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GLOBAL WEAK SOLUTIONS FOR THE RELATIVISTIC VLASOV-KLEIN-GORDON SYSTEM IN TWO DIMENSIONS

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Abstract. This paper is concerned with global existence of weak solutions to the relativistic Vlasov-Klein-Gordon system. The energy of this system is conserved, but the interaction term $\int_{\mathbb{R}^n} \rho\varphi dx$ in it need not be positive. So far existence of global weak solutions has been established only for small initial data [9, 14]. In two dimensions, this paper shows that the interaction term can be estimated by the kinetic energy to the power of $\frac{4q-4}{3q-2}$ for $1 < q < 2$. As a consequence, global existence of weak solutions for general initial data is obtained.

1. Introduction

The relativistic Vlasov-Klein-Gordon system established in [9] describes the evolution of a collisionless ensemble of particles coupled to a quantum mechanical Klein-Gordon field. The relativistic Vlasov-Klein-Gordon system in n dimensions is represented as follows

$$\left\{ \begin{array}{l} \partial_t f + \widehat{v} \cdot \partial_x f - \partial_x \varphi \cdot \partial_v f = 0, \\ \partial_t^2 \varphi - \Delta_x \varphi + \varphi = -\rho(t,x), \end{array} \right. \tag{1}$$

where $f(t,x,v) \geq 0$ denotes the phase-space density of particles at time $t \geq 0$, position $x \in \mathbb{R}^n$ and with momentum $v \in \mathbb{R}^n$, where $\varphi(t,x)$ is a self-consistent quantum mechanical Klein-Gordon field induced by the particles collectively, and where $\rho(t,x) = \int_{\mathbb{R}^n} f(t,x,v)dv$ and $\widehat{v} = \frac{v}{\sqrt{1+|v|^2}}$ denote the charge density of the system and the relativistic velocity with momentum v respectively. We consider this system with the initial data

$$f(0,x,v) = f^{in}(x,v), \quad \varphi(0,x) = \varphi_0^{in}(x), \quad \partial_t \varphi(0,x) = \varphi_1^{in}(x). \tag{2}$$

The relativistic Vlasov-Klein-Gordon system was first investigated in [9,10], where it was noted that formally the system satisfies conservation of mass and

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energy:
$$\frac{d}{dt}\int_{\mathbb{R}^n\times\mathbb{R}^n} f(t,x,v)dxdv = 0, \quad \frac{d}{dt}E(t) = 0,$$
where the energy is defined by
$$\begin{aligned} E(t) = &\int_{\mathbb{R}^n\times\mathbb{R}^n} \sqrt{1+|v|^2} f dxdv \\ &+ \frac{1}{2}\int_{\mathbb{R}^n} (|\partial_t\varphi|^2 + |\partial_x\varphi|^2 + |\varphi|^2)dx + \int_{\mathbb{R}^n} \rho\varphi dx. \end{aligned}$$
And for a smooth solution, local conservation laws were also established:
$$\partial_t\rho + \nabla_x\cdot j = 0, \quad \partial_t e + \nabla_x\cdot\mathfrak{p} = 0,$$
where
$$j(t,x) = \int_{\mathbb{R}^n} \widehat{v} f(t,x,v)dv,$$
$$e(t,x) = \int_{\mathbb{R}^n} \sqrt{1+|v|^2} f dv + \frac{1}{2}(|\partial_t\varphi|^2 + |\partial_x\varphi|^2 + |\varphi|^2) + \rho\varphi$$
and
$$\mathfrak{p}(t,x) = \int_{\mathbb{R}^n} v f(t,x,v)dv - \partial_t\varphi\cdot\partial_x\varphi + j\varphi$$
are the current density, the energy density and the momentum density respectively.

