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KRULL DIMENSION OF HURWITZ POLYNOMIAL RINGS OVER PRÜFER DOMAINS

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Abstract. Let R be a commutative ring with identity and let $R[x]$ be the collection of polynomials with coefficients in R. There are a lot of multiplications in $R[x]$ such that together with the usual addition, $R[x]$ becomes a ring that contains R as a subring. These multiplications are from a class of functions λ from $\mathbb{N}_0$ to $\mathbb{N}$. The trivial case when $\lambda(i) = 1$ for all i gives the usual polynomial ring. Among nontrivial cases, there is an important one, namely, the case when $\lambda(i) = i!$ for all i. For this case, it gives the well-known Hurwitz polynomial ring $R_H[x]$. In this paper, we completely determine the Krull dimension of $R_H[x]$ when R is a Prüfer domain. Let R be a Prüfer domain. We show that $\dim R_H[x] = \dim R+1$ if R has characteristic zero and $\dim R_H[x] = \dim R$ otherwise.

1. Introduction

In this paper, a ring always means a commutative ring with identity. Let R be a ring and let

$$R[x] = \left\{ \sum_{i=0}^{n} a_i x^i \mid n \geq 0, a_i \in R \right\}$$

be the collection of polynomials with coefficients in R. With the usual addition '+' and multiplication '·', $R[x]$ becomes a ring that contains R as a subring. This polynomial ring is an important object in commutative algebra and has been widely studied.

While the usual multiplication in $R[x]$ is usually considered, in general there do exist many other multiplications in $R[x]$ such that together with the usual addition, $R[x]$ is still a ring that contains R as a subring. For example, let $\mathbb{N}_0$ (resp., $\mathbb{N}$) be the set of nonnegative (resp., positive) integers and let $\lambda : \mathbb{N}_0 \to \mathbb{N}$ be any function such that $\lambda(0) = 1$ and $\lambda(i)\lambda(j)$ divides $\lambda(i + j)$ in $\mathbb{N}$ for each i and j. Identifying the positive integer $\alpha_{i,j} = \frac{\lambda(i+j)}{\lambda(i)\lambda(j)}$ with the element $\alpha_{i,j} \cdot 1$

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in R, define a multiplication $*$ in $R[x]$ by

$$\left(\sum_{i=0}^{n} a_i x^i\right) * \left(\sum_{j=0}^{m} b_j x^j\right) = \sum_{k=0}^{n+m} \left(\sum_{i+j=k} \alpha_{i,j} a_i b_j\right) x^k.$$

With this new multiplication, $R[x]$ is also a ring containing R as a subring [10]. We denote this ring by $(R[x], \lambda)$. By this observation, the usual polynomial ring $R[x]$ is the special case of $(R[x], \lambda)$ when λ is trivial, i.e., $\lambda(i) = 1$ for all i (and hence $\alpha_{i,j} = 1$ for all i and j). Among nontrivial cases, there is an important one, namely, the case when $\lambda(i) = i!$ for all i. In this case, $\alpha_{i,j} = \frac{\lambda(i+j)}{\lambda(i)\lambda(j)} = \frac{(i+j)!}{i!j!} = \binom{i+j}{i}$ is a binomial coefficient and the corresponding ring $(R[x], \lambda)$ is the well-known Hurwitz polynomial ring, which is denoted by $R_H[x]$ (the term "H" stands for "Hurwitz"). In fact, a product of two power series can also be defined in the same way, which gives us the Hurwitz power series ring $R_H[\![x]\!]$. This kind of product was first considered by Hurwitz [9] and was further studied in [6, 7, 19]. Closely related to the power series ring, the Hurwitz power series ring has been shown to have many interesting properties, including applications in differential algebra [12, 13]. Noticeably, considered as formal functions, Hurwitz power series provide formal solutions to homogeneous linear ordinary differential equations [13] (see also [14]). Other properties of Hurwitz polynomials and Hurwitz power series can be found in [1–5, 8, 15, 16].

The Hurwitz polynomial ring $R_H[x]$ is very different from the usual polynomial ring $R[x]$. For example, $R_H[x]$ may not be an integral domain even though R is. In [5], it is shown that $R_H[x]$ is never a Noetherian ring if R does not contain the rational numbers, which contrasts to Hilbert Basic Theorem stating that the usual polynomial ring $R[x]$ is always Noetherian if R is. Similarly, $R_H[x]$ is never a unique factorization domain unless R contains the rational numbers [10].

