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FRACTIONAL INTEGRATION AND DIFFERENTIATION OF THE (p, q)-EXTENDED BESSEL FUNCTION

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Abstract. We aim to present some formulas for Saigo hypergeometric fractional integral and differential operators involving (p, q)-extended Bessel function $J_{\nu,\,p,q}(z)$, which are expressed in terms of Hadamard product of the (p, q)-extended Gauss hypergeometric function and the Fox-Wright function ${}_p\Psi_q(z)$. A number of interesting special cases of our main results are also considered. Further, it is emphasized that the results presented here, which are seemingly complicated series, can reveal their involved properties via those of the two known functions in their respective Hadamard product.

1. Introduction and preliminaries

Fractional calculus, which has a long history, is an important branch of mathematical analysis (calculus) where differentiations and integrations can be of arbitrary non-integer order. During the past four decades or so, fractional calculus has been widely and extensively investigated and has gained importance and popularity due mainly to its demonstrated applications in numerous and diverse fields of science and engineering such as turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, and astrophysics (see, for details, [6, 9, 10, 13–15, 21, 25]).

We recall Saigo fractional integral and differential operators involving the hypergeometric function ${}_2F_1$ (see [20]):

$$\left(I_{0+}^{\alpha,\beta,\eta} f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\tfrac{t}{x}\right) f(t)\, \mathrm{d}t, \tag{1}$$

$$\left(I_{-}^{\alpha,\beta,\eta} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\tfrac{x}{t}\right) f(t)\, \mathrm{d}t, \tag{2}$$

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and

$$\left(D_{0+}^{\alpha,\beta,\eta} f\right)(x) = \left(I_{0+}^{-\alpha,-\beta,\alpha+\eta} f\right)(x)$$
$$= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta-n} f\right)(x) \quad (n = [\Re(\alpha)]+1), \tag{3}$$

$$\left(D_{-}^{\alpha,\beta,\eta} f\right)(x) = \left(I_{-}^{-\alpha,-\beta,\alpha+\eta} f\right)(x)$$
$$= (-1)^n \left(\frac{d}{dx}\right)^n \left(I_{-}^{-\alpha+n,-\beta-n,\alpha+\eta} f\right)(x) \quad (n = [\Re(\alpha)]+1). \tag{4}$$

Here and in what follows, $[x]$ denotes the greatest integer less than or equal to a real number x. Setting $\beta = -\alpha$ in (1), (2), (3), and (4) yields the familiar Riemann-Liouville fractional integrals and derivatives of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ and $x \in \mathbb{R}^+$ (see, e.g., [7–9, 21]):

$$\left(I_{0+}^{\alpha,-\alpha,\eta} f\right)(x) = \left(I_{0+}^{\alpha} f\right)(x) \equiv \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t, \tag{5}$$

$$\left(I_{-}^{\alpha,-\alpha,\eta} f\right)(x) = \left(I_{-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)\,\mathrm{d}t, \tag{6}$$

and

$$\left(D_{0+}^{\alpha,-\alpha,\eta} f\right)(x) = \left(D_{0+}^{\alpha} f\right)(x) = \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f(t)\,\mathrm{d}t$$
$$= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{n-\alpha} f\right)(x) \qquad (n = [\Re(\alpha)]+1), \tag{7}$$

$$\left(D_{-}^{\alpha,-\alpha,\eta} f\right)(x) = \left(D_{-}^{\alpha} f\right)(x)$$
$$= (-1)^n \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_x^\infty (t-y)^{n-\alpha-1} f(t)\,\mathrm{d}t \tag{8}$$
$$= (-1)^n \left(\frac{d}{dx}\right)^n \left(I_{-}^{n-\alpha} f\right)(x) \qquad (n = [\Re(\alpha)]+1).$$

Here and in the following, let $\mathbb{C}$, $\mathbb{R}^+$, and $\mathbb{N}$ be the sets of complex numbers, positive real numbers, and positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Setting $\beta = 0$ in (1), (2), (3), and (4) yields the so-called Erdélyi-Kober fractional integrals and derivatives of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ and $x \in \mathbb{R}^+$ (see, e.g., [9, 21]):

$$\left(I_{0+}^{\alpha,0,\eta} f\right)(x) = \left(I_{\eta,\alpha}^{+} f\right)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\eta} f(t)\,\mathrm{d}t, \tag{9}$$

$$\left(I_{-}^{\alpha,0,\eta} f\right)(x) = \left(K_{\eta,\alpha}^{-} f\right)(x) \equiv \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\eta} f(t)\,\mathrm{d}t, \tag{10}$$

and

$$\begin{aligned}\left(D_{0+}^{\alpha,0,\eta} f\right)(x) &= \left(D_{\eta,\alpha}^{+} f\right)(x) \\ &= \left(\frac{d}{dx}\right)^{n} \left(I_{0+}^{-\alpha+n,-\alpha,\alpha+\eta-n} f\right)(x) \quad (n = [\Re(\alpha)] + 1),\end{aligned} \tag{11}$$

$$\begin{aligned}\left(D_{-}^{\alpha,0,\eta} f\right)(x) &= \left(D_{\eta,\alpha}^{-} f\right)(x) \\ &= (-1)^{n} \left(\frac{d}{dx}\right)^{n} \left(I_{-}^{-\alpha+n,-\alpha,\alpha+\eta} f\right)(x) \quad (n = [\Re(\alpha)] + 1),\end{aligned} \tag{12}$$

