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WEIERSTRASS SEMIGROUPS ON DOUBLE COVERS OF PLANE CURVES OF DEGREE SIX WITH TOTAL FLEXES

Seon Jeong Kim and Jiryo Komeda

Abstract. In this paper, we study Weierstrass semigroups of ramification points on double covers of plane curves of degree 6. We determine all the Weierstrass semigroups when the genus of the covering curve is greater than 29 and the ramification point is on a total flex.

1. Introduction

Let C be a smooth irreducible curve of genus g, where a *curve* means a projective 1-dimensional algebraic variety over an algebraically closed field k of characteristic 0. For a point P of C we define the *Weierstrass semigroup* $H(P)$ of P as follows:

$$H(P) = \{n \in \mathbb{N}_0 \mid \text{there is a rational function } f \text{ on } C \text{ such that } (f)_\infty = nP\},$$

where $\mathbb{N}_0$ is the additive monoid of non-negative integers and $(f)_\infty$ means the polar divisor of f. Then $H(P)$ is a *numerical semigroup of genus* g, which means a submonoid of $\mathbb{N}_0$ whose complement is a finite set with cardinality g. The genus of a numerical semigroup H is denoted by $g(H)$. For a numerical semigroup H we denote by $d_2(H)$ the set consisting of the elements h with $2h \in H$, which is a numerical semigroup. If $\pi : \tilde{C} \longrightarrow C$ is a double covering of a curve with a ramification point $\tilde{P}$ over P, then we have $d_2(H(\tilde{P})) = H(P)$. When we treat a double covering $\pi : \tilde{C} \longrightarrow C$ of a curve we assume that C and $\tilde{C}$ are smooth curves.

We will study about the numerical semigroups H which are the Weierstrass semigroups of ramification points on double covers of smooth plane curves of degree d. In this article such a numerical semigroup H is said to be *of double covering type of a plane curve of degree* d, which is abbreviated to *DCP of degree* d. We pose the following problem:

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DCP Hurwitz' Problem. *Let d be a positive integer. Then determine all the Weierstrass semigroups which are DCP of degree d.*

To describe the known facts for DCP Hurwitz' Problem we use the notation $\langle a_1, \ldots, a_s \rangle$ which means the additive monoid generated by $a_1, \ldots, a_s$ for positive integers $a_1, \ldots, a_s$. If $d_2(H)$ is the Weierstrass semigroup of a point on a smooth plane curve of degree $d \leqq 3$, i.e., $d_2(H) = \mathbb{N}_0$ or $\langle 2, 3 \rangle$, then we can show that H is DCP (for example, see [6]). In the case $d = 4$ the papers [1], [2], [3] and [7] gave all the numerical semigroups of DCP of degree 4. In the recent monograph [5] the authors determine the numerical semigroups H which are DCP of degree 5 in the case where $g(H) \geqq 18$ and $d_2(H) = \langle 4, 5 \rangle$ or $\langle 4, 7, 10, 13 \rangle$.

Let C be a smooth plane curve and P its point. Let Z be a plane curve. We denote by $C.Z$ the intersection divisor of C with Z. Moreover, let $\mathrm{ord}_P(C.Z)$ be the multiplicity of $C.Z$ at P. We denote by T_P the tangent line at P on C. If P is a total flex on a smooth plane curve C of degree 6, i.e., $\mathrm{ord}_P\ C.T_P = 6$, then $H(P) = \langle 5, 6 \rangle$. The following is the main result of this article:

Main Theorem. *If H is a numerical semigroup of genus $\geqq 30$ with $d_2(H) = \langle 5, 6 \rangle$, then it is DCP of degree 6.*

2. Preliminary results

In this section we review some known facts which will be used in the proof of Main Theorem.

