGM \times Clayton	-0.999	11.810	0.466	0.046	-21.995	-29.132	0.072	0.067
Gumbel \times AMH	5.108	0.876	0.499	0.999	-21.979	-29.116	0.064	0.112
Max \times Clayton	146.628	13.915	0.483	0.080	-21.801	-28.938	0.072	0.072
Independent \times Frank		9.868	0.440	0.001	-18.546	-23.899	0.083	0.037
Max \times Independent	0.098		0.999	0.999	-18.457	-23.810	0.090	0.012
FGM \times Frank	-0.475	10.201	0.441	0.001	-15.205	-22.342	0.081	0.042
Fourier \times Frank	0.999	9.462	0.496	0.158	-14.897	-22.034	0.121	0.012
Frank \times AMH	9.985	0.053	0.566	0.998	-14.871	-22.008	0.081	0.047
Max \times Frank	2.134	9.869	0.446	0.001	-14.749	-21.885	0.085	0.037
Fourier \times Max	0.549	0.095	0.001	0.001	-14.678	-21.815	0.089	0.022
Max \times FGM	0.098	-0.316	1.000	0.999	-14.677	-21.814	0.090	0.017
Max \times AMH	0.098	-0.316	1.000	0.999	-14.677	-21.814	0.090	0.017
Max \times Min	0.097	1.924	0.999	0.999	-14.675	-21.812	0.089	0.022
Min \times Frank	2.135	9.835	0.453	0.001	-14.674	-21.811	0.087	0.032
Independent \times FGM		1.000	0.001	0.001	-4.200	-9.553	0.223	0.001
Independent \times AMH		0.867	0.005	0.001	-1.393	-6.746	0.207	0.001
FGM \times AMH	0.999	-0.849	1.000	0.998	-0.404	-7.540	0.223	0.001
Min \times FGM	5.103	0.999	0.001	0.001	-0.399	-7.536	0.223	0.001
Fourier \times FGM	0.958	1.000	0.005	0.001	-0.367	-7.504	0.224	0.001
Fourier \times AMH	0.990	0.999	0.350	0.275	-0.254	-7.391	0.280	0.001
Min \times AMH	1.935	0.870	0.001	0.003	2.403	-4.734	0.206	0.001
Fourier \times Independent	1.000		0.324	0.270	6.969	1.617	0.444	0.001
Fourier \times Min	0.999	7.659	0.347	0.256	10.808	3.671	0.446	0.001
Min \times Independent	8.980		0.067	0.001	11.353	6.000	0.462	0.001

BIC = Bayesian information criterion; AIC = Akaike information criterion; S_n = Cramér-von Mises goodness-of-fit statistics.

data.

For the estimation of the hyper-parameters (α and β), one can consider the cross-validation approach instead of the maximum likelihood estimation (MLE) as we did in section 6. After getting the MLE of copula parameters for fixed value of α and β , one can compare the cross validation copula information criterion (CIC) presented by Jordanger and Tjøstheim (2014). Then choose the parameter estimates that have the minimum of CIC. One may consider a Bayesian approach or expectation-maximization algorithm to estimate the hyper-parameters efficiently.

In our future study, we would extend our copula method to a multivariate case, to develop a generalized composite operator of asymmetric copula family as in Louzada and Ferreira (2016), to apply to the direction data from Kim and Kim (2014), and to incorporate time varying component as in Ara *et al.* (2017) to our proposed method. R program and datasets are available upon request from the corresponding author.

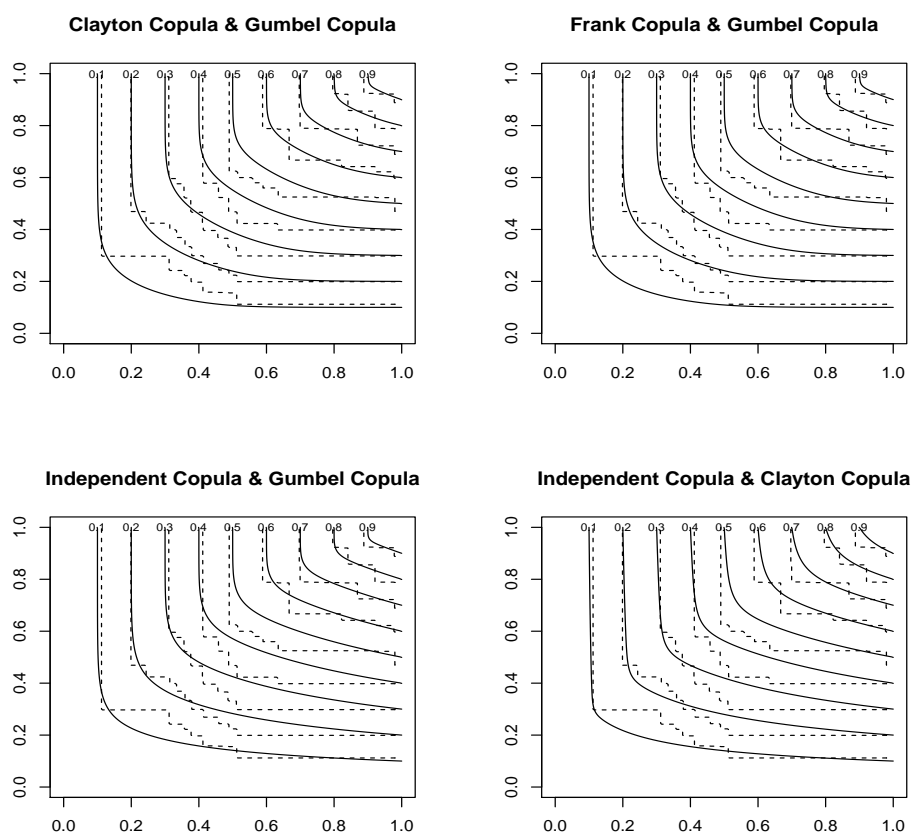


Figure 8: Contour plots of fitted copulas C_{θ_n} (solid line) and empirical copulas C_n (dotted line) for the four constructed asymmetric copulas on the economic indicators data.

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