$$\begin{aligned}\left(D_{\eta,\alpha}^{+} f\right)(x) &= x^{-\eta} \left(\frac{d}{dx}\right)^{n} \frac{1}{\Gamma(n-\alpha)} \int_{0}^{x} t^{\alpha+\eta}(x-t)^{n-\alpha-1} f(t)\, \mathrm{d}t \\ &\quad (n = [\Re(\alpha)] + 1),\end{aligned} \tag{13}$$

$$\begin{aligned}\left(D_{\eta,\alpha}^{-} f\right)(x) &= x^{\eta+\alpha} \left(\frac{d}{dx}\right)^{n} \frac{1}{\Gamma(n-\alpha)} \int_{x}^{\infty} t^{-\eta}(t-x)^{n-\alpha-1} f(t)\, \mathrm{d}t \\ &\quad (n = [\Re(\alpha)] + 1).\end{aligned} \tag{14}$$

Recently, many authors have investigated the (p, q)-variant (when $p = q$, the p-variant) associated with a set of related higher transcendental hypergeometric type special functions (see, for details, [1–4, 11, 17–19, 22]). In particular, Maŝireviĉ et al. [12] introduced and studied the (p, q)-extended Bessel function $J_{\nu,\,p,q}(z)$ of the first kind of order ν with $\min\{p, q\} \geq 0$ and $\Re(\nu) > -\frac{1}{2}$ when $p = q = 0$ in the form:

$$\begin{aligned}J_{\nu,\,p,q}(z) &= \frac{\left(\frac{z}{2}\right)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \sum_{n=0}^{\infty} (-1)^{n} \mathrm{B}\left(n+\frac{1}{2},\, \nu+\frac{1}{2}; p, q\right) \frac{z^{2n}}{(2n)!} \\ &= \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{\mathrm{B}\left(n+\frac{1}{2},\, \nu+\frac{1}{2}; p, q\right)}{\left(\frac{1}{2}\right)_{n} \mathrm{B}\left(\frac{1}{2},\, \nu+\frac{1}{2}\right)\, n!} \left(-\frac{z^{2}}{4}\right)^{n},\end{aligned} \tag{15}$$

where $\mathrm{B}(x, y; p, q)$ is the (p, q)-extended Beta function introduced by Choi et al. [5]

$$\mathrm{B}(x, y; p, q) = \int_{0}^{1} t^{x-1}(1-t)^{y-1}\, \mathrm{e}^{-\frac{p}{t}-\frac{q}{1-t}}\, \mathrm{d}t \tag{16}$$

$$(\min\{\Re(x), \Re(y)\} > 0;\ \min\{\Re(p), \Re(q)\} \geq 0).$$

It is noted that $\mathrm{B}(x, y; 0, 0) = \mathrm{B}(x,\, y)$ is the familiar Beta function (see, e.g., [23, Section 1.1]). They [5] established various properties of $\mathrm{B}(x, y; p, q)$ in (16) such as integral representations, Mellin transforms (see, e.g., [18]), complete monotonicity, Turán type inequality (see, e.g., [16]) and associated nonhomogeneous differential-difference equations. Clearly, the case $p = 0 = q$ in

(15) reduces immediately to the classical Bessel function $J_\nu(z)$ of the first kind (see, e.g., [22, 26])

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)}. \tag{17}$$

Also, for our present investigation, we need the concept of Hadamard product (or convolution) of two analytic functions. It can help us in decomposing newly established series into two known functions. Let

$$f(z) := \sum_{n=0}^{\infty} a_n z^n \ (|z| < R_f) \ \text{ and } \ g(z) := \sum_{n=0}^{\infty} b_n z^n \ (|z| < R_g)$$

be two given power series whose radii of convergence are given by R_f and R_g, respectively. Then their Hadamard product is a power series defined by

$$(f \,*\, g)(z) := \sum_{n=0}^{\infty} a_n\, b_n z^n = (g \,*\, f)(z) \quad (|z| < R), \tag{18}$$

whose radius of convergence R is

$$\frac{1}{R} = \limsup_{n\to\infty}\, (|a_n\, b_n|)^{\frac{1}{n}} \leq \left(\limsup_{n\to\infty}\, (|a_n|)^{\frac{1}{n}}\right)\left(\limsup_{n\to\infty}\, (|b_n|)^{\frac{1}{n}}\right) = \frac{1}{R_f \cdot R_g}$$

and so $R \geqq R_f \cdot R_g$. In particular, if one of the power series defines an entire function, then the Hadamard product series defines an entire function, too.

In this paper, we aim to investigate compositions of the generalized fractional integration and differentiation operators (1), (2), (3) and (4) involving (p, q)-extended Bessel function $J_{\nu,\,p,q}(z)$. Also, those results for the classical Riemann-Liouville and Erdélyi-Kober fractional integral and differential operators involving the (p, q)-extended Bessel function $J_{\nu,\,p,q}(z)$, which correspond to the main identities, are deduced. Further, we show that those compositions are expressed in terms of the Hadamard product (18) of the (p, q)-extended Gauss hypergeometric function (see [5, p. 354, Eq. (7.1)])

$$F_{p,q}(a, b; c; z) = \sum_{n=0}^{\infty} (a)_n \,\frac{\mathrm{B}(b+n,\, c-b\,;\, p, q)}{\mathrm{B}(b,\, c-b)}\, \frac{z^n}{n!} \quad (|z| < 1;\, \Re(c) > \Re(b) > 0) \tag{19}$$