However, in [10], the authors showed that the Krull dimension of $R_H[x]$ is very well behaved. They showed in general that

$$\dim R \leq \dim R_H[x] \leq 2 \dim R + 1,$$

which is very similar to the result for usual polynomial rings (see [17]):

$$\dim R + 1 \leq \dim R[x] \leq 2 \dim R + 1.$$

If R is a Noetherian ring, then so is $R[x]$. In this case, by using Krull's Principal Ideal Theorem, it can be shown that $\dim R[x] = \dim R + 1$ (see, for example, [11]). Unfortunately, $R_H[x]$ is not necessarily a Noetherian ring if R is [5]. Therefore, Krull's Principal Ideal Theorem cannot be applied to determine $\dim R_H[x]$ as in the usual polynomial ring case when R is a Noetherian ring. However, a similar result still holds for $\dim R_H[x]$: the upper bound $2 \dim R + 1$ is reduced to $\dim R + 1$ [10]. This means that if R is a Noetherian ring, then

$$\dim R_H[x] = \dim R \quad \text{or} \quad \dim R_H[x] = \dim R + 1.$$

The purpose of this paper is to calculate the Krull dimension of the Hurwitz polynomial ring $R_H[x]$ when R is a Prüfer domain. We show that the same result holds for $\dim R_H[x]$ when R is a Prüfer domain, i.e.,

$$\dim R_H[x] = \dim R \quad \text{or} \quad \dim R_H[x] = \dim R + 1.$$

We moreover determine when $\dim R_H[x] = \dim R$ or $\dim R_H[x] = \dim R + 1$ holds in this case. More precisely, we prove that

$$\dim R_H[x] = \begin{cases} \dim R + 1 & \text{if } \operatorname{char} R = 0 \\ \dim R & \text{if } \operatorname{char} R \neq 0 \end{cases}$$

when R is a Prüfer domain. For a Prüfer domain R, one has $\dim R[x] = \dim R + 1$ [18]. Hence, our result shows that the behavior of $\dim R_H[x]$ is also very similar to that of $\dim R[x]$ in the case R is a Prüfer domain.

2. Krull dimension of $R_H[x]$

In this section, we study the Krull dimension of the Hurwitz polynomial ring $R_H[x]$ over a Prüfer domain R. Note that if the characteristic of a ring R is nonzero, then $\dim R_H[x] = \dim R$ [5, Section 7]. Hence, when studying the Krull dimension of $R_H[x]$ we can always assume that $\operatorname{char} R = 0$. With this assumption, $R_H[x]$ is an integral domain if R is by the following proposition.

Proposition 1 ([1, Proposition 1])**.** *$R_H[x]$ is an integral domain if and only if R is an integral domain with* $\operatorname{char} R = 0$*.*

We first collect some well-known results about $\dim R_H[x]$ when R is a ring (not necessarily Prüfer) that will be used later in obtaining the main result of the paper. The following theorem says that the Krull dimension of $R_H[x]$ and $R[x]$ are the same if R contains the rational numbers (see [5, Theorem 1.4] or [10, Theorem 6]).

Theorem 2. *If R is a ring such that $\mathbb{Q} \subseteq R$, then $R_H[x] \cong R[x]$ and hence* $\dim R_H[x] = \dim R[x]$.

Lemma 3 ([10, Lemma 7])**.** *If R is a ring, then any three different prime ideals $Q_1 \subset Q_2 \subset Q_3$ in $R_H[x]$ cannot contract to the same prime ideal of R.*

Let $\phi : R_H[x] \to R$ be the natural ring homomorphism mapping each polynomial in $R_H[x]$ to its constant term. Hence, if P is a prime ideal of R, then $\phi^{-1}(P)$ is a prime ideal of $R_H[x]$.

Using the aforementioned results, one can easily obtain the following theorem.

Theorem 4 ([10, Theorem 9])**.** *If R is a finite-dimensional ring, then*

$$\dim R \leq \dim R_H[x] \leq 2 \dim R + 1.$$

Furthermore, if $\mathbb{Q} \subseteq R$ or R is an integral domain with $\operatorname{char} R = 0$*, then*

$$\dim R + 1 \leq \dim R_H[x] \leq 2 \dim R + 1.$$

We now calculate $\dim R_H[x]$ when R is a Prüfer domain. Our purpose is to that $\dim R_H[x] = \dim R + 1$ if $\operatorname{char} R = 0$. We need several lemmas. Our first lemma is for the usual polynomial ring $R[x]$.