Let C be a smooth plane curve and $P_1, \ldots, P_n$ points of C among which we permit the same points. For a positive integer m we denote by $C^*(m; P_1, \ldots, P_n)$ the set of plane curves X of degree m such that $X.C \geqq P_1 + \cdots + P_n$. We consider the set $C(m; P_1, \ldots, P_n) = C^*(m; P_1, \ldots, P_n) \cup \{0\}$ as a k-vector space. The points $P_1, \ldots, P_n$ *impose independent condition on the system of curves of degree m* if

$$\dim C(m; P_1, \ldots, P_n) = \frac{(m+2)(m+1)}{2} - n.$$

The points $P_1, \ldots, P_n$ *fail to impose independent condition on the system of curves of degree m* if

$$\dim C(m; P_1, \ldots, P_n) > \frac{(m+2)(m+1)}{2} - n.$$

Lemma 2.1. i) 2 *points impose independent condition on the system of lines.*

ii) 3 *points fail to impose independent condition on the system of lines if and only if the three points are collinear.*

iii) 3 *points impose independent condition on the system of conics.*

iv) 4 *points fail to impose independent condition on the system of conics if and only if the four points are collinear.*

v) 5 *points fail to impose independent condition on the system of conics if and only if there are four collinear points among them.*

vi) 6 *points fail to impose independent condition on the system of conics if and only if there are four collinear points among them or the six points are on a conic.*

vii) 4 *points impose independent condition on the system of cubics.*

viii) 5 *points fail to impose independent condition on the system of cubics if and only if the five points are collinear.*

ix) 6 *points fail to impose independent condition on the system of cubics if and only if there are five collinear points among them.*

x) 7 *points fail to impose independent condition on the system of cubics if and only if there are five collinear points among them.*

xi) 8 *points fail to impose independent condition on the system of cubics if and only if there are five collinear points among them or the eight points are on a conic.*

Theorem 2.2 (Cayley-Bacharach)**.** *Let C be a non-singular plane curve. Let X_1 and X_2 be two plane curves of degree d and e resp., meeting in a collection Γ of de points of C with multiplicity. Let Y be a curve of degree $d+e-3$ such that the intersection $Y.C$ contains all but one point of Γ. Then $Y.C$ contains that remaining point also.*

Theorem 2.2 of [7] is replaced by the following in our case. Indeed, to show that H is DCP we use this many times.

Theorem 2.3. *Let (C, P) be a pointed non-singular plane curve of degree 6 and H a numerical semigroup with $d_2(H) = H(P)$ and $g(H) \geqq 30$. Set*

$$n = \min\{h \in H \mid h \text{ is odd}\}.$$

We note that

$$g(H) = 20 + \frac{n-1}{2} - r$$

with some non-negative integer r (for example, see Lemma 3.1 in [1]). Let $Q_1, \dots, Q_r$ be points of C different from P with $h^0(Q_1 + \cdots + Q_r) = 1$. Moreover, assume that H has an expression

$$H = 2d_2(H) + \langle n, n+2l_1, \dots, n+2l_s \rangle$$

of generators with positive integers $l_1, \dots, l_s$ such that for any cubic C_3 the inequality $C_3.C \geqq (l_i - 1)P + Q_1 + \cdots + Q_r$ implies that $C_3.C \geqq l_iP + Q_1 + \cdots + Q_r$, i.e.,

$$h^0(K - (l_i - 1)P - Q_1 - \cdots - Q_r) = h^0(K - l_iP - Q_1 - \cdots - Q_r),$$

where K is a canonical divisor on C. Then the complete linear system $|nP - 2Q_1 - \cdots - 2Q_r|$ is base point free and there is a double covering $\pi : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ over P satisfying $H(\tilde{P}) = H$, i.e., H is DCP of degree 6.

3. Proof of Main Theorem

In this section, let H be a numerical semigroup with $g(H) \geqq 30$, $d_2(H) = H_6$ and $n \geqq 25$ where we set $H_6 = \langle 5, 6\rangle$ and $n = \min\{h \in H \mid h \text{ is odd}\}$. Let $r(H)$ be the number of the odd elements of H which are larger than n and less than or equal to $n + 38$. We associate to H the diagram where $\odot$, $\circ$ and $\times$ indicate an integer in $M(H)$, $H\backslash M(H)$ and $\mathbb{N}_0\backslash H$ respectively where $M(H)$ denotes the minimal set of generators for the monoid H. For example, we associate the following diagram with the numerical semigroup $H = 2H_6 + \langle n, n + 4, n + 18\rangle$:

$$\begin{array}{ccccc}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \times & \odot & \times & \times \\
(n) & \bullet & \circ & \circ & \odot \downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}$$

Hence, we get $0 \leqq r(H) \leqq 10$. Moreover, we obtain $g(H) = 20 + \frac{n-1}{2} - r(H)$. Let $t(H)$ be the cardinality of the set

$$\{u \in M(H) \mid u \text{ is an odd integer distinct from } n\}.$$

The proof of Main Theorem is divided into forty two cases classified by the value of $t(H)$ and the generators which are odd.