and Fox-Wright function ${}_p\Psi_q(z)$ $(p,\, q \in \mathbb{N}_0)$ (see, for details, [9, 15]; see also [21, 24]):

$${}_p\Psi_q\left[\begin{array}{l}(\alpha_1, A_1), \ldots, (\alpha_p, A_p);\\ (\beta_1, B_1), \ldots, (\beta_q, B_q);\end{array} z\right] = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 n) \cdots \Gamma(\alpha_p + A_p n)}{\Gamma(\beta_1 + B_1 n) \cdots \Gamma(\beta_q + B_q n)}\, \frac{z^n}{n!} \tag{20}$$

$$\left(A_j \in \mathbb{R}^+ \ (j = 1, \ldots, p);\ B_j \in \mathbb{R}^+ \ (j = 1, \ldots, q);\ 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geqq 0\right),$$

where the equality in the convergence condition holds for

$$|z| < \nabla := \left(\prod_{j=1}^{p} A_j^{-A_j}\right)\left(\prod_{j=1}^{q} B_j^{B_j}\right).$$

Here and in the following, $(\lambda)_\nu$ denotes the Pochhammer symbol defined (for $\lambda,\ \nu \in \mathbb{C}$), in terms of the Gamma function Γ, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}). \end{cases} \tag{21}$$

2. Fractional integrations of the $J_{\nu,\,p,q}(z)$

Here, composition formulas of generalized fractional integrals (1) and (2) involving the (p,q)-extended Bessel function $J_{\nu,\,p,q}(z)$ of the first kind of order ν are presented. To do this, we begin by stating some image formulas regarding (1) and (2) which may be known formulas and are given in the following lemma.

Lemma 2.1. *Let $\alpha,\ \beta,\ \eta \in \mathbb{C}$ and $x \in \mathbb{R}^+$.*

(a) *If $\Re(\alpha) > 0$ and $\Re(\sigma) > \max\{0, \Re(\beta-\eta)\}$, then*

$$(I_{0+}^{\alpha,\beta,\eta} t^{\sigma-1})(x) = \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\alpha+\eta)} x^{\sigma-\beta-1}. \tag{22}$$

In particular, we have

$$(I_{0+}^{\alpha} t^{\sigma-1})(x) = \frac{\Gamma(\sigma)}{\Gamma(\sigma+\alpha)} x^{\sigma+\alpha-1} \qquad (\min\{\Re(\alpha),\ \Re(\sigma)\} > 0) \tag{23}$$

and

$$(I_{\eta,\alpha}^{+} t^{\sigma-1})(x) = \frac{\Gamma(\sigma+\eta)}{\Gamma(\sigma+\alpha+\eta)} x^{\sigma-1} \qquad (\Re(\alpha) > 0,\ \Re(\sigma) > -\Re(\eta)). \tag{24}$$

(b) *If $\Re(\alpha) > 0$ and $\Re(\sigma) < 1 + \min\{\Re(\beta), \Re(\eta)\}$, then*

$$(I_{-}^{\alpha,\beta,\eta} t^{\sigma-1})(x) = \frac{\Gamma(\beta-\sigma+1)\Gamma(\eta-\sigma+1)}{\Gamma(1-\sigma)\Gamma(\alpha+\beta+\eta-\sigma+1)} x^{\sigma-\beta-1}. \tag{25}$$

In particular, for $x > 0$, we have

$$(I_{-}^{\alpha} t^{\sigma-1})(x) = \frac{\Gamma(1-\alpha-\sigma)}{\Gamma(1-\sigma)} x^{\sigma+\alpha-1} \qquad (0 < \Re(\alpha) < 1 - \Re(\sigma)) \tag{26}$$

and

$$(K_{\eta,\alpha}^{-} t^{\sigma-1})(x) = \frac{\Gamma(\eta-\sigma+1)}{\Gamma(\alpha+\eta-\sigma+1)} x^{\sigma-1} \qquad (\Re(\sigma) < 1 + \Re(\sigma)). \tag{27}$$

Theorem 2.2. *Let $\alpha, \beta, \eta, \sigma, \nu, \omega \in \mathbb{C}$ be such that $\min\{\Re(p), \Re(q)\} > 0$, $\Re(\nu) > -1$, $\Re(\alpha) > 0$ and $\Re(\sigma+\nu) > \max\{0, \Re(\beta-\eta)\}$. Then*

$$\left(I_{0+}^{\alpha,\beta,\eta}\left\{t^{\sigma-1} J_{\nu,\,p,q}(\omega t)\right\}\right)(x) \tag{28}$$

$$= \sqrt{\pi}\, x^{\sigma+\nu-\beta-1} \frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)} F_{p,q}\left[\begin{array}{c} 1, \frac{1}{2}; \\ \nu+1; \end{array} -\frac{\omega^2 x^2}{4}\right]$$
$$* \; {}_2\Psi_3 \left[\begin{array}{r} (\sigma+\nu,2),(\sigma+\nu+\eta-\beta,2); \\ (\frac{1}{2},1),(\sigma+\nu-\beta,2),(\sigma+\nu+\alpha+\eta,2); \end{array} -\frac{\omega^2 x^2}{4}\right],$$

where $*$ denotes the Hadamard product in (18) *and whose left-sided hypergeometric fractional integral is assumed to be convergent.*