Lemma 5. *Let R be a Prüfer domain. If P is a height one prime ideal of R, then $P[x]$ is a height one prime ideal of $R[x]$.*

Proof. Suppose on the contrary that $\operatorname{ht} P[x] \geq 2$. Then there exists a chain

$$(0) \subset Q_1 \subset P[x]$$

of prime ideals of $R[x]$. By localizing at $S := R \setminus P$, we get a chain

$$(0) \subset S^{-1}Q_1 \subset S^{-1}(P[x]) = (S^{-1}P)[x] = P_P[x]$$

of prime ideals of $S^{-1}(R[x]) = (S^{-1}R)[x] = R_P[x]$. Since R_P is a 1-dimensional valuation domain, we have $\dim R_P[x] = 2$ by [18, Theorem 4]. It follows that $P_P[x]$ is a maximal ideal of $R_P[x]$. However, this is impossible since $P_P[x]$ is properly contained in the prime ideal $P_P + (x)$ of $R_P[x]$. Therefore, $\operatorname{ht} P[x] = 1$. □

Corollary 6. *If V is a 1-dimensional valuation domain with maximal ideal P, then $P[x]$ is a height one prime ideal of $V[x]$.*

Lemma 7. *If V is a 1-dimensional valuation domain with maximal ideal P such that $\operatorname{char} V/P = 0$, then $P_H[x]$ is a height one prime ideal of $V_H[x]$.*

Proof. First note that $P_H[x]$ is a prime ideal of $V_H[x]$. Indeed, $V_H[x]/P_H[x] \cong (V/P)_H[x]$ is an integral domain since $\operatorname{char} V/P = 0$ (see Proposition 1). Also, $\operatorname{char} V/P = 0$ implies $\operatorname{char} V = 0$. Thus, $V_H[x]$ is also an integral domain. It follows that $\operatorname{ht} P_H[x] \geq 1$ (since (0) is a prime ideal of $V_H[x]$). Now suppose on the contrary that $\operatorname{ht} P_H[x] \geq 2$. Then there exists a chain

$$(0) \subset Q_1 \subset P_H[x]$$

of prime ideals of $V_H[x]$. Consider the ring homomorphism $\varphi : V[x] \to V_H[x]$ defined by $\varphi(\sum_{i=0}^k a_i x^i) = \sum_{i=0}^k i! a_i x^i$. Since V is an integral domain with $\operatorname{char} V = 0$, φ is a ring monomorphism and hence $\varphi(V[x])$ is a subring of $V_H[x]$.

Claim 1. $P_H[x] \cap \varphi(V[x]) = \varphi(P[x])$.

It is clear that $\varphi(P[x]) \subseteq P_H[x] \cap \varphi(V[x])$. For the other containment, let $f = \sum_{i=0}^k b_i x^i \in P_H[x]$ $(b_i \in P)$. If $f \in \varphi(V[x])$, then $f = \sum_{i=0}^k i! a_i x^i$ for some $a_i \in V$. Thus, $i! a_i = b_i \in P$ for all i. Since $\operatorname{char} V/P = 0$, $P \cap \mathbb{Z} = (0)$. It follows that $i! \notin P$ and hence $a_i \in P$ for all i.

Claim 2. $Q_1 \cap \varphi(V[x]) = P_H[x] \cap \varphi(V[x])$.

Consider the chain

$$(0) \subseteq Q_1 \cap \varphi(V[x]) \subseteq P_H[x] \cap \varphi(V[x])$$

of prime ideals of $\varphi(V[x])$. Note that $Q_1 \cap \varphi(V[x]) \neq (0)$. Indeed, taking any $0 \neq f = \sum_{i=0}^{k} b_i x^i \in Q_1$, we have $0 \neq k!f \in Q_1 \cap \varphi(V[x])$. By Corollary 6, $P[x]$ is a height one prime ideal of $V[x]$. Hence, by Claim 1, $P_H[x] \cap \varphi(V[x]) = \varphi(P[x])$ is a height one prime ideal of $\varphi(V[x])$. Since $Q_1 \cap \varphi(V[x]) \neq (0)$, we must have

$$Q_1 \cap \varphi(V[x]) = P_H[x] \cap \varphi(V[x]).$$

Claim 3: $P_H[x] \subseteq Q_1$.

Let $f = \sum_{i=0}^{k} b_i x^i \in P_H[x]$. Then by Claim 2, we have $k!f \in P_H[x] \cap \varphi(V[x]) = Q_1 \cap \varphi(V[x]) \subseteq Q_1$. We have $Q_1 \cap \mathbb{Z} = (Q_1 \cap V) \cap \mathbb{Z} \subseteq P \cap \mathbb{Z} = (0)$. Therefore, $k! \notin Q_1$ and hence $f \in Q_1$.

Claim 3 contradicts the assumption that $Q_1 \subset P_H[x]$. Therefore, $\operatorname{ht} P_H[x] = 1$ and the proof of the lemma is completed. □

Lemma 8. *Let P be a prime ideal of an integral domain R. Then* $\operatorname{char} R/P = \operatorname{char} R_P/P_P$.

