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A NOTE OF LITTLEWOOD-PALEY FUNCTIONS ON TRIEBEL-LIZORKIN SPACES

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Abstract. In this note we prove that several classes of Littlewood-Paley square operators defined by the kernels without any regularity are bounded on Triebel-Lizorkin spaces $F_\alpha^{p,q}(\mathbb{R}^n)$ and Besov spaces $B_\alpha^{p,q}(\mathbb{R}^n)$ for $0 < \alpha < 1$ and $1 < p, q < \infty$.

1. Introduction

It is well known that the theory of the Littlewood-Paley functions has been an important part of harmonic analysis. One can consult [13–15] for its history and significance. The L^p mapping properties for these operators have also been studied extensively by many authors (see [3–8, 11, 12, 19] for example). In this note we shall prove the boundedness of the Littlewood-Paley square functions on Triebel-Lizorkin spaces and Besov spaces. Let $\psi \in L^1(\mathbb{R}^n)$ and satisfy

$$\int_{\mathbb{R}^n} \psi(x)dx = 0.$$

We consider a square function of Littlewood-Paley type

$$g_\psi(f)(x) = \Big(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t}\Big)^{1/2},$$

where $\psi_t(x) = t^{-n}\psi(t^{-1}x)$.

A well-known result for g_ψ proved by Benedek, Calderón and Panzone [2] is the following:

Theorem A. *Suppose that ψ satisfies*

$$|\psi(x)| \leq C(1 + |x|)^{-n-\epsilon} \quad \textit{for some } \epsilon > 0,$$

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$$\int_{\mathbb{R}^n} |\psi(x-y)-\psi(x)|dx \le C|y|^\epsilon \quad \textit{for some } \epsilon>0.$$

Then g_ψ *is bounded on* $L^p(\mathbb{R}^n)$ *for all* $1<p<\infty$.

Later on, Fan and Sato [8] relaxed the conditions imposed on ψ in Theorem A and proved the following result.

Theorem B. *Suppose that the function* ψ *satisfies the following conditions*:
(i) $\int_{|x|\ge 1} |\psi(x)||x|^\epsilon dx<\infty$ *for some* $\epsilon>0$;
(ii) $\big(\int_{|x|<1} |\psi(x)|^u dx\big)^{1/u}<\infty$ *for some* $u>1$;
(iii) $|\psi(x)|\le h(x)\Omega(x')$ *for all* $x\in\mathbb{R}^n\backslash\{0\}$, *where* $x'=x/|x|$, *for some non-negative function* h *on* $(0,\infty)$ *and* Ω *on* S^{n-1} (*the unit sphere in* $\mathbb{R}^n$) *such that*
(a) $h(r)$ *is non-increasing on* $(0,\infty)$ *and* $h(|x|)\in L^1(\mathbb{R}^n)$,
(b) $\Omega\in L^s(\mathrm{S}^{n-1})$ *for some* $1<s\le\infty$.
Then g_ψ *is bounded on* $L^p(\mathbb{R}^n)$ *for all* $1<p<\infty$.

Recently, Sato [12] used a minimum condition on ψ to obtain the following result.

Theorem C. *Suppose that* $|\psi(x)|\le h(x)\Omega(x')$ *for all* $x\in\mathbb{R}^n\backslash\{0\}$, *where* h *is a non-negative, non-increasing function on* $(0,\infty)$ *with supported in* $(0,1]$ *and* Ω *is a non-negative function on* S^{n-1}. *We assume that* $h(|x|)\in L^1(\mathbb{R}^n)$, $\Omega\in L^1(\mathrm{S}^{n-1})$ *and* $\psi\in L^s(\mathbb{R}^n)$ *for some* $1<s\le\infty$. *Put* $m_\psi(x)=h(|x|)\Omega(x')$. *Then*

$$\|g_\psi(f)\|_{L^p(\mathbb{R}^n)}\le C_p(s/(s-1))^{1/2}(\|\psi\|_{L^q(\mathbb{R}^n)}+\|m_\psi\|_{L^1(\mathbb{R}^n)})\|f\|_{L^p(\mathbb{R}^n)}$$

for all $1<p<\infty$, *where the constant* $C_p>0$ *is independent of* s, ψ, h, Ω.

By extrapolation, Theorem C yields a more general result.

Theorem D. *Suppose that* $|\psi(x)|\le h(|x|)\Omega(x')$ *for all* $x\in\mathbb{R}^n\backslash\{0\}$, *where* h *is a non-negative, non-increasing function on* $(0,\infty)$ *with supported in* $(0,1]$ *and* Ω *is a non-negative function on* S^{n-1}. *We further assume that* $\Omega\in L(\log^+ L)^{1/2}(\mathrm{S}^{n-1})$ *and* $h(|x|)\in L^s(\mathbb{R}^n)$ *for some* $1<s\le\infty$. *Then* g_ψ *is bounded on* $L^p(\mathbb{R}^n)$ *for all* $1<p<\infty$.

Here $L(\log^+ L)^\beta(\mathrm{S}^{n-1})$ for $\beta>0$ denotes the class of all functions Ω on S^{n-1} satisfying

$$\|\Omega\|_{L(\log^+ L)^\alpha(\mathrm{S}^{n-1})}:=\int_{\mathrm{S}^{n-1}}|\Omega(\theta)|(\log(2+|\Omega(\theta)|))^\alpha d\sigma(\theta)<\infty,$$

where $d\sigma$ denotes the Lebesgue surface measure on S^{n-1}. Clearly, for any $\alpha>\beta>0$ and $1<q\le\infty$,

$$L^q(\mathrm{S}^{n-1})\subsetneq L(\log^+ L)^\alpha(\mathrm{S}^{n-1})\subsetneq L(\log^+ L)^\beta(\mathrm{S}^{n-1})\subsetneq L^1(\mathrm{S}^{n-1}).$$