Proof. Applying (15) to (1) and changing the order of integration and summation, which is valid under the given conditions here, and using (22), we find

$$(29) \qquad \left(I_{0+}^{\alpha,\beta,\eta}\left\{t^{\sigma-1} J_{\nu,\,p,q}(\omega t)\right\}\right)(x)$$
$$= \frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{\mathrm{B}\left(k+\frac{1}{2},\, \nu+\frac{1}{2}; p,q\right)}{\left(\frac{1}{2}\right)_k \mathrm{B}\left(\frac{1}{2},\, \nu+\frac{1}{2}\right)\, k!} \left(-\frac{\omega^2}{4}\right)^k \left(I_{0+}^{\alpha,\beta,\eta} t^{\sigma+\nu+2k-1}\right)(x)$$
$$= x^{\sigma+\nu-\beta-1} \frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{(1)_k\, \mathrm{B}\left(k+\frac{1}{2},\, \nu+\frac{1}{2}; p,q\right)}{\mathrm{B}\left(\frac{1}{2},\, \nu+\frac{1}{2}\right)\, k!}$$
$$\times \frac{\Gamma(\sigma+\nu+2k)\Gamma(\sigma+\nu+\eta-\beta+2k)}{\left(\frac{1}{2}\right)_k \Gamma(\sigma+\nu-\beta+2k)\Gamma(\sigma+\nu+\alpha+\eta+2k)\, k!} \left(-\frac{\omega^2 x^2}{4}\right)^k.$$

Expressing the last summation in (29) in terms of the Hadamard product (18) with the functions (19) and (20), and considering (21) for $\left(\frac{1}{2}\right)_k$, we obtain the right-hand side of (28). □

Theorem 2.3. *Let $\alpha,\beta,\eta,\sigma,\nu,\omega \in \mathbb{C}$ be such that $\min\{\Re(p),\Re(q)\} > 0$, $\Re(\nu) > -1$, $\Re(\alpha) > 0$ and $\Re(\sigma-\nu) < 1 + \min\{\Re(\beta),\Re(\eta)\}$. Then*

$$(30) \qquad \left(I_{-}^{\alpha,\beta,\eta}\left\{t^{\sigma-1} J_{\nu,\,p,q}\left(\frac{\omega}{t}\right)\right\}\right)(x)$$
$$= \sqrt{\pi}\, x^{\sigma-\nu-\beta-1} \frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)} F_{p,q}\left[\begin{array}{c} 1, \frac{1}{2}; \\ \nu+1; \end{array} -\frac{\omega^2}{4x^2}\right]$$
$$* \; {}_2\Psi_3 \left[\begin{array}{r} (1+\beta-\sigma+\nu,2),(1+\eta-\sigma+\nu,2); \\ (\frac{1}{2},1),(1-\sigma+\nu,2),(1+\alpha+\beta+\eta-\sigma+\nu,2); \end{array} -\frac{\omega^2}{4x^2}\right],$$

where $*$ denotes the Hadamard product in (18) *and whose left-sided hypergeometric fractional integral is assumed to be convergent.*

Proof. Applying (15) to (2) and changing the order of integration and summation, which is valid under the given conditions here, and using (25), we find

$$\left(I_{-}^{\alpha,\beta,\eta}\left\{t^{\sigma-1} J_{\nu,\,p,q}\left(\frac{\omega}{t}\right)\right\}\right)(x)$$
$$= \frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{\mathrm{B}\left(k+\frac{1}{2},\, \nu+\frac{1}{2}; p,q\right)}{\left(\frac{1}{2}\right)_k \mathrm{B}\left(\frac{1}{2},\, \nu+\frac{1}{2}\right)\, k!} \left(-\frac{\omega^2}{4}\right)^k \left(I_{-}^{\alpha,\beta,\eta} t^{\sigma-\nu-2k-1}\right)(x)$$

$$= x^{\sigma-\nu-\beta-1}\,\frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)}\,\sum_{k=0}^{\infty}\frac{(1)_k\,\mathrm{B}\left(k+\frac{1}{2},\,\nu+\frac{1}{2};p,q\right)}{\mathrm{B}\left(\frac{1}{2},\,\nu+\frac{1}{2}\right)\,k!}$$
$$\times\,\frac{\Gamma(\beta-\sigma+\nu+1+2k)\Gamma(\eta-\sigma+\nu+1+2k)}{\left(\frac{1}{2}\right)_k\,\Gamma(1-\sigma+\nu+2k)\Gamma(\alpha+\beta+\eta-\sigma+\nu+1+2k)\,k!}\left(-\frac{\omega^2}{4x^2}\right)^{k}.$$

Applying a similar argument as in the proof of (28) to the last summation leads to the right-hand side of (30). □

The results in Theorems 2.2 and 2.3 can be specialized to give a number of identities involving known fractional integral operators. Here, we present four identities as in the following corollaries, without their proofs.