It is well-known that in order to construct global solutions to nonlinear kinetic equations it is crucial to derive certain a priori bounds for the solutions by the conservation of energy (see e.g., [2,3,5,12,13]). However, from the analysis of [9] we know that the interaction term $\int_{\mathbb{R}^n}\rho\varphi dx$ leads to a significant difficulty. Due to this reason, global weak solutions in three or two dimensions were constructed only for initial data satisfying a size restriction [9, 14]. For classical solutions, the authors in [10] proved local existence and gave a continuation criterion. In addition, they showed global existence of classical solutions in dimension one. On the other hand, Ha and Lee [7] established global classical solution for a damped Vlasov-Klein-Gordon system with small data by the adaptation of the method in [1,4]. So, global existence of classical or weak solutions for general initial data remains unsolved in two and three dimensions.

In this paper, we prove global existence of weak solutions to the two dimensional system (1)-(2) for general initial data. The main ingredient of our method is a new estimate of the interaction term $\int_{\mathbb{R}^2}\rho\varphi dx$. We denote by $L^1_{kin}(\mathbb{R}^4)$ the Banach space consisting of measurable functions $f(x,v)$ defined on $\mathbb{R}^2\times\mathbb{R}^2$ with norm
$$\|(\cdot)\|_{1,kin} = \int_{\mathbb{R}^2\times\mathbb{R}^2} \sqrt{1+|v|^2}(\cdot)dvdx < \infty.$$
Then, our main result is as follows.

Theorem 1.1. *Suppose the triple* $(f^{in}, \varphi_0^{in}, \varphi_1^{in})$ *satisfies*

$$0 \le f^{in} \in L^1_{kin}(\mathbb{R}^4) \cap L^p(\mathbb{R}^4), \quad \varphi_0^{in} \in H^1(\mathbb{R}^2), \quad \varphi_1^{in} \in L^2(\mathbb{R}^2)$$

for some $p \in [2, \infty]$. *Then there exists a global weak solution* (f, φ) *in the sense of distributions to system* (1)-(2) *such that*

$$f(t) \in L^\infty([0,\infty), L^p(\mathbb{R}^4)) \cap L^\infty([0,\infty), L^1_{kin}(\mathbb{R}^4)),$$

$$\varphi \in L^\infty([0,\infty), H^1(\mathbb{R}^2)), \ \ \partial_t\varphi \in L^\infty([0,\infty), L^2(\mathbb{R}^2)).$$

Moveover, the following holds:

(i) *The initial condition* (2) *is satisfied in the following sense*: *the mapping*

$$F : t \mapsto (f(t), \varphi(t), \partial_t\varphi(t)), \quad [0,\infty) \to L^2(\mathbb{R}^4) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$$

is weakly continuous with $F(0) = (f^{in}, \varphi_0^{in}, \varphi_1^{in})$.

(ii) *The total energy* $E(t)$ *is bounded on* $[0, +\infty)$, $\partial_t\rho + \nabla_x \cdot j = 0$ *holds in the sense of distributions and the mass is conserved, i.e.,*

$$\int_{\mathbb{R}^2} \rho(t)dx = \int_{\mathbb{R}^2} \rho(0)dx \quad \textit{for a.e.} \quad t > 0.$$

In this paper, we denote by C a generic positive constant independent of time t, when a constant depends on another parameter, we write it as $C(\cdot)$. We also denote by $B_t(x) = \{y \in \mathbb{R}^2 : |y - x| < t\}$ the disc centered at x and with radius t. The symbol $*$ denotes the convolution referring to x.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. First, we recall some well-known results for the linear Klein-Gordon equation $\partial_t^2\varphi - \Delta_x\varphi + \varphi = u(t,x)$. We know its fundamental solution in two dimensions has the following form [8]

$$\frac{1}{2\pi}\frac{\cos\sqrt{t^2 - |x|^2}}{\sqrt{t^2 - |x|^2}}\chi_{B_t(0)}(x),$$

where $\chi_{B_t(0)}(x)$ is the characteristic function of the disc $B_t(0)$. Consequently, the unique solution to the linear Klein-Gordon equation with initial data $\varphi(0,x) = \varphi_0^{in}, \partial_t\varphi(0,x) = \varphi_1^{in}$ can be represented by