In this paper we focus on the boundedness of the Littlewood-Paley function g_ψ on Triebel-Lizorkin spaces. It is well known that the Triebel-Lizorkin spaces and Besov spaces contain many important function spaces, such as Lebesgue spaces, Hardy spaces, Sobolev spaces and Lipschitz spaces. Let us recall some definitions. For $\alpha \in \mathbb{R}$ and $0 < p,\, q \leq \infty\ (p \neq \infty)$, we define the homogeneous Triebel-Lizorkin spaces $\dot{F}^{p,q}_{\alpha}(\mathbb{R}^n)$ and homogeneous Besov spaces $\dot{B}^{p,q}_{\alpha}(\mathbb{R}^n)$ by

(1.1)
$$\dot{F}^{p,q}_{\alpha}(\mathbb{R}^n) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^n) :\ \|f\|_{\dot{F}^{p,q}_{\alpha}(\mathbb{R}^n)} = \Big\| \Big(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q} |\Psi_i * f|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} < \infty \Big\};$$

(1.2)
$$\dot{B}^{p,q}_{\alpha}(\mathbb{R}^n) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^n) :\ \|f\|_{\dot{B}^{p,q}_{\alpha}(\mathbb{R}^n)} = \Big(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q} \|\Psi_i * f\|^q_{L^p(\mathbb{R}^n)} \Big)^{1/q} < \infty \Big\},$$

where $\mathcal{S}'(\mathbb{R}^n)$ is the tempered distribution class on $\mathbb{R}^n$, $\widehat{\Psi_i}(\xi) = \phi(2^i\xi)$ for $i \in \mathbb{Z}$ and $\phi \in \mathcal{C}^{\infty}_c(\mathbb{R}^n)$ satisfies the conditions:
(i) $0 \leq \phi(x) \leq 1$;
(ii) $\mathrm{supp}(\phi) \subset \{x \in \mathbb{R}^n : 1/2 \leq |x| \leq 2\}$;
(iii) $\phi(x) > c > 0$ if $3/5 \leq |x| \leq 5/3$.

The inhomogeneous versions of Triebel-Lizorkin spaces and Besov spaces, which are denoted by $F^{p,q}_{\alpha}(\mathbb{R}^n)$ and $B^{p,q}_{\alpha}(\mathbb{R}^n)$, respectively, are obtained by adding the term $\|\Theta * f\|_{L^p(\mathbb{R}^n)}$ to the right hand side of (1.1) or (1.2) with $\sum_{i\in\mathbb{Z}}$ replaced by $\sum_{i\geq 1}$, where $\Theta \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz class), $\mathrm{supp}(\hat{\Theta}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$, $\hat{\Theta}(x) > c > 0$ if $|x| \leq 5/3$. The following properties are well known (see [9, 16] for example):

$$\dot{F}^{p,2}_{0}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \text{ for } 1 < p < \infty; \tag{1.3}$$

$$\dot{F}^{p,p}_{\alpha}(\mathbb{R}^n) = \dot{B}^{p,p}_{\alpha}(\mathbb{R}^n) \text{ for } \alpha \in \mathbb{R} \text{ and } 1 < p < \infty; \tag{1.4}$$

$$\begin{aligned} &F^{p,q}_{\alpha}(\mathbb{R}^n) \sim \dot{F}^{p,q}_{\alpha}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \ \text{ and} \\ &\|f\|_{F^{p,q}_{\alpha}(\mathbb{R}^n)} \sim \|f\|_{\dot{F}^{p,q}_{\alpha}(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)}\ (\alpha > 0); \end{aligned} \tag{1.5}$$

$$\begin{aligned} &B^{p,q}_{\alpha}(\mathbb{R}^n) \sim \dot{B}^{p,q}_{\alpha}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \ \text{ and} \\ &\|f\|_{B^{p,q}_{\alpha}(\mathbb{R}^n)} \sim \|f\|_{\dot{B}^{p,q}_{\alpha}(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)}\ (\alpha > 0). \end{aligned} \tag{1.6}$$

In 2009, Zhang and Chen [20] proved the following result.

Theorem E. *Let $\Omega \in H^1(\mathrm{S}^{n-1})$ satisfying $\int_{\mathrm{S}^{n-1}} \Omega(x')d\sigma(x') = 0$ and $\psi(x) = h(|x|)\Omega(x')$. Suppose that there exist ϵ, γ, $C > 0$ such that $|h(t)| \leq Ct^{-n+\epsilon}(1+t)^{-2\epsilon}$ and*

$$\int_{\mathbb{R}} |(t+r)^{n-1}h(t+r) - t^{n-1}h(t)|dt \leq C|r|^{\gamma}.$$

Then g_{ψ} is bounded on $F^{p,q}_{\alpha}(\mathbb{R}^n)$ for all $0 < \alpha < 1$ and $1 < p, q < \infty$.

Here $H^1(\mathrm{S}^{n-1})$ denotes the Hardy space on S^{n-1}, which is the set of all $L^1(\mathrm{S}^{n-1})$ functions Ω satisfying

$$\|\Omega\|_{H^1(\mathrm{S}^{n-1})} := \int_{\mathrm{S}^{n-1}} \sup_{0\le r<1} \Big| \int_{\mathrm{S}^{n-1}} \Omega(\theta) \frac{1-r^2}{|rw-\theta|^n} d\sigma(\theta) \Big| d\sigma(w) < \infty.$$

It was known that

$$H^1(\mathrm{S}^{n-1}) \nsubseteq L(\log^+ L)^\beta(\mathrm{S}^{n-1}) \nsubseteq H^1(\mathrm{S}^{n-1}) \ \text{ for any } 0<\beta<1.$$

It follows from Theorems B-D and (1.3) that the operator g_ψ is bounded on $\dot{F}_0^{p,2}(\mathbb{R}^n)$ for all $1<p<\infty$ under the same assumptions on ψ as in one of Theorems B-D. A natural question is the following:

Question F. *Is the operator g_ψ bounded on $\dot{F}_\alpha^{p,q}(\mathbb{R}^n)$ for some $\alpha \neq 0$ and $q \neq 2$ under the same assumptions on ψ as in one of Theorems* B-D*?*

Question F is the main motivation for this work. This problem will be addressed by our main result.