Corollary 2.4. *Let* $\alpha,\sigma,\nu,\omega\in\mathbb{C}$ *be such that* $\min\{\Re(p),\Re(q)\}>0$, $\Re(\nu)>-1$, $\Re(\alpha)>0$, $\Re(\sigma+\nu)>0$. *Then*

$$\text{(31)}\qquad \left(I_{0+}^{\alpha}\left\{t^{\sigma-1}J_{\nu,\,p,q}(\omega t)\right\}\right)(x)$$
$$= \sqrt{\pi}\,x^{\sigma+\nu+\alpha-1}\,\frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)}$$
$$\times\,F_{p,q}\left[\begin{matrix}1,\frac{1}{2}; \\ \nu+1;\end{matrix}\ -\frac{\omega^2x^2}{4}\right] *\ {}_1\Psi_2\left[\begin{matrix}(\sigma+\nu,2); \\ (\frac{1}{2},1),(\sigma+\nu+\alpha,2);\end{matrix}\ -\frac{\omega^2x^2}{4}\right],$$

where $*$ *denotes the Hadamard product in* (18) *and whose left-sided hypergeometric fractional integral is assumed to be convergent.*

Corollary 2.5. *Let* $\alpha,\eta,\sigma,\nu,\omega\in\mathbb{C}$ *be such that* $\min\{\Re(p),\Re(q)\}>0$, $\Re(\nu)>-1$, $\Re(\alpha)>0$ *and* $\Re(\sigma+\nu)>-\Re(\eta)$. *Then*

$$\text{(32)}\qquad \left(I_{\eta,\alpha}^{+}\left\{t^{\sigma-1}J_{\nu,\,p,q}(\omega t)\right\}\right)(x)$$
$$= \sqrt{\pi}\,x^{\sigma+\nu-1}\,\frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)}$$
$$\times\,F_{p,q}\left[\begin{matrix}1,\frac{1}{2}; \\ \nu+1;\end{matrix}\ -\frac{\omega^2x^2}{4}\right] *\ {}_1\Psi_2\left[\begin{matrix}(\sigma+\nu+\eta,2); \\ (\frac{1}{2},1),(\sigma+\nu+\alpha+\eta,2);\end{matrix}\ -\frac{\omega^2x^2}{4}\right],$$

where $*$ *denotes the Hadamard product in* (18) *and whose left-sided hypergeometric fractional integral is assumed to be convergent.*

Corollary 2.6. *Let* $\alpha,\sigma,\nu,\omega\in\mathbb{C}$ *be such that* $\min\{\Re(p),\Re(q)\}>0$, $\Re(\nu)>-1$, $0<\Re(\alpha)<1-\Re(\sigma-\nu)$. *Then*

$$\text{(33)}\qquad \left(I_{-}^{\alpha}\left\{t^{\sigma-1}J_{\nu,\,p,q}\left(\frac{\omega}{t}\right)\right\}\right)(x)$$
$$= \sqrt{\pi}\,x^{\sigma-\nu+\alpha-1}\,\frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)}$$
$$\times\,F_{p,q}\left[\begin{matrix}1,\frac{1}{2}; \\ \nu+1;\end{matrix}\ -\frac{\omega^2}{4x^2}\right] *\ {}_1\Psi_2\left[\begin{matrix}(1-\alpha-\sigma+\nu,2); \\ (\frac{1}{2},1),(1-\sigma+\nu,2);\end{matrix}\ -\frac{\omega^2}{4x^2}\right],$$

where $*$ denotes the Hadamard product in (18) *and whose left-sided hypergeometric fractional integral is assumed to be convergent.*

Corollary 2.7. *Let $\alpha, \eta, \sigma, \nu, \omega \in \mathbb{C}$ be such that* $\min\{\Re(p), \Re(q)\} > 0$, $\Re(\nu) > -1$, $\Re(\alpha) > 0$ *and* $\Re(\sigma - \nu) < 1 + \Re(\eta)$. *Then*

$$(34)\quad \left(K^{-}_{\eta,\alpha}\left\{t^{\sigma-1} J_{\nu,\,p,q}\left(\frac{\omega}{t}\right)\right\}\right)(x)$$
$$= \sqrt{\pi}\, x^{\sigma-\nu-1} \frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)}$$
$$\times F_{p,q}\left[\begin{array}{c} 1, \frac{1}{2}; \\ \nu+1; \end{array} -\frac{\omega^2}{4x^2}\right] * {}_1\Psi_2\left[\begin{array}{r} (1+\eta-\sigma+\nu, 2); \\ (\frac{1}{2}, 1), (1+\alpha+\eta-\sigma+\nu, 2); \end{array} -\frac{\omega^2}{4x^2}\right],$$

where $*$ denotes the Hadamard product in (18) *and whose left-sided hypergeometric fractional integral is assumed to be convergent.*

3. Fractional differentiations of the $J_{\nu,\,p,q}(z)$

Here, composition formulas of the generalized fractional differentiations (3) and (4) involving (p, q)-extended Bessel function $J_{\nu,\,p,q}(z)$ of the first kind of order ν are presented. To do this, we begin by giving the following image formulas asserted in Lemma 3.1, without their proofs.