Proof. The natural homomorphism $R/P \to R_P/P_P$ is a ring monomorphism. Hence, R/P can be considered as a subring of R_P/P_P. Thus they have the same characteristic. □

Lemma 9. *Let R be a Prüfer domain. If P is a height one prime ideal of R such that* $\operatorname{char} R/P = 0$*, then $P_H[x]$ is a height one prime ideal of $R_H[x]$.*

Proof. As in the proof of Lemma 7, we can show that $P_H[x]$ is a prime ideal of $R_H[x]$ and that $R_H[x]$ is an integral domain. Hence, $\operatorname{ht} P_H[x] \geq 1$. Now suppose on the contrary that $\operatorname{ht} P_H[x] \geq 2$. Then there exists a chain

$$(0) \subset Q_1 \subset P_H[x]$$

of prime ideals of $R_H[x]$. By localizing at $S := R \setminus P$, we get a chain

$$(0) \subset S^{-1}Q_1 \subset S^{-1}(P_H[x]) = (S^{-1}P)_H[x] = (P_P)_H[x]$$

of prime ideals of $S^{-1}(R_H[x]) = (S^{-1}R)_H[x] = (R_P)_H[x]$. Hence, $\operatorname{ht}(P_P)_H[x] \geq 2$. Note that R_P is a 1-dimensional valuation domain with maximal ideal P_P. Since $\operatorname{char} R/P = 0$, we have $\operatorname{char} R_P/P_P = 0$ by Lemma 8. By Lemma 7, $(P_P)_H[x]$ is a height one prime ideal of $(R_P)_H[x]$. This contradiction finishes the proof of the lemma. □

We say that a ring R is a finite-dimensional ring if the Krull dimension of R is finite, i.e., $\dim R = n < \infty$.

Theorem 10. *If R is a finite-dimensional Prüfer domain with* $\operatorname{char} R = 0$*, then* $\dim R_H[x] = \dim R + 1$.

Proof. Since R is an integral domain with $\operatorname{char} R = 0$, we have $\dim R_H[x] \geq \dim R + 1$ (Theorem 4). We now show that $\dim R_H[x] \leq \dim R + 1$ by using induction on $\dim R$. If $\dim R = 0$, then $\dim R_H[x] \leq 1$ by Theorem 4. Suppose that $\dim R = n \geq 1$ and that the result holds for any Prüfer domain with dimension $< n$. We show that a chain of prime ideals of length $n+2$ in $R_H[x]$ never exists. Suppose on the contrary that such a chain exists, says,

$$(0) \subset Q_1 \subset Q_2 \subset \cdots \subset Q_{n+2}.$$

Let $P = Q_2 \cap R$. Since $(0) \subset Q_1 \subset Q_2$ cannot contract to the same prime ideal of R, $\operatorname{ht} P \geq 1$. We have a ring epimorphism $R_H[x] \to R_H[x]/P_H[x] \cong (R/P)_H[x]$. Let

$$\overline{Q}_2 \subset \cdots \subset \overline{Q}_{n+2}$$

be the images of $Q_2 \subset \cdots \subset Q_{n+2}$ in $(R/P)_H[x]$.

Case 1. $\operatorname{char} R/P \neq 0$**.**

In this case, $\dim(R/P)_H[x] = \dim(R/P) \leq \dim R - \operatorname{ht} P \leq n-1$. This is a contradiction since the chain $\overline{Q}_2 \subset \cdots \subset \overline{Q}_{n+2}$ has length n.

Case 2. $\operatorname{char} R/P = 0$**.**

By induction hypothesis, we have $\dim(R/P)_H[x] \leq \dim(R/P)+1 \leq \dim R - \operatorname{ht} P + 1 \leq \dim R = n$. Since the chain $\overline{Q}_2 \subset \cdots \subset \overline{Q}_{n+2}$ has length n and $(R/P)_H[x]$ is a domain, we must have $\operatorname{ht} P = 1$ and $\overline{Q}_2 = (0)$. The later equality means $P_H[x] = Q_2$ and hence $\operatorname{ht} P_H[x] \geq 2$. However, this is impossible by Lemma 9.

Therefore, every chain of prime ideals of $R_H[x]$ must have length $\leq n+1$. This finishes the proof of $\dim R_H[x] \leq \dim R+1$ and hence of the desired result $\dim R_H[x] = \dim R + 1$. □

If $\operatorname{char} R \neq 0$, then $\dim R_H[x] = \dim R$. Adding this to Theorem 10, we get the main theorem of the paper.

Theorem 11. *Let R be a finite-dimensional Prüfer domain. Then*

$$\dim R_H[x] = \begin{cases} \dim R + 1 & \textit{if } \operatorname{char} R = 0 \\ \dim R & \textit{if } \operatorname{char} R \neq 0. \end{cases}$$

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