Theorem 1.1. (i) *Suppose that ψ satisfies the condition of Theorem* B*. Then*

$$\|g_\psi(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)}$$

for all $0<\alpha<1$ and $1<p,q<\infty$, where the constant $C>0$ depends on s, ψ, h, Ω.

(ii) *Suppose that ψ satisfies the condition of Theorem* C*. Then*

$$\|g_\psi(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \le C(s/(s-1))^{1/2}(\|\psi\|_{L^s(\mathbb{R}^n)} + \|m_\psi\|_{L^1(\mathbb{R}^n)})\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)}$$

for all $0<\alpha<1$ and $1<p,q<\infty$, where the constant $C>0$ is independent of s, ψ, h, Ω.

Actually, Theorem 1.1 will be derived from the following more abstract one.

Theorem 1.2. *Let $A>0$ and $v\ge 1$. Suppose that ψ satisfies the following conditions*:

(i) *there exist ϵ, $\delta>0$ and $C>0$ independent of A, v such that*

$$\int_{2^{kv}}^{2^{(k+1)v}} |\hat{\psi}(t\xi)|^2 \frac{dt}{t} \le CA^2 v \min(1, |2^{kv}\xi|^{\epsilon/v}, |2^{kv}\xi|^{-\delta/v}) \tag{1.7}$$

for all $k\in\mathbb{Z}$ and $\xi\in\mathbb{R}^n$;

(ii) *there exists a constant $C>0$ independent of A, v such that*

$$\Big\|\Big(\sum_{l\in\mathbb{Z}} \Big\|\sup_{t>0} ||\psi_t| * g_{l,\zeta}|\Big\|_{L^r(\mathfrak{R}_n)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \le CA\Big\|\Big(\sum_{l\in\mathbb{Z}} \|g_{l,\zeta}\|_{L^r(\mathfrak{R}_n)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \tag{1.8}$$

for all $1<p,q,r<\infty$, where $\mathfrak{R}_n = \{\zeta\in\mathbb{R}^n;\ 1/2<|\zeta|\le 1\}$.

Then for all $0 < \alpha < 1$ *and* $1 < p, q < \infty$, *there exists a constant* $C > 0$ *independent of* A, v *such that*

$$\|g_\psi(f)\|_{\dot{F}^{p,q}_\alpha(\mathbb{R}^n)} \le CAv^{1/2}\|f\|_{\dot{F}^{p,q}_\alpha(\mathbb{R}^n)}.$$

Applying Theorem 1.1, Theorems B-C and (1.5), we can get the following result immediately.

Theorem 1.3. (i) *Suppose that* ψ *satisfies the condition of Theorem* B. *Then*

$$\|g_\psi(f)\|_{F^{p,q}_\alpha(\mathbb{R}^n)} \le C\|f\|_{F^{p,q}_\alpha(\mathbb{R}^n)}$$

for all $0 < \alpha < 1$ *and* $1 < p, q < \infty$, *where the constant* C *depends on* s, ψ, h, Ω.

(ii) *Suppose that* ψ *satisfies the condition of Theorem* C. *Then*

$$\|g_\psi(f)\|_{F^{p,q}_\alpha(\mathbb{R}^n)} \le C(s/(s-1))^{1/2}(\|\psi\|_{L^s(\mathbb{R}^n)} + \|m_\psi\|_{L^1(\mathbb{R}^n)})\|f\|_{F^{p,q}_\alpha(\mathbb{R}^n)}$$

for all $0 < \alpha < 1$ *and* $1 < p, q < \infty$, *where the constant* C *is independent of* s, ψ, h, Ω.

By (ii) of Theorems 1.1 and 1.3 and applying extrapolation argument, we can prove the following result.

Corollary 1.1. *Let* $\Omega \in L(\log^+ L)^{1/2}(\mathrm{S}^{n-1})$ *and* $\Omega \ge 0$. *Suppose that* $|\psi(x)| \le h(|x|)\Omega(x')$ *for all* $x \in \mathbb{R}^n\backslash\{0\}$, *where* h *is a non-negative, non-increasing function on* $(0,\infty)$ *with supported in* $(0,1]$ *and* $h(|x|) \in L^2(\mathbb{R}^n)$. *Then* g_ψ *is bounded on* $\dot{F}^{p,q}_\alpha(\mathbb{R}^n)$ *and* $F^{p,q}_\alpha(\mathbb{R}^n)$ *for all* $0 < \alpha < 1$ *and* $1 < p, q < \infty$. *Moreover,*

$$\|g_\psi f\|_{\dot{F}^{p,q}_\alpha(\mathbb{R}^n)} \le C\|H\|_{L^2(\mathbb{R}^n)}(1 + \|\Omega\|_{L(\log^+ L)^{1/2}(\mathrm{S}^{n-1})})\|f\|_{\dot{F}^{p,q}_\alpha(\mathbb{R}^n)}$$

for all $0 < \alpha < 1$ *and* $1 < p, q < \infty$, *where* $H(x) = h(|x|)$ *and the constant* C *is independent of* ψ, h, Ω.

In particular, when $\psi(x) = \Omega(x')|x|^{\rho-n}\chi_{\{|x|<1\}}$ with $\rho > 0$, $\Omega \in L^1(\mathrm{S}^{n-1})$ and $\int_{\mathrm{S}^{n-1}} \Omega(x')d\sigma(x') = 0$, the Littlewood-Paley operator g_ψ reduces to the classical parametric Marcinkiewicz integral operator μ^ρ_Ω. As an application of Corollary 1.1, we have:

Corollary 1.2. *Let* $\rho > 0$, $\Omega \in L(\log^+ L)^{1/2}(\mathrm{S}^{n-1})$ *and satisfy* $\int_{\mathrm{S}^{n-1}} \Omega(x')d\sigma(x') = 0$. *Then* μ^ρ_Ω *is bounded on* $\dot{F}^{p,q}_\alpha(\mathbb{R}^n)$ *and* $F^{p,q}_\alpha(\mathbb{R}^n)$ *for all* $0 < \alpha < 1$ *and* $1 < p, q < \infty$.