Lemma 3.1. *Let α, β, $\eta \in \mathbb{C}$ and $x \in \mathbb{R}^{+}$. Then*

(a) *If $\Re(\alpha) > 0$ and $\Re(\sigma) > -\min\{0, \Re(\alpha+\beta+\eta)\}$, then*

$$(35)\quad (D^{\alpha,\beta,\eta}_{0+} t^{\sigma-1})(x) = \frac{\Gamma(\sigma)\Gamma(\sigma+\alpha+\beta+\eta)}{\Gamma(\sigma+\beta)\Gamma(\sigma+\eta)} x^{\sigma+\beta-1}.$$

In particular, we have

$$(36)\quad (D^{\alpha}_{0+} t^{\sigma-1})(x) = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\alpha)} x^{\sigma-\alpha-1} \quad (\Re(\alpha) > 0,\ \Re(\sigma) > 0)$$

and

$$(37)\quad (D^{+}_{\eta,\alpha} t^{\sigma-1})(x) = \frac{\Gamma(\sigma+\alpha+\eta)}{\Gamma(\sigma+\eta)} x^{\sigma-1}\ (\Re(\alpha) > 0,\ \Re(\sigma) > -\Re(\alpha+\eta)).$$

(b) *If $\Re(\alpha) > 0, \Re(\sigma) < 1+\min\{\Re(-\beta-n), \Re(\alpha+\eta)\}$ and $n = [\Re(\alpha)]+1$, then*

$$(38)\quad (D^{\alpha,\beta,\eta}_{-} t^{\sigma-1})(x) = \frac{\Gamma(1-\sigma-\beta)\Gamma(1-\sigma+\alpha+\eta)}{\Gamma(1-\sigma)\Gamma(1-\sigma+\eta-\beta)} x^{\sigma+\beta-1}.$$

In particular, we have

$$(39)\quad (D^{\alpha}_{-} t^{\sigma-1})(x) = \frac{\Gamma(1-\sigma+\alpha)}{\Gamma(1-\sigma)} x^{\sigma-\alpha-1} \quad (\Re(\alpha) > 0,\ \Re(\sigma) < 1+\Re(\alpha)-n)$$

and

(40)
$$(D^{-}_{\eta,\alpha} t^{\sigma-1})(x) = \frac{\Gamma(1-\sigma+\alpha+\eta)}{\Gamma(1-\sigma-\eta)} x^{\sigma-1} \quad (\Re(\alpha) > 0,\ \Re(\sigma) < 1+\Re(\alpha+\eta)-n).$$

Theorem 3.2. *Let $\alpha,\beta,\eta,\sigma,\nu,\omega \in \mathbb{C}$ be such that* $\min\{\Re(p),\Re(q)\} > 0$, $\Re(\nu) > -1$, $\Re(\alpha) \geq 0$ *and* $\Re(\sigma+\nu) > -\min\{0,\Re(\alpha+\beta+\eta)\}$. *Then*

$$\begin{aligned}(41)\qquad &\left(D_{0+}^{\alpha,\beta,\eta}\left\{t^{\sigma-1}J_{\nu,\,p,q}(\omega t)\right\}\right)(x)\\ &= \sqrt{\pi}\,x^{\sigma+\nu+\beta-1}\frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)}\\ &\quad\times F_{p,q}\left[\begin{array}{c}1,\frac{1}{2};\\ \nu+1;\end{array}\;-\frac{\omega^2x^2}{4}\right] * {}_2\Psi_3\left[\begin{array}{c}(\sigma+\nu,2),(\sigma+\nu+\alpha+\beta+\eta,2);\\ (\frac{1}{2},1),(\sigma+\nu+\beta,2),(\sigma+\nu+\eta,2);\end{array}\;-\frac{\omega^2x^2}{4}\right],\end{aligned}$$

where $*$ denotes the Hadamard product in (18) *and whose left-sided hypergeometric fractional derivative is assumed to be convergent.*

Proof. Applying the fractional differential operator (3) to (15) and conducting term-by-term fractional differentiation, which is valid under the given conditions here, with the aid of (35), we have

$$\begin{aligned}(42)\qquad &\left(D_{0+}^{\alpha,\beta,\eta}\left\{t^{\sigma-1}J_{\nu,\,p,q}(\omega t)\right\}\right)(x)\\ &= \frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)}\sum_{k=0}^{\infty}\frac{\mathrm{B}\left(k+\frac{1}{2},\,\nu+\frac{1}{2};p,q\right)}{\mathrm{B}\left(\frac{1}{2},\,\nu+\frac{1}{2}\right)\,k!}\left(-\frac{\omega^2}{4}\right)^{k}\left(D_{0+}^{\alpha,\beta,\eta}t^{\sigma+\nu+2k-1}\right)(x)\\ &= x^{\sigma+\nu+\beta-1}\frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)}\sum_{k=0}^{\infty}\frac{(1)_k\,\mathrm{B}\left(k+\frac{1}{2},\,\nu+\frac{1}{2};p,q\right)}{\mathrm{B}\left(\frac{1}{2},\,\nu+\frac{1}{2}\right)\,k!}\\ &\quad\times\frac{\Gamma(\sigma+\nu+2k)\Gamma(\sigma+\nu+\alpha+\beta+\eta+2k)}{\left(\frac{1}{2}\right)_k\Gamma(\sigma+\nu+\beta+2k)\Gamma(\sigma+\nu+\eta+2k)\,k!}\left(-\frac{\omega^2x^2}{4}\right)^{k}.\end{aligned}$$

Expressing the last summation in (42) in terms of the Hadamard product (18) together with the two functions (19) and (20) yields the desired formula (41). □