$$\begin{aligned}
\varphi(t,x) = &\frac{1}{2\pi}\int_{|x-y|<t}\frac{\cos\sqrt{t^2 - |x-y|^2}}{\sqrt{t^2 - |x-y|^2}}\varphi_1^{in}dy \\
&+ \frac{1}{2\pi}\frac{\partial}{\partial t}\left[\int_{|x-y|<t}\frac{\cos\sqrt{t^2 - |x-y|^2}}{\sqrt{t^2 - |x-y|^2}}\varphi_0^{in}dy\right] \\
&- \frac{1}{2\pi}\int_0^t\int_{|x-y|<t-\tau}\frac{\cos\sqrt{(t-\tau)^2 - |x-y|^2}}{\sqrt{(t-\tau)^2 - |x-y|^2}}u(\tau,y)dyd\tau.
\end{aligned}$$

Let $\gamma \in \mathbb{N}_0^2$ be any multi-index and ∂^γ be the spatial derivative, by a straightforward calculation we get

$$\begin{aligned}(1)\quad &\|\partial^\gamma \varphi(t)\|_\infty \\ &\le C(1+t)^2\bigg(\|\partial^\gamma \varphi_0^{in}\|_\infty + \|\nabla\partial^\gamma \varphi_0^{in}\|_\infty + \|\partial^\gamma \varphi_1^{in}\|_\infty + \int_0^t \|\partial^\gamma u(\tau)\|_\infty d\tau\bigg).\end{aligned}$$

Proof of Theorem 1.1. The proof is carried out in several steps.

Step 1. The approximate equations. For any $n \in \mathbb{N}$, let $0 \le \delta_n \in C_c^\infty(\mathbb{R}^2)$ be a mollifier such that $\int \delta_n(x)dx = 1$ and $\delta_n(-x) = \delta_n(x)$, and let $\chi_n(x) \in C_c^\infty(\mathbb{R}^2)$ be a cutoff function such that $\chi_n = 1$ on $\{x : |x| \le n\}$ and $\chi_n = 0$ on $\{x : |x|^2 > n^2+1\}$. Choose $0 \le f_n^{in}(x,v) \in C_c^\infty(\mathbb{R}^4)$ such that

$$\|f_n^{in}\|_p \le \|f^{in}\|_p, \quad f_n^{in} \to f^{in} \text{ in } L^1_{kin} \cap L^p(\mathbb{R}^4),\ n \to \infty.$$

We construct approximate equations of system (1)-(2) as follows:

$$(2)\qquad \begin{cases} \partial_t f_n + \widehat{v}\cdot\partial_x f_n - \partial_x\varphi_n\cdot\partial_v f_n = 0, \quad f_n(0,x,v) = f_n^{in}(x,v), \\ \partial_t^2\varphi_n - \Delta_x\varphi_n + \varphi_n = -\rho_n * \delta_n * \delta_n, \\ \varphi_n(0,x) = \varphi_{0,n}^{in}(x), \quad \partial_t\varphi_n(0,x) = \varphi_{1,n}^{in}(x), \end{cases}$$

where $\varphi_{0,n}^{in}(x) = (\varphi_0^{in}\chi_n) * \delta_n * \delta_n$, $\varphi_{1,n}^{in}(x) = (\varphi_1^{in}\chi_n) * \delta_n * \delta_n$. Hence,

$$\varphi_{0,n}^{in}(x) \to \varphi_0^{in} \text{ in } H^1(\mathbb{R}^2), \quad \varphi_{1,n}^{in}(x) \to \varphi_1^{in} \text{ in } L^2(\mathbb{R}^2),\ n \to \infty.$$

For the approximate equations, the charge is still conserved. Hence for any $\gamma \in \mathbb{N}_0^2$ with $|\gamma| \le 2$, (1) concludes that $\|\partial^\gamma\varphi_n(t)\|_\infty$ is uniformly bounded for t on bounded interval. This is sufficient to reveal that the approximate system (2) has a unique global smooth solution (f_n, φ_n) (for the details we refer the readers to [9]). We also have $\|f_n(t)\|_p = \|f_n^{in}\|_p \le \|f^{in}\|_p$ and $\partial_t\rho_n + \nabla_x\cdot j_n = 0$ with $\rho_n = \int_{\mathbb{R}^2} f_n dv$ and $j_n = \int_{\mathbb{R}^2} \widehat{v} f_n dv$.