Observing that

$$|\Delta_\zeta(g_\psi(f))(x)| \le g_\psi(\Delta_\zeta(f))(x) \tag{1.9}$$

for all $x, \zeta \in \mathbb{R}^n$, where $\Delta_\zeta(f)$ denotes the difference of f, i.e., $\Delta_\zeta(f)(x) = f(x+\zeta) - f(x)$ for all $x, \zeta \in \mathbb{R}^n$. By (1.9) and (ii) of Lemma 2.1, the L^p boundedness of g_ψ automatically implies its boundedness on $\dot{B}^{p,q}_\alpha(\mathbb{R}^n)$. This together with Theorems B-D and (1.6) yields the following result immediately.

Theorem 1.4. *Under the same conditions of Theorem* 1.1 *and Corollaries* 1.1-1.2*, the operator* g_ψ *is bounded on* $\dot{B}_\alpha^{p,q}(\mathbb{R}^n)$ *and* $B_\alpha^{p,q}(\mathbb{R}^n)$ *for all* $0 < \alpha < 1$ *and* $1 < p, q < \infty$.

The rest of this paper is organized as follows. After presenting some auxiliary lemmas following from [18], we shall prove Theorems 1.1-1.2 and Corollary 1.1 in Section 3. We would like to remark that the main method employed in this paper is a combination of ideas and arguments from [1], [11], [12], [18], among others. Compare our main results with Theorem E, our main results and proofs are greatly different from Theorem E and its proof. In [20] the proof of Theorem E relies heavily on BCP's method developed in [10] and the rotation method developed in [17], but the above methods do not work for our main results.

Throughout the paper, we denote p' by the conjugate index of p, which satisfies $1/p + 1/p' = 1$. The letter C or c, sometimes with certain parameters, will stand for positive constants not necessarily the same one at each occurrence, but are independent of the essential variables. In what follows, we set $\mathfrak{R}_n = \{\zeta \in \mathbb{R}^n;\ 1/2 < |\zeta| \le 1\}$.

2. Preliminary lemmas

To prove our main results, we need some useful characterizations of Triebel-Lizorkin spaces and Besov spaces, which are followed from [18].

Lemma 2.1. (i) *Let* $0 < \alpha < 1$, $1 < p < \infty$, $1 < q \le \infty$ *and* $1 \le r < \min(p, q)$. *Then*

$$\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \sim \Big\|\Big(\sum_{l\in\mathbb{Z}} 2^{lq\alpha}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-l}\zeta}(f)|^r d\zeta\Big)^{q/r}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.$$

(ii) *Let* $1 \le p < \infty$, $1 \le q \le \infty$ *and* $1 \le r \le p$. *Then*

$$\|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^n)} \sim \Big(\sum_{l\in\mathbb{Z}} 2^{lq\alpha}\Big\|\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-l}\zeta}(f)|^r d\zeta\Big)^{1/r}\Big\|_{L^p(\mathbb{R}^n)}^q\Big)^{1/q}.$$

The result for the following vector-valued inequalities of the Hardy-Littlewood maximal functions followed from [18] will also be needed.

Lemma 2.2. *Let* $y' \in \mathrm{S}^{n-1}$ *and* $M_{y'}$ *be the directional Hardy-Littlewood maximal function along* θ *defined by*

$$M_{y'}(f)(x) = \sup_{r>0} \frac{1}{2r}\int_{|t|<r} |f(x - ty')|dt.$$

Then

$$\Big\|\Big(\sum_{j\in\mathbb{Z}} \|M_{y'}(f_{j,\zeta})\|_{L^r(\mathfrak{R}_n)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \le C\Big\|\Big(\sum_{j\in\mathbb{Z}} \|f_{j,\zeta}\|_{L^r(\mathfrak{R}_n)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}$$

for all $1 < p,\ q,\ r < \infty$, *where* $C > 0$ *is independent of* y'.

3. Proofs of main results

This section is devoted to proving our main results. Let us begin with the proof of Theorem 1.2.

Proof of Theorem 1.2. By (1.9) and (i) of Lemma 2.1, we have

$$\begin{aligned}
&\|g_\psi(f)\|_{\dot F^{p,q}_\alpha(\mathbb{R}^n)}\\
&\le C\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}|\Delta_{2^{-l}\zeta}(g_\psi(f))|d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\le C\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}|g_\psi(\Delta_{2^{-l}\zeta}(f))|d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \qquad (3.1)\\
&\le C\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\int_0^\infty|\psi_t*\Delta_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}
\end{aligned}$$

for all $0<\alpha<1$ and $1<p,\,q<\infty$. Therefore, to prove Theorem 1.2, it suffices to show that there exists a constant $C>0$ independent of A and v such that

$$\begin{aligned}
&\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\int_0^\infty|\psi_t*\Delta_{2^{-l}\zeta}f|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \qquad (3.2)\\
&\le CAv^{1/2}\|f\|_{\dot F^{p,q}_\alpha(\mathbb{R}^n)}
\end{aligned}$$

for all $0<\alpha<1$ and $1<p,\,q<\infty$.