(I) *The case* $t(H) = 0$. Then $H = 2H_6 + \langle n\rangle$, which is DCP by Proposition 2.3 in [4].

(II) *The case* $t(H) = 1$. There are ten kinds of numerical semigroups.

II-1) $H = 2H_6 + \langle n, n + 38\rangle$. By (i) in Theorem 2.5 in [5] H is DCP.
II-2) $H = 2H_6 + \langle n, n + 28\rangle$. By (i) in Theorem 2.5 in [5] H is DCP.
II-3) $H = 2H_6 + \langle n, n + 26\rangle$. By (ii) in Theorem 2.5 in [5] H is DCP.
II-4) $H = 2H_6 + \langle n, n + 18\rangle$. By (i) in Theorem 2.5 in [5] H is DCP.
II-5) $H = 2H_6 + \langle n, n + 16\rangle$. By (iii) in Theorem 2.5 in [5] H is DCP.
II-6) $H = 2H_6 + \langle n, n + 14\rangle$. By (ii) in Theorem 2.5 in [5] H is DCP.
II-7) $H = 2H_6 + \langle n, n + 8\rangle$. By (i) in Theorem 2.5 in [5] H is DCP.
II-8) $H = 2H_6 + \langle n, n + 6\rangle$. We have the following diagram.

$$\begin{array}{ccccc}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \times & \times & \odot & \times \\
(n) & \bullet & \times & \circ & \circ \downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}$$

Let C be a plane curve of degree 6 defined by an equation

$$(yz^2 - x^3)\left(\frac{1}{2}z^3 + ax^3\right) + (yz^2 + x^3 - 2y^3)\left(\frac{1}{2}z^3 + by^3\right) = 0,$$

that is to say,

$$z^3(yz^2 - y^3) + ax^3(yz^2 - x^3) + by^3(yz^2 + x^3 - 2y^3) = 0,$$

where a and b are general constants. Then the curve C is non-singular. Let C_{31} and C_{32} be cubics defined by $yz^2 - x^3 = 0$ and $yz^2 + x^3 - 2y^3 = 0$ respectively. The former one has only one singularity $(0 : 1 : 0)$ and it is irreducible. The latter one is non-singular. Let $P = (0 : 0 : 1) \in C$. Then $T_P.C = 6P$, where the tangent line T_P is defined by the equation $y = 0$. Let ζ be a primitive 6-th root of unity. For $i = 1, \dots, 6$ we set $Q_i = (\zeta^i : \zeta^{3i} : 1) \in C$. Then we have

$$C_{31}.C_{32} = 3P + Q_1 + \cdots + Q_6 \text{ and } C_{3i}.C \geqq 3P + Q_1 + \cdots + Q_6 \text{ for } i = 1, 2.$$

Since C_{3i}'s are irreducible, any four points of $P, Q_1, \dots, Q_6$ are not collinear and the seven points $P, Q_1, \dots, Q_6$ are not on a conic. Hence, by Lemma 2.1 xi) we get

$$h^0(K - 2P - Q_1 - \cdots - Q_6) = 10 - 8 = 2 = h^0(K - 3P - Q_1 - \cdots - Q_6).$$

Hence, H is DCP by Theorem 2.3.

From now on, let us take a pointed curve (C, P) with $H(P) = H_6$ and distinct points $Q_1, \dots, Q_r$ of C with $r = r(H)$ different from P, which are defined in each item. We set $E_r = Q_1 + \cdots + Q_r$.