Step 2. Uniform bounds. It is worth noting that the energy of the approximate equations (2) is not conserved any more. However, if we denote by $\widetilde{\varphi}_n$ the solution to the problem

$$(3)\qquad \begin{cases} \partial_t^2\widetilde{\varphi}_n - \Delta_x\widetilde{\varphi}_n + \widetilde{\varphi}_n = -\rho_n * \delta_n, \\ \widetilde{\varphi}_n(0,x) = (\varphi_0^{in}\chi_n) * \delta_n, \quad \partial_t\widetilde{\varphi}_n(0,x) = (\varphi_0^{in}\chi_n) * \delta_n, \end{cases}$$

then we directly have the following lemma:

Lemma 2.1. *Let (f_n, φ_n) be the solution of (2) and $\widetilde{\varphi}_n$ be the solution of (3), then we have $\varphi_n = \widetilde{\varphi}_n * \delta_n$ and the energy*

$$\begin{aligned}\widetilde{E}_n(t) = &\int_{\mathbb{R}^2\times\mathbb{R}^2} \sqrt{1+|v|^2} f_n dxdv \\ &+ \frac{1}{2}\int_{\mathbb{R}^2} (|\partial_t\widetilde{\varphi}_n|^2 + |\partial_x\widetilde{\varphi}_n|^2 + |\widetilde{\varphi}_n|^2)dx + \int_{\mathbb{R}^2} \rho_n\varphi_n dx\end{aligned}$$

is constant.

The main difficulty occurs from the interaction term $\int_{\mathbb{R}^2}\rho_n\varphi_n dx$. Then the following lemma will help us to overcome the difficulty and obtain the desired uniform bounds.

Lemma 2.2. *Suppose that (f_n,φ_n) is the solution of (2) and $\widetilde{\varphi}_n$ is the solution of (3), for any $q\in(1,2)$ and $\epsilon>0$ we have*

$$\left|\int_{\mathbb{R}^2}\rho_n\varphi_n dx\right| \leq \frac{C^2(q)\|f^{in}\|_q^{\frac{2q}{3q-2}}}{4\epsilon}\|f_n(t)\|_{1,kin}^{\frac{4q-4}{3q-2}}+\epsilon C(\beta)\left(\|\widetilde{\varphi}_n(t)\|_2^2+\|\partial_x\widetilde{\varphi}_n(t)\|_2^2\right), \tag{4}$$

where $C(q)=\pi^{\frac{q-1}{3q-2}}\cdot(\frac{3q-2}{2q-2})\cdot(\frac{q-1}{2q})^{\frac{q}{3q-2}}$, $C(\beta)=\left[\beta^{1-\frac{2}{\beta}}(\beta-1)^{-1+\frac{1}{\beta}}(\frac{\beta-2}{8\pi})^{\frac{1}{2}-\frac{1}{\beta}}\right]^2$ *and* $\beta=\frac{3q-2}{q-1}$.

In addition, there exists a constant $C>0$ such that

$$\max\{\|f_n(t)\|_{1,kin},\ \|\varphi_n(t)\|_2,\ \|\partial_t\varphi_n(t)\|_2,\ \|\partial_x\varphi_n(t)\|_2\}\leq C,\quad t\geq 0. \tag{5}$$

Proof. For any $r>0$

$$\begin{aligned}\rho_n(t,x)&=\int_{|v|\leq r}f_n(t,x,v)dv+\int_{|v|>r}f_n(t,x,v)dv\\&\leq(\pi r^2)^{\frac{q-1}{q}}\|f_n(t,x,\cdot)\|_q+r^{-1}\int_{\mathbb{R}^2}\sqrt{1+|v|^2}f_n dv.\end{aligned}$$