Let $\eta_0\in\mathcal{C}^\infty(\mathbb{R})$ be an even function satisfying $0\le\eta_0(t)\le1$, $\eta_0(0)=1$ and $\eta_0(t)=0$ for $|t|\ge1$. Set $\eta(\xi)=1$ for $|\xi|\le1$, $\eta(\xi)=\eta_0(\frac{|\xi|-1}{2^v-1})$, where $a>1$. Then, η satisfies $\chi_{\{|\xi|\le1\}}(\xi)\le\eta(\xi)\le\chi_{\{|\xi|\le2^v\}}(\xi)$ and $|\partial^\alpha\eta(\xi)|\le c_\alpha(2^v-1)^{-|\alpha|}$ for $\xi\in\mathbb{R}^n$ and $\alpha\in\mathbb{N}^n$, where c_α is independent of v. We define the sequence of functions $\{\phi_k\}_{k\in\mathbb{Z}}$ on $\mathbb{R}^n$ by

$$\phi_k(\xi)=\eta(2^{-(k+1)v}\xi)-\eta(2^{-kv}\xi),\quad \xi\in\mathbb{R}^n.$$

Observing that $\mathrm{supp}(\phi_k)\subset\{2^{kv}\le|\xi|\le2^{(k+2)v}\}$, $\mathrm{supp}(\phi_k)\cap\mathrm{supp}(\phi_j)=\emptyset$ for $|j-k|\ge2$ and $\sum_{k\in\mathbb{Z}}\phi_k(\xi)=1$ for $\xi\in\mathbb{R}^n\backslash\{0\}$. Define the multiplier operator Γ_k on $\mathbb{R}^n$ by

$$\widehat{\Gamma_k f}(\xi)=\phi_k(|\xi|)\hat f(\xi).$$

It follows from [18, Lemma 2.5] that for $1<p,\,q,\,r<\infty$, there exists a constant $C>0$ independent of v such that

$$\begin{aligned}
&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|\Gamma_k f_{j,\zeta}|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \qquad (3.3)\\
&\le C\Big(\frac{2^v}{2^v-1}\Big)^{n+2}\Big\|\Big(\sum_{j\in\mathbb{Z}}\|f_{j,\zeta}\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

By Minkowski's inequality, it follows that for all $0<\alpha<1$ and $1<p,\ q<\infty$,

(3.4)

$$\begin{aligned}
&\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\int_0^\infty|\psi_t*\Delta_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&=\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\Big|\psi_t*\sum_{j\in\mathbb{Z}}\Gamma_{j-k}\Delta_{2^{-l}\zeta}(f)\Big|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\le\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}|\psi_t*\Gamma_{j-k}\Delta_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Define the mixed norm $\|\cdot\|_{E^\alpha_{p,q}}$ for measurable functions on $\mathbb{R}^n\times\mathfrak{R}_n\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{R}_+$ by

$$\|g\|_{E^\alpha_{p,q}}:=\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\sum_{k\in\mathbb{Z}}\int_0^\infty|g(t,x,\zeta,l,k)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.$$

For any $j\in\mathbb{Z}$, let

$$A_j(f)(t,x,\zeta,l,k):=\psi_t*\Gamma_{j-k}\Delta_{2^{-l}\zeta}(f)(x)\chi_{[2^{kv},2^{(k+1)v}]}(t).$$

Thus, we have

(3.5)

$$\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\int_0^\infty|\psi_t*\Delta_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\le\sum_{j\in\mathbb{Z}}\|A_j(f)\|_{E^\alpha_{p,q}}$$

for all $0<\alpha<1$ and $1<p,\ q<\infty$.

By our assumption (1.7), (ii) of Lemma 2.1, Hölder's inequality, Minkowski's inequality, Fubini's theorem and Plancherel's theorem, it yields that

(3.6)

$$\begin{aligned}
&\|A_j(f)\|^2_{E^\alpha_{2,2}}\\
&=\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{2l\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}|\psi_t*\Gamma_{j-k}\Delta_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^2\Big)^{1/2}\Big\|^2_{L^2(\mathbb{R}^n)}\\
&=\int_{\mathbb{R}^n}\sum_{l\in\mathbb{Z}}2^{2l\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}|\psi_t*\Gamma_{j-k}\Delta_{2^{-l}\zeta}(f)(x)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^2dx\\
&\le C\sum_{l\in\mathbb{Z}}2^{2l\alpha}\int_{\mathfrak{R}_n}\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\int_{\mathbb{R}^n}|\psi_t*\Gamma_{j-k}\Delta_{2^{-l}\zeta}(f)(x)|^2dx\frac{dt}{t}d\zeta\\
&\le C\sum_{l\in\mathbb{Z}}2^{2l\alpha}\int_{\mathfrak{R}_n}\sum_{k\in\mathbb{Z}}\int_{E_{j-k}}\int_{2^{kv}}^{2^{(k+1)v}}|\widehat{\psi}(tx)|^2\frac{dt}{t}|\widehat{\Delta_{2^{-l}\zeta}(f)}(x)|^2dxd\zeta\\
&\le CA^2v2^{-c|j|}\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big\|\Big(\int_{\mathfrak{R}_n}|\Delta_{2^{-l}\zeta}(f)|^2d\zeta\Big)^{1/2}\Big\|^2_{L^2(\mathbb{R}^n)}\\
&\le CA^2v2^{-c|j|}\|f\|^2_{\dot{B}^{2,2}_\alpha(\mathbb{R}^n)},
\end{aligned}$$

where $E_{j-k} = \{x \in \mathbb{R}^n : 2^{(j-k)v} \le |x| \le 2^{(j-k+2)v}\}$ and $c = \min(\epsilon, \delta)$. Here the constant $C > 0$ is independent of A and v. (3.6) together with (1.4) yields

$$\|A_j(f)\|_{E^\alpha_{2,2}} \le CAv^{1/2}2^{-c|j|/2}\|f\|_{\dot{F}^{2,2}_\alpha(\mathbb{R}^n)}. \tag{3.7}$$