II-9) $H = 2H_6 + \langle n, n+4 \rangle$. We have the following diagram.

$$\begin{array}{ccccc}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \times & \odot & \times & \times \\
(n) & \bullet & \circ & \circ & \times \downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}$$

Let us take distinct lines L_1 and L_2 through P such that the intersection of C and L_1 (resp. L_2) contains distinct points Q_1, Q_2 and Q_3 (resp. Q_4, Q_5 and Q_6). Let C_3 be a cubic with $C_3.C \geqq P + Q_1 + \cdots + Q_6$. Then we get $C_3 = L_1 L_2 L$ with a line L which means that

$$h^0(K - P - Q_1 - \cdots - Q_6) = h^0(K - 2P - Q_1 - \cdots - Q_6) = 3.$$

Hence, H is DCP by Theorem 2.3.

II-10) $H = 2H_6 + \langle n, n+2 \rangle$. By (ii) in Theorem 2.5 in [5] H is DCP.

(III) *The case* $t(H) = 2$. There are twenty kinds of numerical semigroups

III-1) $H = 2H_6 + \langle n, n+26, n+28 \rangle$. By (iv) in Theorem 2.5 in [5] H is DCP.

III-2) $H = 2H_6 + \langle n, n+14, n+28 \rangle$. By (iv) in Theorem 2.5 in [5] H is DCP.

III-3) $H = 2H_6 + \langle n, n+2, n+28 \rangle$. By (iv) in Theorem 2.5 in [5] H is DCP.

III-4) $H = 2H_6 + \langle n, n+18, n+26 \rangle$. By (v) in Theorem 2.5 in [5] H is DCP.

III-5) $H = 2H_6 + \langle n, n+8, n+26 \rangle$. By (v) in Theorem 2.5 in [5] H is DCP.

III-6) $H = 2H_6 + \langle n, n+16, n+18\rangle$. By (vii) in Theorem 2.5 in [5] H is DCP.

III-7) $H = 2H_6 + \langle n, n+14, n+18\rangle$. By (vi) in Theorem 2.5 in [5] H is DCP.

III-8) $H = 2H_6 + \langle n, n+4, n+18\rangle$. See the beginning of this section for the diagram. Let us take distinct lines L_1 and L_2 through P such that the intersection of C and L_1 (resp. L_2) contains distinct points Q_1, Q_2 and Q_3 (resp. Q_4, Q_5 and Q_6). Take a point Q_7 of C which does not lie on $L_1 \cup L_2$. Let C_3 be a cubic through the eight points P, $Q_1, \ldots, Q_7$. Then C_3 should be a reduced curve L_1L_2L with a line $L \ni Q_7$. In this case we get $C_3.C \geqq 2P + E_7$. Hence, we get $h^0(K - P - E_7) = h^0(K - 2P - E_7) = 2$. Moreover, we have $h^0(K - 4P - E_7) = 0$. H is DCP by Theorem 2.3.

III-9) $H = 2H_6 + \langle n, n+2, n+18\rangle$.

$$
\begin{array}{ccccc}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \odot & \times & \times & \times \\
(n) & \bullet & \circ & \times & \odot \downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}
$$

Let us take lines L_1 and L_2 with $L_1 \ni P$ and $L_2 \not\ni P$ such that the intersection of C and L_1 (resp. L_2) contains distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5 and Q_6). Let C_3 be a cubic through the seven points P, $Q_1, \ldots, Q_6$. Then C_3 must be a curve L_1C_2 with a conic $C_2 \ni Q_5$ and Q_6. Hence, by Lemma 2.1 iii) we get

$$h^0(K - E_6) = h^0(K - P - E_6) = 6 - 2 = 4.$$

Let C'_3 be a cubic with $C'_3.C \geqq 8P + E_6$. Then C'_3 must be a curve $L_1T_PL_2$ which implies that $C'_3.C \not\geqq 8P$. This is a contradiction. Hence, we obtain $h^0(K - 8P - E_6) = 0$.

III-10) $H = 2H_6 + \langle n, n+14, n+16\rangle$. By (viii) in Theorem 2.5 in [5] H is DCP.