Setting $r=\Big(\frac{\int_{\mathbb{R}^2}\sqrt{1+|v|^2}f_n dv}{\frac{2(q-1)}{q}\pi^{\frac{q-1}{q}}\|f_n(t,x,\cdot)\|_q}\Big)^{\frac{q}{3q-2}}$ we obtain

$$\rho_n(t,x)\leq C(q)\|f_n(t,x,\cdot)\|_q^{\frac{q}{3q-2}}\left(\int_{\mathbb{R}^2}\sqrt{1+|v|^2}f_n dv\right)^{\frac{2q-2}{3q-2}},$$

where $C(q)=\pi^{\frac{q-1}{3q-2}}\cdot(\frac{3q-2}{2q-2})\cdot(\frac{q-1}{2q})^{\frac{q}{3q-2}}$. If we take this estimate to the power $\frac{3q-2}{2q-1}$ and integrate in x, then Hölder's inequality deduces

$$\|\rho_n(t)\|_{\frac{3q-2}{2q-1}}\leq C(q)\|f_n^{in}\|_q^{\frac{q}{3q-2}}\|f_n(t)\|_{1,kin}^{\frac{2q-2}{3q-2}}\leq C(q)\|f^{in}\|_q^{\frac{q}{3q-2}}\|f_n(t)\|_{1,kin}^{\frac{2q-2}{3q-2}}. \tag{6}$$

Combining Hölder's inequality, (6) with Cauchy's inequality with ϵ, we deduce

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\rho_n\varphi_n dx\right|&\leq\|\rho_n(t)\|_{\frac{3q-2}{2q-1}}\cdot\|\widetilde{\varphi}_n(t)\|_{\frac{3q-2}{q-1}}\\&\leq C(q)\|f^{in}\|_q^{\frac{q}{3q-2}}\|f_n(t)\|_{1,kin}^{\frac{2q-2}{3q-2}}\|\widetilde{\varphi}_n(t)\|_{\frac{3q-2}{q-1}}\\&\leq\frac{C^2(q)\|f^{in}\|_q^{\frac{2q}{3q-2}}}{4\epsilon}\|f_n(t)\|_{1,kin}^{\frac{4q-4}{3q-2}}+\epsilon\|\widetilde{\varphi}_n(t)\|_{\frac{3q-2}{q-1}}^2.\end{aligned} \tag{7}$$

By [11, Theorem 8.5(ii)] and noting $\beta > 4$ we have

$$\|\widetilde{\varphi}_n(t)\|_\beta^2 \le C(\beta)\Big(\|\widetilde{\varphi}_n(t)\|_2^2 + \|\partial_x\widetilde{\varphi}_n(t)\|_2^2\Big). \tag{8}$$

Inserting (8) into (7) the assertion (4) follows. Taking $\epsilon = \frac{1}{4C(\beta)}$ and using Lemma 2.1, it is easy to verify that

$$\begin{aligned}
&\widetilde{E}_n(0) = \widetilde{E}_n(t)\\
&\ge \|f_n(t)\|_{1,kin} + \frac{1}{2}\Big(\|\widetilde{\varphi}_n(t)\|_2^2 + \|\partial_t\widetilde{\varphi}_n(t)\|_2^2 + \|\partial_x\widetilde{\varphi}_n(t)\|_2^2\Big) - \Big|\int_{\mathbb{R}^2}\rho_n\varphi_n dx\Big|\\
&\ge \|f_n(t)\|_{1,kin} - C(\beta)C^2(q)\|f^{in}\|_q^{\frac{2q}{3q-2}}\|f_n(t)\|_{1,kin}^{\frac{4q-4}{3q-2}} + \frac{1}{2}\|\partial_t\widetilde{\varphi}_n(t)\|_2^2\\
&\quad + \frac{1}{4}\Big(\|\widetilde{\varphi}_n(t)\|_2^2 + \|\partial_x\widetilde{\varphi}_n(t)\|_2^2\Big)\\
&\ge \|f_n(t)\|_{1,kin} - C(\beta)C^2(q)\|f^{in}\|_1^{\frac{2(p-q)}{(p-1)(3q-2)}}\|f^{in}\|_p^{\frac{2p(q-1)}{(p-1)(3q-2)}}\|f_n(t)\|_{1,kin}^{\frac{4q-4}{3q-2}}\\
&\quad + \frac{1}{2}\|\partial_t\widetilde{\varphi}_n(t)\|_2^2 + \frac{1}{4}\Big(\|\widetilde{\varphi}_n(t)\|_2^2 + \|\partial_x\widetilde{\varphi}_n(t)\|_2^2\Big)\\
&\ge \|f_n(t)\|_{1,kin} - C(\beta)C^2(q)\|f^{in}\|_1^{\frac{2(p-q)}{(p-1)(3q-2)}}\|f^{in}\|_p^{\frac{2p(q-1)}{(p-1)(3q-2)}}\|f_n(t)\|_{1,kin}^{\frac{4q-4}{3q-2}}\\
&\quad + \frac{1}{2}\|\partial_t\varphi_n(t)\|_2^2 + \frac{1}{4}\Big(\|\varphi_n(t)\|_2^2 + \|\partial_x\varphi_n(t)\|_2^2\Big).
\end{aligned}$$