Theorem 3.3. *Let $\alpha,\beta,\eta,\sigma,\nu,\omega \in \mathbb{C}$ be such that* $\min\{\Re(p),\Re(q)\} > 0$, $\Re(\nu) > -1$, $\Re(\alpha) \geq 0$ *and* $\Re(\sigma-\nu) < 1+\min\{\Re(-\beta-n),\Re(\alpha+\eta)\}$, $n = [\Re(\alpha)]+1$. *Then*

$$\begin{aligned}(43)\qquad &\left(D_{-}^{\alpha,\beta,\eta}\left\{t^{\sigma-1}J_{\nu,\,p,q}\left(\frac{\omega}{t}\right)\right\}\right)(x)\\ &= \sqrt{\pi}\,x^{\sigma-\nu+\beta-1}\frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)}F_{p,q}\left[\begin{array}{c}1,\frac{1}{2};\\ \nu+1;\end{array}\;-\frac{\omega^2}{4x^2}\right]\\ &\quad * {}_2\Psi_3\left[\begin{array}{c}(1-\sigma+\nu-\beta,2),(1-\sigma+\nu+\alpha+\eta,2);\\ (\frac{1}{2},1),(1-\sigma+\nu,2),(1-\sigma+\nu+\eta-\beta,2);\end{array}\;-\frac{\omega^2}{4x^2}\right],\end{aligned}$$

where $*$ denotes the Hadamard product in (18) *and whose left-sided hypergeometric fractional derivative is assumed to be convergent.*

Proof. The proof would run parallel to that of Theorem 3.2. We omit the details. □

Four interesting special cases of the results in Theorems 3.2 and 3.3 are given in the following corollaries, without their proofs.

Corollary 3.4. *Let* $\alpha, \sigma, \nu, \omega \in \mathbb{C}$ *be such that* $\min\{\Re(p), \Re(q)\} > 0$*,* $\Re(\nu) > -1$ *and* $\Re(\alpha) \geq 0$*,* $\Re(\sigma+\nu) > 0$*. Then*

$$(44) \quad \left(D_{0+}^{\alpha}\left\{t^{\sigma-1}J_{\nu,\,p,q}(\omega t)\right\}\right)(x)$$
$$= \sqrt{\pi}\, x^{\sigma+\nu-\alpha-1}\frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)}$$
$$\times F_{p,q}\left[\begin{array}{c} 1, \frac{1}{2}; \\ \nu+1; \end{array} -\frac{\omega^2 x^2}{4}\right] * {}_1\Psi_2\left[\begin{array}{r} (\sigma+\nu, 2); \\ (\frac{1}{2}, 1), (\sigma+\nu-\alpha, 2); \end{array} -\frac{\omega^2 x^2}{4}\right],$$

where $*$ *denotes the Hadamard product in* (18) *and whose left-sided hypergeometric fractional derivative is assumed to be convergent.*

Corollary 3.5. *Let* $\alpha, \eta, \sigma, \nu, \omega \in \mathbb{C}$ *be such that* $\min\{\Re(p), \Re(q)\} > 0$*,* $\Re(\nu) > -1$*,* $\Re(\alpha) \geq 0$ *and* $\Re(\sigma+\nu) > -\Re(\eta+\alpha)$*. Then*

$$(45) \quad \left(D_{\eta,\alpha}^{+}\left\{t^{\sigma-1}J_{\nu,\,p,q}(\omega t)\right\}\right)(x)$$
$$= \sqrt{\pi}\, x^{\sigma+\nu-1}\frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)}$$
$$\times F_{p,q}\left[\begin{array}{c} 1, \frac{1}{2}; \\ \nu+1; \end{array} -\frac{\omega^2 x^2}{4}\right] * {}_1\Psi_2\left[\begin{array}{c} (\sigma+\nu+\alpha+\eta, 2); \\ (\frac{1}{2}, 1), (\sigma+\nu+\eta, 2); \end{array} -\frac{\omega^2 x^2}{4}\right],$$

where $*$ *denotes the Hadamard product in* (18) *and whose left-sided hypergeometric fractional derivative is assumed to be convergent.*

Corollary 3.6. *Let* $\alpha, \sigma, \nu, \omega \in \mathbb{C}$ *be such that* $\min\{\Re(p), \Re(q)\} > 0$*,* $\Re(\nu) > -1$*,* $\Re(\alpha) \geq 0$*,* $\Re(\sigma-\nu) < \Re(\alpha) - [\Re(\alpha)]$*. Then*

$$(46) \quad \left(D_{-}^{\alpha}\left\{t^{\sigma-1}J_{\nu,\,p,q}\left(\frac{\omega}{t}\right)\right\}\right)(x)$$
$$= \sqrt{\pi}\, x^{\sigma-\nu-\alpha-1}\frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)}$$
$$\times F_{p,q}\left[\begin{array}{c} 1, \frac{1}{2}; \\ \nu+1; \end{array} -\frac{\omega^2}{4x^2}\right] * {}_1\Psi_2\left[\begin{array}{c} (1-\sigma+\nu+\alpha, 2); \\ (\frac{1}{2}, 1), (1-\sigma+\nu, 2); \end{array} -\frac{\omega^2}{4x^2}\right],$$

where $*$ *denotes the Hadamard product in* (18) *and whose left-sided hypergeometric fractional derivative is assumed to be convergent.*