III-11) $H = 2H_6 + \langle n, n+8, n+16\rangle$.

$$
\begin{array}{ccccc}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \times & \times & \times & \odot \\
(n) & \bullet & \times & \odot & \circ \downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}
$$

Let us take distinct lines L_1 and L_2 not through P such that the intersection of C and L_1 (resp. L_2) contains distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5 and Q_6). Let C_3 be a cubic with $C.C_3 \geqq 3P + Q_1 + \cdots + Q_6$. Then C_3 must be the curve $L_1T_PL_2$, which implies that

$$h^0(K-3P-E_6)=\cdots=h^0(K-6P-E_6)=1 \text{ and } h^0(K-7P-E_6)=0.$$

III-12) $H=2H_6+\langle n,n+2,n+16\rangle$.

$$\begin{array}{ccccc}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \odot & \times & \times & \times \\
(n) & \bullet & \circ & \odot & \times\downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}$$

Let us take distinct lines L_1 and L_2 through P such that the intersection of C and L_1 (resp. L_2) contains distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5 and Q_6). Let C_3 be a cubic with $C_3.C \geqq E_6$. Then $C_3=L_1C_2$ with a conic $C_2 \ni Q_5, Q_6$. Hence, by Lemma 2.1 iii) we get

$$h^0(K-E_6)=h^0(K-P-E_6)=6-2=4.$$

Moreover, let $C_3.C \geqq 4P+E_6$. Then we obtain $C_3=L_1L_2T_P$, which implies that

$$h^0(K-4P-E_6)=h^0(K-8P-E_6)=1.$$

III-13) $H=2H_6+\langle n,n+8,n+14\rangle$.

$$\begin{array}{ccccc}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \times & \times & \times & \odot \\
(n) & \bullet & \odot & \times & \circ\downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}$$

Let us take lines L_1 and L_2 with $L_1 \not\ni P$ and $L_2 \ni P$ such that the intersection of C and L_1 (resp. L_2) contains distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5 and Q_6). Let C_3 be a cubic through the seven points $P, Q_1, \ldots, Q_6$. Then $C_3=L_1L_2L$ with a line L, which means that

$$h^0(K-3P-E_6)=h^0(K-7P-E_9)=1.$$

III-14) $H=2H_6+\langle n,n+6,n+14\rangle$.

$$\begin{array}{ccccc}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \times & \times & \odot & \times \\
(n) & \bullet & \odot & \circ & \circ\downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}$$

Let us take distinct lines L_1, L_2 and L_3 through P such that the intersection of C and L_1 (resp. L_2, L_3) contains distinct points Q_1, Q_2 and Q_3 (resp. Q_4 and Q_5, Q_6 and Q_7). Let C_3 be a cubic with $C_3.C \geqq 2P + E_7$. Then $C_3 = L_1L_2L_3$, which implies that

$$h^0(K-2P-E_7) = h^0(K-3P-E_7) = 1 \text{ and } h^0(K-4P-Q_1-\cdots-Q_7) = 0.$$

III-15) $H = 2H_6 + \langle n, n+6, n+8 \rangle$.

$$\begin{array}{ccccl}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \times & \times & \odot & \odot \\
(n) & \bullet & \times & \circ & \circ \downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}$$

Let us take distinct lines L_1 and L_2 not through P such that the intersection of C and L_1 (resp. L_2) contains distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5, Q_6 and Q_7). Let C_3 be a cubic through the seven points $Q_1, \ldots, Q_7$. Then C_3 must be the curve L_1L_2L with a line L. If a cubic C_3 satisfies $C_3.C \geqq 2P + E_7$, then $C_3 = L_1L_2T_P$, which implies that

$$h^0(K-2P-E_7) = \cdots = h^0(K-6P-E_7) = 1.$$

III-16) $H = 2H_6 + \langle n, n+4, n+8 \rangle$.

$$\begin{array}{ccccl}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \times & \odot & \times & \odot \\
(n) & \bullet & \circ & \circ & \circ \downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}$$

Let us take distinct lines L_1, L_2 and L_3 with $L_1 \not\ni P$, $L_2 \ni P$ and $L_3 \ni P$ such that the intersection of C and L_1 (resp. L_2, L_3) contains distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5 and Q_6, Q_7 and Q_8). Let C_3 be a cubic through the nine points $P, Q_1, \ldots, Q_8$. Then C_3 is the curve $L_1L_2L_3$, which implies that

$$h^0(K-P-E_8) = h^0(K-2P-E_8) = 1 \text{ and } h^0(K-3P-E_8) = 0.$$

III-17) $H = 2H_6 + \langle n, n+2, n+8 \rangle$.