On the other hand, by our assumption (1.8) we have

$$\begin{aligned}
&\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\sup_{k\in\mathbb{Z}}\sup_{t\in[1,2^v]}|\psi_{2^{kv}t}*g_{l,\zeta,k}|\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\le \Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\sup_{t>0}\Big||\psi_t|*\Big(\sup_{k\in\mathbb{Z}}|g_{l,\zeta,k}|\Big)\Big|\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\le CA\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\sup_{k\in\mathbb{Z}}|g_{l,\zeta,k}|\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}
\end{aligned} \tag{3.8}$$

for any $1 < p, q, r < \infty$. Fix $1 < p, q, r < \infty$, by the duality, Fubini's theorem, Hölder's inequality and our assumption (1.8) again, there exists a sequence of functions $\{f_{l,\zeta}\}_{l,\zeta}$ such that $\|\{f_{l,\zeta}\}\|_{L^{p'}(\ell^{q'}(L^{r'}(\mathfrak{R}_n),\mathbb{R}^n))} = 1$ and

$$\begin{aligned}
&\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\sum_{k\in\mathbb{Z}}\int_1^{2^v}|\psi_{2^{kv}t}*g_{l,\zeta,k}|dt\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&= \Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\sum_{k\in\mathbb{Z}}\int_1^{2^v}|\psi_{2^{kv}t}*g_{l,\zeta,k}|dt\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&= \sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^n}\int_{\mathfrak{R}_n}\sum_{k\in\mathbb{Z}}\int_1^{2^v}|\psi_{2^{kv}t}*g_{l,\zeta,k}|dt|f_{l,\zeta}(x)|d\zeta dx\\
&= \sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^n}\int_{\mathfrak{R}_n}\sum_{k\in\mathbb{Z}}|g_{l,\zeta,k}|\int_1^{2^v}|\psi_{2^{kv}t}|*|\tilde{f_{l,\zeta}}|(-x)|dtd\zeta dx\\
&\le Cv\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^n}\int_{\mathfrak{R}_n}\sum_{k\in\mathbb{Z}}|g_{l,\zeta,k}|\sup_{t>0}\big||\psi_t|*(|\tilde{f_{l,\zeta}}|)\big|(-x)d\zeta dx\\
&\le Cv\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\sum_{k\in\mathbb{Z}}|g_{l,\zeta,k}|\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\qquad\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\sup_{t>0}||\psi_t|*(|\tilde{f_{l,\zeta}}|)|\Big\|^{q'}_{L^{r'}(\mathfrak{R}_n)}\Big)^{1/q'}\Big\|_{L^{p'}(\mathbb{R}^n)}\\
&\le CAv\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\sum_{k\in\mathbb{Z}}|g_{l,\zeta,k}|\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

where $\tilde{f_{l,\zeta}}(x) = f_{l,\zeta}(-x)$. This together with (3.8) yields

$$\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\Big(\int_1^{2^v}|\psi_{2^{kv}t}*g_{l,\zeta,k}|^2dt\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \tag{3.9}$$

$$\le CAv^{1/2}\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{l,\zeta,k}|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}$$

for any $1 < p, q, r < \infty$, where C is independent of A and v.

Fix $1 < p, q < \infty$, we can choose $1 < r < \min(p, q)$. By Lemma 2.1, (3.3), (3.9) and Hölder's inequality, we obtain

(3.10)

$$\begin{aligned}
&\|A_j(f)\|_{E^\alpha_{p,q}}\\
&= \Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}|\psi_t*\Gamma_{j-k}\Delta_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&= \Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\sum_{k\in\mathbb{Z}}\int_{1}^{2^{v}}|\psi_{2^{kv}t}*\Gamma_{j-k}\Delta_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\le \Big\|\Big(\sum_{l\in\mathbb{Z}}2^{2q\alpha}\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{1}^{2^{v}}|\psi_{2^{kv}t}*\Gamma_{j-k}\Delta_{2^{-l}\zeta}(f)|^2dt\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\le CAv^{1/2}\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|\Gamma_{j-k}(2^{l\alpha}\Delta_{2^{-l}\zeta}(f))|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\le CAv^{1/2}\Big(\frac{2^v}{2^v-1}\Big)^{n+2}\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\|\Delta_{2^{-l}\zeta}(f)\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\le CAv^{1/2}\|f\|_{\dot F^{p,q}_\alpha(\mathbb{R}^n)},
\end{aligned}$$

where the constant $C > 0$ is independent of A, v.

Interpolation between (3.7) and (3.10) implies that for any $1 < p, q < \infty$, there exists $\theta \in (0, 1]$ and a constant $C > 0$ independent of A, v such that

$$\|A_j(f)\|_{E^\alpha_{p,q}} \le CAv^{1/2}2^{-c\theta|j|/2}\|f\|_{\dot F^{p,q}_\alpha(\mathbb{R}^n)}. \tag{3.11}$$

Combining (3.11) with (3.5) yields (3.2) and completes the proof of Theorem 1.2. □

Proof of Theorem 1.1. It was shown in Lemmas 1-3 of [11] that there exists a constant C depending on ψ, h, Ω such that

$$\int_{2^k}^{2^{k+1}}|\hat\psi(t\xi)|^2\frac{dt}{t} \le C\min(1, |2^k\xi|^\epsilon, |2^k\xi|^{-\epsilon}) \tag{3.12}$$

for all $k \in \mathbb{Z}$, $\xi \in \mathbb{R}^n$ and some $\epsilon > 0$ if ψ satisfies the condition of Theorem B. It follows from [12, Lemma 2] that there exists $C > 0$ independent of s, ψ, h, Ω such that

$$\int_{2^{ks'}}^{2^{(k+1)s'}}|\hat\psi(t\xi)|^2\frac{dt}{t} \le Cs'\|\psi\|^2_{L^s(\mathbb{R}^n)}\min\{1, |2^{ks'}\xi|^{1/(2s')}, |2^{ks'}\xi|^{-1/(2s')}\} \tag{3.13}$$

if $\psi \in L^s(\mathbb{R}^n)$ for some $s > 1$.