Then, since $0 < \frac{4q-4}{3q-2} < 1$ we can easily deduce (5). □

Step 3. The weak limits. Besides (5), we also have that $\|f_n(t)\|_2 = \|f_n^{in}\|_2$ is bounded. Applying diagonal argument (up to a subsequence), we get that there are $f(t) \in L^\infty([0,\infty), L^2(\mathbb{R}^4)), \varphi \in L^\infty([0,\infty), H^1(\mathbb{R}^2))$ with $\partial_t\varphi \in L^\infty([0,\infty), L^2(\mathbb{R}^2))$ such that for any $T > 0$,

$$f_n \rightharpoonup f \text{ in } L^2([0,T]\times\mathbb{R}^4), \quad \varphi_n \rightharpoonup \varphi \text{ in } H^1([0,T]\times\mathbb{R}^2),$$

$$\partial_t\varphi_n \rightharpoonup \partial_t\varphi \text{ in } L^2([0,T]\times\mathbb{R}^2), \quad \partial_x\varphi_n \rightharpoonup \partial_x\varphi \text{ in } L^2([0,T]\times\mathbb{R}^2).$$

On the one hand, (6) and (5) imply $\|\rho_n(t)\|_{\frac{3q-2}{2q-1}}$ is bounded. In consideration of boundedness of kinetic energy $\|f_n(t)\|_{1,kin}$ we may assume up to a subsequence that $\rho_n \rightharpoonup \rho$ in $L^{\frac{3q-2}{2q-1}}([0,T]\times\mathbb{R}^2)$. Based on this result and usual method we can show $\rho_n * \delta_n * \delta_n \to \rho$ in distributional sense. Hence φ satisfies the inhomogeneous Klein-Gordon equation in (1) in the sense of distributions.

Step 4. Momentum averaging. This step is devoted to dealing with the Vlasov equation in (1), where continuity of the nonlinear term $\partial_x\varphi\cdot\partial_v f$ is the crucial part in the proof. Actually, we need to show for any $\zeta \in C_c^\infty((0,\infty)\times\mathbb{R}^4)$

$$\int_0^\infty dt\int_{\mathbb{R}^2\times\mathbb{R}^2} f_n\partial_x\varphi_n\cdot\partial_v\zeta dxdv \to \int_0^\infty dt\int_{\mathbb{R}^2\times\mathbb{R}^2} f\partial_x\varphi\cdot\partial_v\zeta dxdv. \tag{9}$$