$$\begin{array}{ccccl}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \odot & \times & \times & \odot \\
(n) & \bullet & \circ & \times & \circ \downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}$$

Let us take distinct lines L_1 and L_2 with $L_1 \not\ni P$ and $L_2 \ni P$ such that the intersection of C and L_1 (resp. L_2) contains distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5, Q_6 and Q_7). Let C_3 be a cubic through the seven points $Q_1, \dots, Q_7$. Then C_3 must be a curve L_1L_2L with a line L. Hence, we obtain

$$\begin{gathered} h^0(K - Q_1 - \cdots - Q_7) = h^0(K - P - Q_1 - \cdots - Q_7) = 3, \\ h^0(K - 3P - Q_1 - \cdots - Q_7) = h^0(K - 7P - Q_1 - \cdots - Q_7) = 1. \end{gathered}$$

III-18) $H = 2H_6 + \langle n, n+4, n+6 \rangle$.

$$\begin{array}{ccccc} \to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\ \bullet & \times & \odot & \odot & \times \\ (n) & \bullet & \circ & \circ & \circ \downarrow +10 \\ \searrow +12 & (n+12) & \bullet & \circ & \circ \\ & & (n+24) & \bullet & \circ \\ & & & (n+36) & \bullet \\ & & & & (n+48) \end{array}$$

Let us take distinct lines L_1, L_2 and L_3 through P such that the intersection of C and L_1 (resp. L_2, L_3) contains distinct points Q_1, Q_2 and Q_3 (resp. Q_4, Q_5 and Q_6, Q_7 and Q_8). Let C_3 be a cubic with $C_3.C \geqq P + E_8$. Then we get $C_3 = L_1L_2L_3$, which implies that

$$h^0(K - P - E_8) = h^0(K - 3P - E_8) = 1.$$

III-19) $H = 2H_6 + \langle n, n+2, n+6 \rangle$.

$$\begin{array}{ccccc} \to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\ \bullet & \odot & \times & \odot & \times \\ (n) & \bullet & \circ & \circ & \circ \downarrow +10 \\ \searrow +12 & (n+12) & \bullet & \circ & \circ \\ & & (n+24) & \bullet & \circ \\ & & & (n+36) & \bullet \\ & & & & (n+48) \end{array}$$

Let us take distinct lines L_1, L_2 and L_3 through P such that the intersection of C and L_1 (resp. L_2, L_3) contains distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5 and Q_6, Q_7 and Q_8). Let C_3 be a cubic with $C_3.C \geqq E_8$. Then we get $C_3 = L_1C_2$ with $C_2 \ni Q_5, Q_6, Q_7, Q_8$, which means that

$$h^0(K - E_8) = h^0(K - P - E_8) = 6 - 4 = 2$$

from by Lemma 2.1 iv). Moreover, let $C_3.C \geqq 2P + E_8$. Then $C_3 = L_1L_2L_3$, which implies that

$$h^0(K - 2P - E_8) = h^0(K - 3P - E_8) = 1.$$

III-20) $H = 2H_6 + \langle n, n+2, n+4 \rangle$.

$$\begin{array}{ccccl}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \odot & \odot & \times & \times \\
(n) & \bullet & \circ & \circ & \times \downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}$$

Let us take distinct lines L_1 and L_2 through P such that the intersection of C and L_1 (resp. L_2) contains distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5, Q_6 and Q_7). Let C_3 be a cubic with $C_3.C \geqq E_7$. Then we get $C_3 = L_1L_2L$ with a line L, which means that

$$h^0(K - E_7) = h^0(K - 2P - E_7) = 3.$$

(IV) *The case* $t(H) = 3$. There are ten kinds of numerical semigroups.