On the other hand, when ψ satisfies the condition of Theorem B or C, as in Stein [14, pp. 63–64], we can show that

$$\sup_{t>0}\big||\psi_t|*f(x)\big| \le \|h\|_{L^1(\mathbb{R}^n)} v_n^{-1} M_\Omega(f)(x), \tag{3.14}$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$ and

$$M_\Omega(f)(x) = \sup_{t>0} t^{-n}\int_{|y|<t}|f(x-y)|\Omega(\frac{y}{|y|})dy.$$

One can easily check that

$$M_\Omega(f)(x) \le \int_{S^{n-1}}\Omega(y')M_{y'}(f)(x)d\sigma(y'), \tag{3.15}$$

where $y'=\frac{y}{|y|}$ and $M_{y'}$ is the directional Hardy-Littlewood maximal function along y'. We notice that $\|m_\psi\|_{L^1(\mathbb{R}^n)} = \|h\|_{L^1(\mathbb{R}^n)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}(nv_n)^{-1}$. By (3.14)-(3.15), Lemma 2.2 and Minkowski's inequality, we can obtain

$$\begin{aligned}&\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\sup_{t>0}\big||\psi_t|*f_{l,\zeta}\big|\Big\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\le C\|m_\psi\|_{L^1(\mathbb{R}^n)}\Big\|\Big(\sum_{l\in\mathbb{Z}}\|f_{l,\zeta}\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\end{aligned} \tag{3.16}$$

for any $1<p,q,r<\infty$, where $C>0$ is independent of s, ψ, h, Ω.

By (3.12), (3.16) and Theorem 1.2 with $v=A=1$, there exists a constant $C>0$ depending on ψ, h, Ω such that

$$\|g_\psi(f)\|_{\dot{F}^{p,q}_\alpha(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^{p,q}_\alpha(\mathbb{R}^n)}$$

for all $0<\alpha<1$ and $1<p,q<\infty$ if ψ satisfies the condition of Theorem B.

It follows also from (3.13), (3.16) and Theorem 1.2 with $v=s'$ and $A=\|\psi\|_{L^s(\mathbb{R}^n)}+\|m_\psi\|_{L^1(\mathbb{R}^n)}$ that there exists a constant $C>0$ independent of s, ψ, h, Ω such that

$$\|g_\psi(f)\|_{\dot{F}^{p,q}_\alpha(\mathbb{R}^n)} \le C(\|\psi\|_{L^s(\mathbb{R}^n)}+\|m_\psi\|_{L^1(\mathbb{R}^n)})\|f\|_{\dot{F}^{p,q}_\alpha(\mathbb{R}^n)}$$

for all $0<\alpha<1$ and $1<p,q<\infty$ if ψ satisfies the condition of Theorem C. This finishes the proof of Theorem 1.1. □

Proof of Corollary 1.1. One can easily check that $H\in L^r(\mathbb{R}^n)$ for any $1<r\le 2$. Employing the notation in [12], let $F_1=\{\theta\in \mathrm{S}^{n-1}: |\Omega(\theta)|\le 2\}$ and $F_k=\{\theta\in\mathrm{S}^{n-1}: 2^{k-1}<|\Omega(\theta)|\le 2^k\}$ for any $k\ge 2$. Let $E_k=\{x\in B(0,1): x\ne 0, x'\in F_k\}$ for all $k\ge 1$, $B(0,1)=\{x\in\mathbb{R}^n: |x|\le 1\}$ and $\Omega_k=\Omega\chi_{F_k}$. We decompose ψ as $\psi=\sum_{k=1}^\infty \psi_k$, where

$$\psi_k=\psi\chi_{E_k}-|B(0,1)|^{-1}\int_{E_k}\psi(x)dx\chi_{B(0,1)}.$$

Note that

$$\int_{\mathbb{R}^n}\psi_k(x)dx=0;$$

$$|\psi(x)\chi_{E_k}| \leq H(x)\Omega_k(x')\chi_{B(0,1)};$$

$$\Big||B(0,1)|^{-1}\int_{E_k}\psi(x)dx\chi_{B(0,1)}\Big| \leq \|H\|_{L^1(\mathbb{R}^n)}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\chi_{B(0,1)}.$$

Let

$h^*(|x|) = (H(x) + \|H\|_{L^1(\mathbb{R}^n)})\chi_{B(0,1)}(x)$ and $\Omega_k^*(x') = \Omega_k(x') + \|\Omega_k\|_{L^1(\mathrm{S}^{n-1})}$.

One can easily check that

$$|\psi_k(x)| \leq h^*(|x|)\Omega_k^*(x'),$$

where h^* is a nonnegative, nonincreasing function on $(0,\infty)$ with supported in $(0,1]$ and Ω_k^* is a nonnegative function on S^{n-1}. Specially, $h^*(|x|) \in L^1(\mathbb{R}^n)$, $\Omega_k^* \in L^1(\mathrm{S}^{n-1})$ and $\psi_k \in L^s(\mathbb{R}^n)$ for any $1 < s \leq 2$. Let $\psi_{k,t} = t^{-n}\psi_k(t^{-1}x)$ and $m_{\psi_k} = h^*(|x|)\Omega_k^*(x')$. By (3.1) and Minkowski's inequality,

$$\begin{aligned}
&\|g_\psi(f)\|_{\dot F_\alpha^{p,q}(\mathbb{R}^n)}\\
&\leq C\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\int_0^\infty|\psi_t*\Delta_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leq C\sum_{k=1}^\infty\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\int_0^\infty|\psi_{k,t}*\Delta_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}
\end{aligned}\tag{3.17}$$

for all $0 < \alpha < 1$ and $1 < p,\, q < \infty$. By the proofs of Theorems 1.1-1.2, there exists a constant $C > 0$ independent of $s, \psi_k, h^*, \Omega_k^*$ such that

$$\begin{aligned}
&\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_n}\Big(\int_0^\infty|\psi_t*\Delta_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leq C(s/(s-1))^{1/2}(\|\psi_k\|_{L^s(\mathbb{R}^n)} + \|m_{\psi_k}\|_{L^1(\mathbb{R}^n)})\|f\|_{\dot F_\alpha^{p,q}(\mathbb{R}^n)}
\end{aligned}\tag{3.18}$$