Corollary 3.7. *Let* $\alpha, \eta, \sigma, \nu, \omega \in \mathbb{C}$ *be such that* $\min\{\Re(p), \Re(q)\} > 0$*,* $\Re(\nu) > -1$*,* $\Re(\alpha) \geq 0$ *and* $\Re(\sigma-\nu) < \Re(\alpha+\eta) - [\Re(\alpha)]$*. Then*

$$(47) \quad \left(D_{\eta,\alpha}^{-}\left\{t^{\sigma-1}J_{\nu,\,p,q}\left(\frac{\omega}{t}\right)\right\}\right)(x)$$
$$= \sqrt{\pi}\, x^{\sigma-\nu-1}\frac{\left(\frac{\omega}{2}\right)^{\nu}}{\Gamma(\nu+1)}$$

$$\times F_{p,q}\left[\begin{array}{c} 1, \frac{1}{2}; \\ \nu+1; \end{array} -\frac{\omega^2}{4x^2}\right] * {}_1\Psi_2\left[\begin{array}{c} (1-\sigma+\nu+\alpha+\eta, 2); \\ (\frac{1}{2}, 1), (1-\sigma+\nu+\eta, 2); \end{array} -\frac{\omega^2}{4x^2}\right],$$

where $*$ *denotes the Hadamard product in* (18) *and whose left-sided hypergeometric fractional derivative is assumed to be convergent.*

Concluding remarks

The results presented in Theorems 2.2-3.3 together with corollaries are sure to be new and potentially useful, mainly because they are expressed in terms of the Hadamard product with two known functions. At least, what seemingly complicated resulting series are expressed in terms of two known functions *means* that certain properties involved in the complicated resulting series can be revealed via those of the two known functions in their respective Hadamard product.

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References

[1] M. A. Chaudhry, A. Qadir, M. Rafique, and S. M. Zubair, *Extension of Euler's beta function*, J. Comput. Appl. Math. **78** (1997), no. 1, 19–32.

[2] M. A. Chaudhry, A. Qadir, H. M. Srivastava, and R. B. Paris, *Extended hypergeometric and confluent hypergeometric functions*, Appl. Math. Comput. **159** (2004), no. 2, 589–602.

[3] M. A. Chaudhry and S. M. Zubair, *On a Class of Incomplete Gamma Functions with Applications*, Chapman & Hall/CRC, Boca Raton, FL, 2002.

[4] J. Choi, R. K. Parmar, and T. K. Pogány, *Mathieu-type series built by (p,q)-extended Gaussian hypergeometric function*, Bull. Korean Math. Soc. **54** (2017), no. 3, 789–797.

[5] J. Choi, A. K. Rathie, and R. K. Parmar, *Extension of extended beta, hypergeometric and confluent hypergeometric functions*, Honam Math. J. **36** (2014), no. 2, 357–385.

[6] A. A. Kilbas and M. Saigo, *H-Transforms*, Analytical Methods and Special Functions, **9**, Chapman & Hall/CRC, Boca Raton, FL, 2004.

[7] A. A. Kilbas and N. Sebastian, *Generalized fractional differentiation of Bessel function of the first kind*, Math. Balkanica (N.S.) **22** (2008), no. 3-4, 323–346.

[8] ______, *Generalized fractional integration of Bessel function of the first kind*, Integral Transforms Spec. Funct. **19** (2008), no. 11-12, 869–883.

[9] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, **204**, Elsevier Science B.V., Amsterdam, 2006.

[10] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Pitman Research Notes in Mathematics Series, **301**, Longman Scientific & Technical, Harlow, 1994.

[11] M.-J. Luo, R. K. Parmar, and R. K. Raina, *On extended Hurwitz-Lerch zeta function*, J. Math. Anal. Appl. **448** (2017), no. 2, 1281–1304.

[12] D. J. Maŝireviĉ, R. K. Parmar, and T. K. Pogány, *(p,q)-extended Bessel and modified Bessel functions of the first kind*, Results Math. **72** (2017), no. 1-2, 617–632.

[13] A. M. Mathai and R. K. Saxena, *Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences*, Lecture Notes in Mathematics, Vol. 348, Springer-Verlag, Berlin, 1973.

[14] ______, *The H-Function with Applications in Statistics and Other Disciplines*, Halsted Press, New York, 1978.

[15] A. M. Mathai, R. K. Saxena, and H. J. Haubold, *The H-Function*, Springer, New York, 2010.
[16] K. S. Nisar, S. R. Mondal, and J. Choi, *Certain inequalities involving the* k*-Struve function*, J. Inequal. Appl. **2017** (2017), Paper No. 71, 8 pp.
[17] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.
[18] R. B. Paris and D. Kaminski, *Asymptotics and Mellin-Barnes Integrals*, Encyclopedia of Mathematics and its Applications, **85**, Cambridge University Press, Cambridge, 2001.
[19] R. K. Parmar and T. K. Pogány, *Extended Srivastava's triple hypergeometric $H_{A,p,q}$ function and related bounding inequalities*, J. Contemp. Math. Anal. (2017) (to appear).
[20] M. Saigo, *A remark on integral operators involving the Gauss hypergeometric functions*, Math. Rep. Kyushu Univ. **11** (1977/78), no. 2, 135–143.
[21] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *[Integrals and derivatives of fractional order and some of their applications]* (Russian), "Nauka i Tekhnika", Minsk, 1987.
[22] L. J. Slater, *Confluent Hypergeometric Functions*, Cambridge University Press, New York, 1960.
[23] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier, Inc., Amsterdam, 2012.
[24] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd., Chichester, 1985.
[25] H. M. Srivastava and R. K. Saxena, *Operators of fractional integration and their applications*, Appl. Math. Comput. **118** (2001), no. 1, 1–52.
[26] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, England, 1944.

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