IV-1) $H = 2H_6 + \langle n, n+14, n+16, n+18\rangle$.

$$\begin{array}{ccccl}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \times & \times & \times & \times \\
(n) & \bullet & \odot & \odot & \odot \downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}$$

Let $Q_1, \ldots, Q_6$ be general six points of C. Since we have $h^0(K - 6P) = h^0(12P) = 6$, we obtain $h^0(K - 6P - E_6) = 0$. H is DCP by Theorem 2.3.

IV-2) $H = 2H_6 + \langle n, n+2, n+16, n+18\rangle$.

$$\begin{array}{ccccl}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \odot & \times & \times & \times \\
(n) & \bullet & \circ & \odot & \odot \downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}$$

Let us take distinct lines L_1 and L_2 with $L_1 \ni P$ and $L_2 \not\ni P$ such that the intersection of C and L_1 (resp. L_2) contains distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5 and Q_6). Take a point Q_7 of C which does not lie on $L_1 \cup L_2$. Let C_3 be a cubic through the seven points $Q_1, \ldots, Q_7$. Then C_3 must be the reduced curve L_1C_2, where C_2 is a conic containing Q_5, Q_6 and Q_7. Hence, by Lemma 2.1 iii) we obtain

$$h^0(K - E_7) = h^0(K - P - E_7) = 6 - 3 = 3.$$

We get $h^0(K - 8P - E_7) = 0$, because there are no cubics C_3 such that $C_3.C \geqq 8P + E_7$.

IV-3) $H = 2H_6 + \langle n, n+2, n+4, n+18\rangle$.

$$\begin{array}{ccccc}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \odot & \odot & \times & \times \\
(n) & \bullet & \circ & \circ & \odot \downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}$$

Let us take distinct lines L_1 and L_2 through P such that the intersection of C and L_1 (resp. L_2) contains distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5, Q_6 and Q_7). Take a point Q_8 of C which does not lie on $L_1 \cup L_2$. Let C_3 be a cubic through the eight points $Q_1, \ldots, Q_8$. Then C_3 must be a curve L_1L_2L with a line $L \ni Q_8$. Hence, we get

$$h^0(K-E_8) = h^0(K-P-E_8) = h^0(K-2P-E_8) = 2 \text{ and } h^0(K-4P-E_8) = 0.$$

IV-4) $H = 2H_6 + \langle n, n+8, n+14, n+16\rangle$.

$$\begin{array}{ccccc}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \times & \times & \times & \odot \\
(n) & \bullet & \odot & \odot & \circ \downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}$$

Let us take distinct lines L_1 and L_2 with $L_1 \not\ni P$ and $L_2 \ni P$ such that the intersection of C and L_1 (resp. L_2) contains distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5 and Q_6). Take a point Q_7 of C which does not lie on $L_1 \cup L_2$. Let C_3 be a cubic through the eight points $P, Q_1, \ldots, Q_7$. Then we get $C_3 = L_1L_2L$ with $L \ni Q_7$, which implies that $h^0(K-3P-E_7) = 0$.

IV-5) $H = 2H_6 + \langle n, n+2, n+8, n+16\rangle$.

$$\begin{array}{ccccc}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \odot & \times & \times & \odot \\
(n) & \bullet & \circ & \odot & \circ \downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}$$

Let us take distinct lines L_1 and L_2 with $L_1 \not\ni P$ and $L_2 \ni P$ such that the intersection of C and L_1 (resp. L_2) contains distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5, Q_6 and Q_7). Take a point Q_8 of C which does not lie on $L_1 \cup L_2$. Let C_3 be a cubic through the eight points $Q_1, \ldots, Q_8$. Then C_3 must be a curve L_1L_2L with a line $L \ni Q_8$. Hence, we get

$$h^0(K-E_8) = h^0(K-P-E_8) = 2 \text{ and } h^0(K-3P-E_8) = 0.$$

IV-6) $H = 2H_6 + \langle n, n+6, n+8, n+14\rangle$.

$$\begin{array}{ccccc}
\rightarrow +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \times & \times & \odot & \odot \\
(n) & \bullet & \odot & \circ & \circ \downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}$$

Let us take distinct lines L_1 and L_2 not through P such that the intersection of C and L_1 (resp. L_2) contains distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5, Q_6 and Q_7). Take a point Q_8 of C not lying on the union $L_1 \cup L_2$. Let C_3 be a cubic through the eight points $Q_1, \ldots, Q_8$. Then C_3 must be the curve L_1L_2L with a line L containing Q_8, which implies that $h^0(K-2P-Q_1-\cdots-Q_8)=0$.