for all $0 < \alpha < 1$ and $1 < p,q < \infty$. Taking $s = 1 + 1/k$. It follows from (3.17)-(3.18) that

$$\begin{aligned}
&\|g_\psi(f)\|_{\dot F_\alpha^{p,q}(\mathbb{R}^n)}\\
&\leq C\sum_{k=1}^\infty k^{1/2}(\|\psi_k\|_{L^{1+1/k}(\mathbb{R}^n)} + \|m_{\psi_k}\|_{L^1(\mathbb{R}^n)})\|f\|_{\dot F_\alpha^{p,q}(\mathbb{R}^n)}
\end{aligned}\tag{3.19}$$

for all $0 < \alpha < 1$ and $1 < p,q < \infty$, where $C > 0$ is independent of k, ψ_k, h, Ω. Note that

$$\|m_{\psi_k}\|_{L^1(\mathbb{R}^n)} \leq C\|h^*\|_{L^1(\mathbb{R}^n)}\|\Omega_k^*\|_{L^1(\mathrm{S}^{n-1})} \leq C\|H\|_{L^2(\mathbb{R}^n)}\|\Omega_k\|_{L^{1+1/k}(\mathrm{S}^{n-1})};$$

$$\|\psi_k\|_{L^{1+1/k}(\mathbb{R}^n)} \leq C\|H\|_{L^2(\mathbb{R}^n)}\|\Omega_k\|_{L^{1+1/k}(\mathrm{S}^{n-1})};$$

$$\|\Omega_k\|_{L^{1+1/k}(\mathrm{S}^{n-1})} = \begin{cases} C\|\Omega_k\|_{L^1(\mathrm{S}^{n-1})}, & \text{if } \|\Omega_k\|_{L^1(\mathrm{S}^{n-1})} \geq 2^{-k};\\ 2^{-k^2/(k+1)}, & \text{if } \|\Omega_k\|_{L^1(\mathrm{S}^{n-1})} < 2^{-k}.\end{cases}$$

It follows from (3.19) that

$$\begin{aligned}
&\|g_\psi f\|_{\dot{F}^{p,q}_\alpha(\mathbb{R}^n)} \\
&\le C\|H\|_{L^2(\mathbb{R}^n)} \sum_{k=1}^{\infty} (k^{1/2} 2^{-k^2/(k+1)} + k^{1/2}\|\Omega_k\|_{L^1(\mathrm{S}^{n-1})})\|f\|_{\dot{F}^{p,q}_\alpha(\mathbb{R}^n)} \\
&\le C\|H\|_{L^2(\mathbb{R}^n)}(1+\|\Omega\|_{L(\log^+ L)^{1/2}(\mathrm{S}^{n-1})})\|f\|_{\dot{F}^{p,q}_\alpha(\mathbb{R}^n)}.
\end{aligned}$$

This yields the conclusion of Corollary 1.1. □

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References

[1] H. Al-Qassem and Y. Pan, *On certain estimates for Marcinkiewicz integrals and extrapolation*, Collect. Math. **60** (2009), no. 2, 123–145.

[2] A. Benedek, A. P. Calderón, and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), no. 3, 356–365.

[3] L. C. Cheng, *On Littlewood-Paley functions*, Proc. Amer. Math. Soc. **135** (2007), no. 10, 3241–3247.

[4] Y. Ding, D. Fan, and Y. Pan, *On Littlewood-Paley functions and singular integrals*, Hokkaido Math. J. **29** (2000), no. 3, 537–552.

[5] Y. Ding and S. Sato, *Littlewood-Paley functions on homogeneous groups*, Forum Math. **28** (2016), no. 1, 43–55.

[6] J. Duoandikoetxea, *Sharp L^p boundedness for a class of square functions*, Rev. Mat. Complut. **26** (2013), no. 2, 535–548.

[7] J. Duoandikoetxea and E. Seijo, *Weighted inequalities for rough square functions through extrapolation*, Studia Math. **149** (2002), no. 3, 239–252.

[8] D. Fan and S. Sato, *Remarks on Littlewood-Paley functions and singular integrals*, J. Math. Soc. Japan **54** (2002), no. 3, 565–585.

[9] M. Frazier, B. Jawerth, and G. Weiss, *Littlewood-Paley theory and the study of function spaces*, CBMS Regional Conference Series in Mathematics, **79**, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1991.

[10] S. Korry, *A class of bounded operators on Sobolev spaces*, Arch. Math. (Basel) **82** (2004), no. 1, 40–50.

[11] S. Sato, *Remarks on square functions in the Littlewood-Paley theory*, Bull. Austral. Math. Soc. **58** (1998), no. 2, 199–211.

[12] ______, *Estimates for Littlewood-Paley functions and extrapolation*, Integral Equations Operator Theory **62** (2008), no. 3, 429–440.

[13] E. M. Stein, *On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz*, Trans. Amer. Math. Soc. **88** (1958), 430–466.

[14] ______, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, NJ, 1970.

[15] ______, *Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, **43**, Princeton University Press, Princeton, NJ, 1993.

[16] H. Triebel, *Theory of Function Spaces*, Monographs in Mathematics, **78**, Birkhäuser Verlag, Basel, 1983.

[17] H. Xu, J. Chen, and Y. Ying, *A note on Marcinkiewicz integrals with H^1 kernels*, Acta Math. Sci. Ser. B Engl. Ed. **23** (2003), no. 1, 133–138.

[18] K. Yabuta, *Triebel-Lizorkin space boundedness of Marcinkiewicz integrals associated to surfaces*, Appl. Math. J. Chinese Univ. Ser. B **30** (2015), no. 4, 418–446.

[19] ______, *Remarks on L^2 boundedness of Littlewood-Paley operators*, Analysis (Berlin) **33** (2013), no. 3, 209–218.

[20] C. Zhang and J. Chen, *Boundedness of g-functions on Triebel-Lizorkin spaces*, Taiwanese J. Math. **13** (2009), no. 3, 973–981.

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