Since the tensor product $C_c^\infty((0,\infty)\times\mathbb{R}^2)\bigotimes C_c^\infty(\mathbb{R}^2)$ is dense in $C_c^\infty((0,\infty)\times\mathbb{R}^4)$, it is sufficient to show (9) for $\zeta(t,x,v)=\zeta_1(t,x)\zeta_2(v)$ with $\zeta_1\in C_c^\infty((0,\infty)\times\mathbb{R}^2)$ and $\zeta_2\in C_c^\infty(\mathbb{R}^2)$. Suppose that supp$\zeta_1\subset[0,T]\times B_r(0)$ and supp$\zeta_2\subset B_r(0)$ for some $T>0$ and $r>0$, in consideration of the weak convergence $\partial_x\varphi_n\rightharpoonup\partial_x\varphi$ in $L^2([0,T]\times\mathbb{R}^2)$, it remains to prove that for $\xi(v)=\partial_v\zeta_2(v)\in C_c^\infty(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2}\xi(v)f_n(\cdot,\cdot,v)dv\to\int_{\mathbb{R}^2}\xi(v)f(\cdot,\cdot,v)dv\quad\text{in}\quad L^2([0,T]\times B_r(0)).\tag{10}$$

For this purpose, we take $\psi(t)\in C_c^\infty(R)$ such that $0\le\psi\le1$, $\psi=1$ on $[\varepsilon,T]$ for small $\varepsilon>0$. Setting $f_{n,\alpha}=\frac{f_n}{1+\alpha f_n}$ for fixed small constant $\alpha>0$ and defining $\widetilde{f}_{n,\alpha}=f_{n,\alpha}(t,x,v)\psi(t)$. Then we have

$$\partial_t\widetilde{f}_{n,\alpha}+\widehat{v}\cdot\partial_x\widetilde{f}_{n,\alpha}=w_0+\mathrm{div}_v w_1$$

in distributional sense, where $w_0=\frac{f_n}{1+\alpha f_n}\psi'$, $w_1=\partial_x\varphi_n f_{n,\alpha}\psi$. Since $\widetilde{f}_{n,\alpha}$, w_0, w_1 are bounded in $L^2([0,\infty)\times\mathbb{R}^2\times B_r(0))$, the velocity averaging lemma in [2,6,13] deduces that $\int_{\mathbb{R}^2}\xi(v)\widetilde{f}_{n,\alpha}(\cdot,\cdot,v)dv$ is bounded in $H^{\frac14}(\mathbb{R}\times\mathbb{R}^2)$. By the compact embedding theorem, we have $\int_{\mathbb{R}^2}\frac{f_n}{1+\alpha f_n}\xi(v)dv$ is relatively compact in $L^2([0,T]\times B_r(0))$ for $r>0$. Then, sending $n\to\infty$ and $\alpha\to0$ in succession, we can obtain (10) by renormalization method ([2,13]).

Thus, (f,φ) is a global weak solution to the system (1)-(2).

Step 5. The proof of (i) *and* (ii). (i) Multiplying both sides of the Vlasov equation in system (2) by a test function $\eta(x,v)\in C_c^\infty(\mathbb{R}^4)$, and integrating in t,x and v, we find

$$\begin{aligned}&\int_{\mathbb{R}^2\times\mathbb{R}^2}\eta f_n dxdv\\&=\int_{\mathbb{R}^2\times\mathbb{R}^2}\eta f_n^{in}dxdv+\int_0^t\int_{\mathbb{R}^2\times\mathbb{R}^2}(\widehat{v}\cdot\partial_x\eta-\partial_x\varphi_n\cdot\partial_v\eta)f_n(\tau,x,v)dxdvd\tau.\end{aligned}$$

The convergence of f_n and φ_n is enough to pass to limits in this equation. Dropping n in this equation we can define a time dependent distribution $f(t)\in\mathcal{D}'(\mathbb{R}^4)$. Hence $f(t)$ is continuous and satisfies the initial condition $f(0)=f^{in}$ in distributional sense. By a density argument, we can show that $t\mapsto f(t):[0,\infty)\to L^2(\mathbb{R}^4)$ is weak continuous. Similarly, we can show the desired continuity for φ.

For (ii), the boundedness of the total energy follows directly from Lemma 2.2 and its proof. The proofs of others are the same as these in [9,14]. □

Remark 2.3. In the two dimensional case, we can establish the estimate (5) due to (8), which is crucial in constructing weak solutions by compactness. Nevertheless, this kind of estimate is not available in three dimensions by the method of this paper.

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