IV-7) $H = 2H_6 + \langle n, n+4, n+6, n+8\rangle$.

$$\begin{array}{ccccc}
\rightarrow +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \times & \odot & \odot & \odot \\
(n) & \bullet & \circ & \circ & \circ \downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}$$

Let us take distinct lines L_1, L_2 and L_3 not through P such that the intersection of C and L_1 (resp. L_2, L_3) contains distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5, Q_6 and Q_7, Q_8 and Q_9). Let C_3 be a cubic through the nine points $Q_1, \ldots, Q_9$. Then $C_3 = L_1L_2L_3$, which means that $h^0(K-P-E_9)=0$.

IV-8) $H = 2H_6 + \langle n, n+2, n+6, n+8\rangle$.

$$\begin{array}{ccccc}
\rightarrow +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \odot & \times & \odot & \odot \\
(n) & \bullet & \circ & \circ & \circ \downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}$$

Let us take distinct lines L_1, L_2 and L_3 with $L_1 \not\ni P$, $L_2 \not\ni P$ and $L_3 \ni P$ such that the intersection of C and L_1 (resp. L_2, L_3) contains distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5, Q_6 and Q_7, Q_8 and Q_9). Let C_3 be a cubic through the nine points $Q_1, \ldots, Q_9$. Then C_3 must be the curve $L_1L_2L_3$, which means that

$$h^0(K-E_9) = h^0(K-P-E_9) = 1 \text{ and } h^0(K-2P-E_9) = 0.$$

IV-9) $H = 2H_6 + \langle n, n+2, n+4, n+8\rangle$.

$$\begin{array}{ccccc}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \odot & \odot & \times & \odot \\
(n) & \bullet & \circ & \circ & \circ \downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}$$

Let us take distinct lines L_1, L_2 and L_3 with $L_1 \ni P$, $L_2 \ni P$ and $L_3 \not\ni P$ such that the intersection of C and L_1 (resp. L_2, L_3) contains distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5, Q_6 and Q_7, Q_8 and Q_9). Let C_3 be a cubic through the nine points $Q_1, \ldots, Q_9$. Then C_3 must be the curve $L_1 L_2 L_3$. Hence, we obtain

$$h^0(K - E_9) = h^0(K - 2P - E_9) = 1 \text{ and } h^0(K - 3P - E_9) = 0.$$

IV-10) $H = 2H_6 + \langle n, n+2, n+4, n+6 \rangle$.

$$\begin{array}{ccccc}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) \\
\bullet & \odot & \odot & \odot & \times \\
(n) & \bullet & \circ & \circ & \circ \downarrow +10 \\
\searrow +12 & (n+12) & \bullet & \circ & \circ \\
 & & (n+24) & \bullet & \circ \\
 & & & (n+36) & \bullet \\
 & & & & (n+48)
\end{array}$$

Let us take distinct lines L_1, L_2 and L_3 through P such that the intersection of C and L_1 (resp. L_2, L_3) contains distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5, Q_6 and Q_7, Q_8 and Q_9). Let C_3 be a cubic with $C_3.C \geqq E_9$. Then $C_3 = L_1 L_2 L_3$, which implies that

$$h^0(K - E_9) = h^0(K - 3P - E_9) = 1.$$

(V) *The case* $t(H) = 4$. Then $H = 2H_6 + \langle n, n+2, n+4, n+6, n+8 \rangle$. By Corollary 2.8 in [4] H is DCP.

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Seon Jeong Kim
Department of Mathematics and RINS
Gyeongsang National University
Jinju 660-701, Korea
Email address: skim@gnu.ac.kr

Jiryo Komeda
Department of Mathematics
Center for Basic Education and Integrated Learning
Kanagawa Institute of Technology
Atsugi 243-0292, Japan
Email address: komeda@gen.kanagawa-it.ac.jp